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LINEARIZATION OF HOLOMORPHIC MAPPINGS ON INFINITE DIMENSIONAL SPACES

JORGE MUJICA

Introduction

Several authors have obtained linearization theorems for various classes of holomorphic mappings. It seems that the first general result of this kind is due to Mazet [6], who obtained a linearization theorem for holomorphic mappings on locally convex spaces, thus improving previous results of Schottenloher [13] and Ryan [12]. In a recent paper Nachbin and the author [10] gave a new proof and several applications of the Mazet linearization theorem. Very recently Dineen, Galindo, Garcia and Maestre [1] solved two of the problems left open in [10].

By specializing to smaller classes of mappings, the author [8] obtained a linearization theorem for bounded holomorphic mappings, whereas Galindo, Garcia and Maestre [2] obtained a linearization theorem for holomorphic mappings of bounded type. In a very recent paper the author [9] showed that the last two classes of mappings are intimately connected.

In this lecture we begin with a survey of the main results concerning linearization of bounded holomorphic mappings on Banach spaces, taken mainly from [8]. We next apply these results to the study of interpolating sequences. At the end we restrict our attention to the case of the open unit disc and show the connection of the preceding results with the classical Hardy spaces.

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1. Notation and Terminology

The letters E and F always represent complex Banach spaces, and $L(E;F)$ denotes the vector space of all continuous linear operators from E into F. Unless stated otherwise, $L(E;F)$ is endowed with its natural norm topology. We write E' instead of $L(E;\mathbb{C})$ for the dual of E.

Let U be a nonvoid open subset of E. A mapping $f : U \to F$ is said to be *holomorphic* if f is continuous and the function $\lambda \to \psi \circ f(a + \lambda b)$ is holomorphic on an open neigborhood of the origin in $\mathbb{C}$ for every $a \in U, b \in F$ and $\psi \in F'$. Let $H(U;F)$ denote the vector space of all holomorphic mappings from U into F, and let $H^\infty(U;F)$ denotes the subspace of all bounded members of $H(U;F)$. Unless stated otherwise, $H^\infty(U;F)$ is equipped with the sup norm topology. When $F = \mathbb{C}$ we write $H(U)$ instead of $H(U;\mathbb{C})$, and $H^\infty(U)$ instead of $H^\infty(U;\mathbb{C})$. We refer to the author's book [7] for the properties of holomorphic mappings on Banach spaces.

2. Linearization of Bounded Holomorphic Mappings

The following linearization theorem is taken from [8].

2.1. Theorem *[8]. Let U be an open subset of a Banach space E. Then there are a Banach space $G^\infty(U)$ and a mapping $\delta_U \in H^\infty(U; G^\infty(U))$ with the following universal property: For each Banach space F and each mapping $f \in H^\infty(U; F)$, there is a unique operator $T_f \in L(G^\infty(U); F)$ such that $T_f \circ \delta_U = f$. The mapping*

$$f \in H^\infty(U; F) \to T_f \in L(G^\infty(U); F)$$

is an isometric isomorphism. These properties characterize $G^\infty(U)$ uniquely up to an isometric isomorphism.

The following proposition shows the connection between E and $G^\infty(U)$.

2.2. Proposition *[8]. Let U be a bounded open subset of a Banach space E. Then E is topologically isomorphic to a complemented subspace of $G^\infty(U)$.*

The following proposition shows the connection between properties of a mapping $f \in H^\infty(U; F)$ and properties of the corresponding operator $T_f \in L(G^\infty(U); F)$.

2.3. Proposition *[8]. Let E and F be a Banach spaces, and let U be an open subset of E.*

(a) The range of a mapping $f \in H^\infty(U; F)$ is contained in a finite dimensional subspace of F if and only if the corresponding operator $T_f \in L(G^\infty(U); F)$ has finite rank.

(b) The range of a mapping $f \in H^\infty(U; F)$ is relatively compact (resp. relatively weakly compact) if and only if the corresponding operator $T_f \in L(G^\infty(U); F)$ is compact (resp. weakly compact).

We denote by $H^\infty_K(U; F)$ (resp. $H^\infty_{wK}(U; F)$) the subspace of all $f \in H^\infty(U; F)$ which have a relatively compact range (resp. relatively weakly compact range).

3. Another Topology on $H^\infty(U; F)$

Theorem 2.1 tells us that the mapping

$$f \in H^\infty(U; F) \to T_f \in L(G^\infty(U); F)$$

is an isometric isomorphism. If τ_c denotes the compact-open topology, then it is very useful to consider the unique locally convex topology τ_γ on $H^\infty(U; F)$ such that the mapping

$$f \in (H^\infty(U; F), \tau_\gamma) \to T_f \in (L(G^\infty(U); F), \tau_c)$$

is a topological isomorphism. The following theorem furnishes an explicit description of τ_γ in terms of seminorms.

3.1. Theorem *[8]. Let E and F be Banach spaces, and let U be an open subset of E. Then τ_γ is the locally convex topology on $H^\infty(U;F)$ generated by all the seminorms of the form*

$$p(f) = \sup_j \alpha_j ||f(x_j)||,$$

with $(x_j) \subset U$ and $(\alpha_j) \in c_0, \alpha_j > 0$.

With the aid of Theorem 3.1 we can describe $G^\infty(U)$ as follows.

3.2. Theorem *[8][9]. Let U be an open subset of a Banach space E. Then $G^\infty(U)$ consists of all linear functionals $u \in H^\infty(U)'$ of the form*

$$u = \sum_{j=1}^\infty \beta_j \delta_{x_j},$$

with $(x_j) \subset U$ and $(\beta_j) \in l^1$. Moreover,

$$||u|| = \inf \sum_{j=1}^\infty |\beta_j|,$$

where the infimum is taken over all such representations of u.

4. The Approximation Property

We recall that a Banach space E is said to have the *approximation property*, introduced by Grothendieck [4], if the identity operator on E lies in the τ_c-closure of $E' \otimes E$ in $L(E;E)$. The following results give several characterizations of the approximation property in terms of bounded holomorphic mappings.

4.1. Theorem *[8] Let U be a balanced, bounded, open subset of a Banach space E. Then the following conditions are equivalent:*

(a) E has the approximation property.

(b) For each Banach space F, $H^\infty(U) \otimes F$ is τ_γ-dense in $H^\infty(U;F)$.

(c) The inclusion mapping $U \hookrightarrow E$ lies in the τ_γ-closure of $H^\infty(U) \otimes E$ in $H^\infty(U;E)$.

(d) The mapping δ_U lies in the τ_γ-closure of $H^\infty(U) \otimes G^\infty(U)$ in $H^\infty(U, G^\infty(U))$.

(e) $G^\infty(U)$ has the approximation property.

(f) For each Banach space F and each open set $V \subset F$, $H^\infty(V) \otimes E$ is τ_γ-dense in $H^\infty(V;E)$.

(g) For each Banach space F and each open set $V \subset F$, $H^\infty(V) \otimes E$ is norm-dense in $H^\infty_K(V;E)$.

4.2. Proposition *[8]. Let U be an open subset of a Banach space E. Then $H^\infty(U)$ has the approximation property if and only if, for each Banach space F, $H^\infty(U) \otimes F$ is norm-dense in $H^\infty_K(U;F)$.*

Since it is still unknown whether $H^\infty(\Delta)$ has the approximation property, where Δ denotes the open unit disc, Proposition 4.2 may be of some use in this connection.

5. Interpolating Sequences

Let $B(X)$ denote the Banach algebra of all bounded complex-valued functions on a nonvoid set X, with the sup norm. We will need the following result from the book of Garnett [3, Theorem VII.2.2].

5.1. Theorem *[3]. Let A be a subalgebra of $B(X)$ which contains the constants and separates the points of X. Let $x_1, \dots, x_n$ be distinct points of X and let*

$$M = \sup_{\|\eta\|_\infty \leq 1} \inf\{\|f\| : f \in A, f(x_k) = \eta_k \text{ for } k = 1, \dots, n\}.$$

Then for each $\varepsilon > 0$ there are $f_1, \dots, f_n \in A$ such that $f_j(x_k) = \delta_{jk}$ for $j, k = 1, \dots, n$ and

$$\sup_{x \in X} \sum_{j=1}^{n} |f_j(x)| \leq M^2 + \varepsilon.$$

Let U be an open subset of a Banach space E. A sequence $(x_k) \subset U$ is said to be an *interpolating sequence* if the mapping $f \in H^\infty(U) \to (f(x_k)) \in l^\infty$ is surjective. The following theorem gives several characterizations of interpolating sequences.

5.2. Theorem. *Let U be an open subset of a Banach space E. For a sequence $(x_k) \subset U$ the following conditions are equivalent:*

(a) The mapping

$$Q : f \in H^\infty(U) \to (f(x_k)) \in l^\infty$$

is surjective.

(b) There exists $P \in L(l^\infty; H^\infty(U))$ such that $Q \circ P\eta = \eta$ for every $\eta \in l^\infty$.

(c) There exists $P \in L(c_o; H^\infty(U))$ such that $Q \circ P\eta = \eta$ for every $\eta \in c_o$.

(d) The mapping

$$S : (\xi_k) \in l^1 \to \sum_{k=1}^{\infty} \xi_k \delta_{x_k} \in G^\infty(U)$$

is an embedding.

(e) There exists $T \in L(G^\infty(U); l^1)$ such that $T \circ S\xi = \xi$ for every $\xi \in l^1$.

(f) There exists $\varphi \in H^\infty(U; l^1)$ such that $\varphi(x_k) = e_k$ for every $k \in \mathbb{N}$.

(g) For each Banach space F, the mapping

$$Q_F : f \in H^\infty(U; F) \to (f(x_k)) \in l^\infty(F)$$

is surjective.

(h) For each Banach space F, there exists $P_F \in L(l^\infty(F); H^\infty(U; F))$ such that $Q_F \circ P_F y = y$ for every $y \in l^\infty(F)$.

Proof. (a) $\Rightarrow$ (f): By the open mapping theorem, Q is open and therefore induces a topological isomorphism

$$R : H^\infty(U)/KerQ \longrightarrow l^\infty.$$

If $M = ||R^{-1}||$, then

$$M = \sup_{||\eta||_\infty \leq 1} inf\{||f|| : f \in H^\infty(U), f(x_k) = \eta_k \text{ for every } k \in \mathbb{N}\}.$$

By Theorem 5.1, for each $n \in \mathbb{N}$ there are functions $\varphi_{n1}, \ldots, \varphi_{nn} \in H^\infty(U)$ such that $\varphi_{nj}(x_k) = \delta_{jk}$ for $j, k = 1, \ldots, n$, and

$$\sup_{x \in U} \sum_{j=1}^{n} |\varphi_{nj}(x)| \leq M^2 + \frac{1}{n}.$$

Then a compactness argument yields a sequence (φ_j) in $H^\infty(U)$ such that $\varphi_j(x_k) = \delta_{jk}$ for all $j, k \in \mathbb{N}$ and

$$\sup_{x \in U} \sum_{j=1}^{\infty} |\varphi_j(x)| \leq M^2.$$

If we define $\varphi(x) = (\varphi_j(x))$ for $x \in U$, then φ verifies (f).

(f) $\Rightarrow$ (e): By (f) there exists $\varphi \in H^\infty(U; l^1)$ such that $\varphi(x_k) = e_k$ for every $k \in \mathbb{N}$. By Theorem 2.1 there exists $T \in L(G^\infty(U); l^1)$ such that $T \circ \delta_U = \varphi$. Whence T verifies (e).

(e) $\Rightarrow$ (d): This is obvious.

(d) $\Rightarrow$ (a): Since we can readily verify that $S' = Q$, the desired conclusion follows at once (see [14, Theorem 4.7 - A]).

Thus we have shown that conditions (d), (e) and (f) are equivalent to (a).

(e) $\Rightarrow$ (b): Since $T \circ S\xi = \xi$ for every $\xi \in l^1$, it follows that $S' \circ T'\eta = \eta$ for every $\eta \in l^\infty$. We already know that $S' = Q$. If we define $P = T'$, then P verifies (b).

(b) $\Rightarrow$ (c): This is obvious.

(c) $\Rightarrow$ (e): By (c) there exists $P \in L(c_0; H^\infty(U))$ such that $P\eta(x_k) = \eta_k$ for every $\eta = (\eta_k) \in c_o$ and every $k \in \mathbb{N}$. Whence it follows that $P'\delta_{x_k} = e_k$ for every $k \in \mathbb{N}$. If we define $T = P'|G^\infty U$, then T verifies (e).

Thus we have shown that conditions (b) and (c) are also equivalent to (a).

(f) $\Rightarrow$ (h): By (f) there exists $(\varphi_j) \subset H^\infty(U)$ such that $\varphi_j(x_k) = \delta_{jk}$ for all $j, k \in \mathbb{N}$ and $\sup_{x \in U} \sum_j |\varphi_j(x)| < \infty$. Given $y = (y_j) \in l^\infty(F)$, let $P_F y \in H^\infty(U; F)$ be defined by $P_F y(x) = \sum_j \varphi_j(x) y_j$ for $x \in U$. Then P_F verifies (h).

Since the implications (h) $\Rightarrow$ (g) and (g) $\Rightarrow$ (f) are obvious, the proof of the theorem is complete.

6. Connection between $G^\infty(\Delta)$ and $L^1(\partial\Delta)/H_0^1(\Delta)$.

For $1 \leq p < \infty$ let $H^p(\Delta)$ denote the Banach space of all $f \in H(\Delta)$ such that

$$||f||_p = \lim_{r\to 1}(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta)^{1/p} < \infty.$$

Let $1 < p \leq \infty$ and let $f \in H^p(\Delta)$. By a theorem of Fatou, the radial limits $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exist almost everywhere and

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\theta})}{1 - ze^{-i\theta}}d\theta$$

for every $z \in \Delta$. The same conclusion is true for $p = 1$, by a theorem of F. and M. Riesz. Thus for $1 \leq p \leq \infty$ we may identify $H^p(\Delta)$ with a closed subspace of $L^p(\partial\Delta)$, namely

$$H^p(\Delta) = \{f \in L^p(\partial\Delta) : \int_{-\pi}^{\pi} f(e^{i\theta})e^{in\theta}d\theta = 0 \text{ for } n = 1, 2, 3, \ldots\}.$$

If we set $H_0^1(\Delta) = \{f \in H^1(\Delta) : f(0) = 0\}$, then

$$H_0^1(\Delta) = \{f \in L^1(\partial\Delta) : \int_{-\pi}^{\pi} f(e^{i\theta})e^{in\theta}d\theta = 0 \text{ for } n = 0, 1, 2, \ldots\}.$$

Whence it follows that the canonical isometric isomorphism between $L^\infty(\partial\Delta)$ and the dual of $L^1(\partial\Delta)$ induces an isometric isomorphism between $H^\infty(\Delta)$ and the dual of $L^1(\partial\Delta)/H_0^1(\Delta)$. All this is well known and can be found in the book of Hoffman [5].

Since $L^1(\partial\Delta) / H_0^1(\Delta)$ is the unique predual of $H^\infty(\Delta)$, up to an isometric isomorphism (see [3, Theorem V.5.4]), it follows that $G^\infty(\Delta)$ is isometrically isomorphic to $L^1(\partial\Delta)/H_0^1(\Delta)$. But it is illustrative to give a direct proof of this fact.

6.1 .Theorem. *The spaces $G^\infty(\Delta)$ and $L^1(\partial\Delta)/H_0^1(\Delta)$ are isometrically isomorphic.*

Proof. For every $f \in H^\infty(\Delta)$ and $z \in \Delta$ we have that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\theta})}{1 - ze^{-i\theta}}d\theta. \tag{6.1}$$

Let $g : z \in \Delta \to g_z \in L^1(\partial\Delta)$ be defined by $g_z(e^{i\theta}) = (1 - ze^{-i\theta})^{-1}$, and let $\pi : L^1(\partial\Delta) \to L^1(\partial\Delta)/H_0^1(\Delta)$ denote the quotient mapping. If we identify $H^\infty(\Delta)$ with the dual of $L^1(\partial\Delta)/H_0^1(\Delta)$, then (6.1) tells us that $\langle f, \pi g_z\rangle = f(z)$ for every $f \in H^\infty(\Delta)$ and $z \in \Delta$. By using [7, Theorem 8.12] we easily see that $\pi \circ g \in H^\infty(\Delta; L^1(\partial\Delta)/H_0^1(\Delta))$. Then by Theorem 2.1 there exists $T \in L(G^\infty(\Delta); L^1(\partial\Delta)/H_0^1(\Delta))$ such that $T\delta_z = \pi g_z$ for every $z \in \Delta$. Since we can readily verify that the dual mapping T' is the identity on $H^\infty(\Delta)$, we can conclude that T is an isometric isomorphism, as we wanted.

Since the space $L^1(\partial\Delta)/H_0^1(\Delta)$ is weakly sequentially complete and has the Dunford-Pettis property (see [3, Theorem V.5.2] and [11, Corollary 8.1]), it is natural to pose the following problems.

6.2. Problem. Let U be a bounded open subset of a weakly sequentially complete Banach space E. Is $G^\infty(U)$ weakly sequentially complete?

6.3. Problem. Let U be a bounded open subset of a Banach space E with the Dunford-Pettis property. Does $G^\infty(U)$ have the Dunford-Pettis property?

Observe that in each of these problems the hypotheses on E are necessary, in view of Proposition 2.2.

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Instituto de Matematica
Universidade Estadual de Campinas
Caixa Postal 6065
13081 Campinas, SP, Brazil