Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

WAVE PHENOMENA, HUYGENS' PRINCIPLE AND 3-D ELASTIC WAVES

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Abstract

We briefly discuss some important results on Huygens' principle in the sense of Hadamard's minor premise. We also indicate a negative result we obtained concerning the above principle and the system of elastic waves in the presence of an impurity of the medium. We also mention some open problems in the subject as well as possible interesting generalizations.

I. INTRODUCTION

It is a fact of nature that waves propagate in very different manners depending whether the spatial dimension is **two** or **three**. Suppose that a little stone falls in water at a certain point x_0, then, we observe that the initial ripple on a circle around x_0 will be followed by subsequent ripples. Evidently, if x_1 is another point (not very far from x_0) then x_1 will be hit by **residual waves**. In three dimensions, the situation is very different. If you produce a bang, there **will be no after-sounding**. We have a pure propagation without residual waves. The above examples suggest us that all wave phenomena could be divided into two classes: Those for which the **Huygens' principle** holds (that is, there are no after effects) and the ones for which the principle fails (that is, there are **always** some after effects).

Christian Huygens wrote his Traité de la Lumiere in 1690 where he discussed the propagation of light based on a new principle, which nowadays is known as **Huygens' principle.** Unfortunately it took quite a long time before Huygens ideas on wave propagation receive adequate recognition. Around 1818, A. Fresnel by using the Huygens' principle, discovered significant facts on quantitative wave optics and found the real cause of diffraction. Several difficulties encountered in Fresnel's theory were overcome by G. Kirchhoff who used Helmholtz's formulation of Huygens' principle for monochromatic phenomena. Since the original formulation, the term Huygens' principle has suffered an evolution. We shall follow J. Hadamard [5] who described the principle in the following form nowadays known as **Hadamard's minor premise**: "If at the instant $t = t_0$ or more precisely, in the short interval $t_0 - \varepsilon \leq t \leq t_0 + \varepsilon$, we produce a sound disturbance localized in the immediate neighborhood of a point x_0, the effect at the subsequent instant $t = t_1$ is localized in a very thin spherical shell with center x_0 and radius $c(t_1 - t_0)$ where c is the velocity of sound". In this Lecture, we describe several aspects and contributions (old and new) concerning Hadamard's minor premise.

This is an extended version of an invited Lecture given by the first author at the X ELAM in Cordoba, Argentina. We would like to thank the Organizing Committe for the kind invitation and for the pleasent hospitality during the time that the event was held.

In Section 2 we give the precise definition of a Huygens' operator and discuss some illustrative examples. In Section 3 we outline the proof of a recent (negative) result we obtained [12] concerning the system of elastic waves in the presence of an impurity. Some open problems on the subject are also indicated.

II. SOME EXAMPLES AND HADAMARD's PROBLEM

In order to simplify our discussion we shall restrict our attention to second order differential operators. Let L be a linear partial differential operator of hyperbolic type

(1) $$L[u] = \frac{\partial^2 u}{\partial t^2} - L_0[u]$$

where L_0 is a linear elliptic operator, say for instance, $L_0[u] = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^n a_j(x) \frac{\partial u}{\partial x_j} + b(x)u$. The bicharacteristics of L issuing from (x_0, t_0) are assume to generate the characteristic conoid K with vertex at (x_0, t_0). We say that u satisfies the **Huygens' principle** or, that L is a **Huygens' operator**, if the solution u of the Cauchy problem $L[u] = 0$ with respect to any space like hypersurface S depends (at the point (x_0, t_0)) only on the Cauchy data $u\big|_S$ and $\frac{\partial u}{\partial \eta}\big|_S$ taken on the intersection of K with S.

Example 1. (Scalar Wave Equation) Consider the Cauchy problem

(2) $$\begin{cases} u_{tt} - a^2 \Delta u = 0 \\ u(x,0) = f(x), \qquad u_t(x,0) = g(x) \end{cases}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and n is **odd** ≥ 3. Also $a > 0$, $f \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$ and $g \in C^{\frac{n+1}{2}}(\mathbb{R}^n)$. In this case $S = \mathbb{R}^n$. It is well known that, the solution is given (for $t > 0$) by

$$u(x,t) = c_n \left[\frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} t^{n-2} \int_{|y|=1} f(x + aty) dS_y + \right.$$
$$\left. + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} t^{n-2} \int_{|y|=1} g(x + aty) dS_y \right]$$

(see [3]) where $c_n = \frac{1}{3 \cdot 5 \cdot \dots \cdot (n-2) w_n a^2}$ and $w_n = \int_{|y|=1} dS_y$.

Observe that at each (x_0, t_0) the solution of (2) depends only on the spherical means of f and g respectively. In this case the characteristic conoid with vertex at (x_0, t_0) is

$$K: \quad |x - x_0|^2 - a^2 (t - t_0)^2 = 0$$

It follows that the solution u of (2) satisfies the Huygens' principle.

It is easy to prove that if n is **even** ≥ 2 then the Huygens' principle is **false** for the solutions of (2).

Example 2. (System of Elastic Waves) Let $a > b > 0$. Consider the system

$$(3) \qquad \begin{cases} u_{tt} - b^2\Delta u - (a^2 - b^2)\text{Grad}\,(\text{Div}\ u) = 0 \\ u\,(x,0) = F(x), \quad u_t(x,0) = G(x) \end{cases}$$

where $x \in \mathbb{R}^3$, t denotes time, $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$, Δ denotes the (vector) Laplace operator, that is $\Delta u = (\Delta u^1, \Delta u^2, \Delta u^3)$. For simplicity we assume that each component of F and G belongs to $C_0^\infty(\mathbb{R}^3)$, that is, the space of functions of class C^∞ with compact support. It is easy to verify that the solution u of (3) can be written as the superposition of two waves

$$(4) \qquad u = \text{Grad}\ v + \text{Curl}\ w$$

where v and w are the solutions of the following initial value problems:

$$(5) \qquad \begin{cases} v_{tt} - a^2\Delta v = 0 \qquad \text{(scalar wave equation)} \\ v\,(x,0) = F_1(x),\ v_t(x,0) = G_1(x), \quad x \in \mathbb{R}^3,\ t \in \mathbb{R} \end{cases}$$

$$(6) \qquad \begin{cases} w_{tt} - b^2\Delta w = 0 \qquad \text{(vector wave equation)} \\ w\,(x,0) = F_2(x),\ w_t(x,0) = G_2(x), \quad x \in \mathbb{R}^3,\ t \in \mathbb{R} \end{cases}$$

where F_j and G_j are obtain using the decompositions $F = \text{Grad}\ F_1 + \text{Curl}\ F_2$ and $G = \text{Grad}\ G_1 + \text{Curl}\ G_2$ respectively. As we already saw in example 1, v and w do satisfy the Huygens' principle. Using (4) we deduce that the solution u of (3) satisfies the Huygens' principle.

Other Examples. 3) **Maxwell's equations** (propagation of electromagnetic waves) [8], 4) Some modified wave equations on special homogeneous spaces [6]. Clearly, if the solution u satisfies the Huygens' principle, then, the following property will be satisfy: If the initial data have compact support then, the solution u vanishes identically in a (forward) cone (and also in a backward cone). For example, if the initial data have compact support contained in the ball $\{|x| \leq M\}$ then the solution u of Example 1 will vanish identically in the space-time cone

$$\left\{(x,t), \quad |x| \leq a|t| - M, \quad |t| \geq \frac{M}{a}\right\}$$

In Example 2, if all components of F and G have support contained in $\{|x| \leq M\}$, then $u \equiv 0$ in the cone

$$\left\{(x,t), \quad |x| \leq b|t| - M, \quad |t| \geq \frac{M}{b}\right\}$$

because $a > b > 0$.

Definition. If L is a Huygens operator, then any other operator $\widetilde{L}$ obtained from L by one of the following transformations: 1) Gauge transformations; 2) conformal transformations or 3) non-singular transformations of the independent variables, will be said to be **essentially equivalent to** L. If a Huygens operator $\widetilde{L}$ is essentially equivalent to a wave operator (like in Examples 1) or 2)) then $\widetilde{L}$ is called a **trivial operator**. J. Hadamard conjecture that every Huygens operator was trivial, but his conjecture turned out to be **false.** A famous counterexample was given by K. Stellmacker [14].

Problem 1. Can we find all Huygens operators?

J. Hadamard found a criterion for solving the above problem: Huygens' principle holds for u if and only if a certain two-point field $V = V(x, y)$ vanishes identically. This V is called **coefficient of the Logarithmic part of Hadamard's elementary solution of the formal adjoint.** It turns out that when V is expanded as a formal power series, the coefficients $V_j = V_j(x, y)$ obey a recursive first order system of geodesic propagation equations. The criterion is necessary and sufficient but **highly implicit**. Several authors contributed to the subject trying to develop more explicit necessary conditions for the validity of Huygens' principle. See [9], [4] and the references therein.

Despite all of the above contributions the above problem remains unsolved. Perhaps a simpler problem may be more accessible.

Problem 2. Suppose that an operator L of type (1) is a Huygens operator, can we characterize a class of linear perturbations P so that $L + P$ ceases to be a Huygens operator?

In the next section we briefly discuss an answer for Problem 2 for small (linear) perturbations of the system of elastic waves.

III. PERTURBED ELASTIC WAVES

In this section we shall consider "small" perturbations of the Example 2 given in Section II. Our aim is to present a simple proof showing that the Huygens' principle fails for such class of perturbations.

Let us define the operator P as follows: Let $\vec{h} : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$

$$\begin{aligned} P\vec{h}(x,t) = &\frac{1}{4\pi b^2} \int_{|z|\leq bt} |z|^{-1} \left\{ \vec{h}\left(x+z, t-\frac{|z|}{b}\right) - |z|^{-2} z \left[z \bullet \vec{h}\left(x+z, t-\frac{|z|}{b}\right)\right]\right\} dz + \\ &+ \frac{1}{4\pi a^2} \int_{|z|\leq at} |z|^{-3} z \left[z \bullet \vec{h}\left(x+z, t-\frac{|z|}{a}\right)\right] dz + \\ &+ \frac{1}{4\pi} \int_0^t s \int_{bs\leq|z|\leq as} |z|^{-3} \left[3z|z|^{-2}\left\{z \bullet \vec{h}(x+z, t-s)\right\} - \vec{h}(x+z,t-s)\right] dz\, ds \end{aligned}$$

where $a > b > 0$ and the dot $\bullet$ denotes the usual inner product in $\mathbb{R}^3$. It can be shown that P is actually the following operator:

$$P\vec{h}(x,t) = \int_0^t R \star \vec{h}\, ds$$

where R denotes the Riemann matrix associated with the free system (3) and $\star$ denotes spatial convolution. We have the following result.

Theorem 1. Let $G(x) = (G_1(x), G_2(x), G_3(x))$ with $G_j \in C_0^\infty(\mathbb{R}^3)$ and **supp** $G_j \subseteq \{x \in \mathbb{R}^3,\ |x| \leq M\}$. Let $v = v(x,t)$ be the solution of (3) with $F(x) \equiv 0$. Let $q: \mathbb{R}^3 \to \mathbb{R}$ satisfying the following conditions:

a) q is "smooth", except at most in a finite number of points in $\mathbb{R}^3$ and $q(x) \geq 0$ for all x where it is defined.

b) $q \in L^2(\mathbb{R}^3) \cap W^{1,\frac{2p}{2-p}}(\mathbb{R}^3)$ for some $1 < p < 2$.

c) There exists γ, $0 < \gamma < 1$ and $\alpha > 1$ such that

$$\sup_{x \in R^3} \int_{|z| \leq a\gamma} |z|^{-2\alpha} q(x+z)dz < +\infty$$

d) There exists at least one point (x_0, t_0) belonging to the cone

$$\left\{(x,t), \quad |x| \leq b|t| - M, \quad |t| \geq \frac{M}{b}\right\}$$

such that $P(qv)(x_0,t_0) \neq (0,0,0)$ where P is given by (7). Then the solution $u^\varepsilon = u^\varepsilon(x,t)$ of the initial value problem

$$(8) \qquad \begin{cases} u_{tt} - b^2 \Delta u - (a^2 - b^2)\text{Grad}\,(\text{Div}\ u) + \varepsilon q(x) u = 0, \ x \in \mathbb{R}^3,\ t \in \mathbb{R} \\ u\ (x,0) = 0, \qquad u_t(x,0) = G(x) \end{cases}$$

does not vanish identically on the above cone. In particular the (strong) Huygens' principle is not valid for such perturbations.

General Comments on the Proof of Theorem 1.

i) We proceed as follows: the solution u of (8) can be written as follows

$$(9) \qquad \begin{cases} u(x,t) & = v(x,t) - \varepsilon P(qu)(x,t) = \\ & = v(x,t) - \varepsilon P(qv)(x,t) + \varepsilon^2 P(qP(qu))(x,t) \end{cases}$$

Suppose by contradiction that u vanish identically on the above cone, then using (9) with $(x,t) = (x_0, t_0)$ we obtain

$$P(qv)(x_0,t_0)=\varepsilon P(q(P(qu)))(x_0,t_0) \tag{10}$$

because v satisfies the Huygens' principle. Hence, to prove the result would be enough to show that the right hand side of (10) approaches zero as $\varepsilon \to 0$. Estimates on the right hand side of (10) were done using the energy method together with apriori estimates over characteristic cones. Here, the structure of the operator P was explore in detail. Complete proof will appear elsewhere [12]

ii) Observe that assumptions b) and c) have to do with the behavior of q at $+\infty$ and at zero respectively. In case $a = b$ that is, when we consider perturbations of the wave equation (2) it is well known that there exist some singular perturbations for which the Huygens' principle is still valid, see [14]. We suspect that our assumption c) is essential for the conclusion of Theorem 1. Our assumption d) may not be so simple to verify directly due to the complicated structure of the (matrix) Riemann function for (3). However, there are some situations where we can check directly that condition d) holds. Further details can be found in [12].

Some Open Problems.

1) What can be say about Problems 1 or 2 for linear hyperbolic operators of higher order? Say, for instance, for hyperbolic operators in the sense of Garding or Petrowsky. The notion of Huygens' principle is subordinated to the notion of "lacunas" (see [1]) but we don't know satisfactory answers for this problem.

2) Replace the Cauchy problem (in the definition of the Huygens' principle) by a characteristic initial value problem, say, the Goursat problem.

3) What can we say about the Huygens' principle and solutions of semilinear hyperbolic equations? As far as we know only in a very special case Problem 2 has recently been considered [11].

4) The exact validity of the Huygens' principle could be (perhaps) substitute by an "approximate validity", which would allow us to use perturbation techniques. J.J. Duistermat recently suggested that instead of requiring that u vanishes identically on the Huygens' cone, it may be more appropriate saying that u has a **"singularity at the wave front"** (see [2]).

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