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Multipliers for $H^p(G)$-spaces

C. W. ONNEWEER

Abstract

In this paper we present some multiplier theorems for Hardy spaces defined on a locally compact Vilenkin group. We discuss how these theorems compare with some known multiplier theorems for Lebesgue and Hardy spaces defined on $\mathbf{R}^n$ and we conclude with a brief description of a few open questions.

In 1947 N. Ya. Vilenkin [10] introduced a class of compact Abelian topological groups G, with G containing a strictly decreasing sequence of open subgroups $(G_n)_0^\infty$ such that $\bigcup_0^\infty G_n = G$ and $\bigcap_0^\infty G_n = \{0\}$. Vilenkin studied the character groups of such groups and proved a number of results, many of them similar to known results for the Fourier series of functions defined on the circle group $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$. An example of a group G as defined by Vilenkin is the dyadic group D_0 consisting of all sequences $(x_n)_0^\infty$, with each $x_n \in \{0, 1\}$ and with addition defined coordinate wise modulo 2. In this case we may take $G_n = \{(x_n) \in D_0 : x_i = 0 \text{ for } 0 \leq i < n\}$. As Vilenkin observed, the elements of the character group of D_0 may be identified with the Walsh functions, introduced in 1923 by J. L. Walsh [11]. This same observation was made, independently, by N. J. Fine in 1949 [4]. An extensive and up-to-date introduction to the Fourier theory of Walsh functions and of some of its extensions to compact Vilenkin groups can be found in the recent book by F. Schipp, W. R. Wade and P. Simon [7].

In this paper we discuss some results in the harmonic analysis of groups that are the locally compact analogue of the groups introduced by Vilenkin in 1947. From here on, G will denote a locally compact Vilenkin group. This means that G is a locally compact Abelian (LCA) topological group containing a sequence of subgroups $(G_n)_{-\infty}^{\infty}$, such that

(i) each G_n is an open compact subgroup of G,

(ii) $G_{n+1} \subsetneq G_n$ for all $n \in \mathbf{Z}$,

(ii) $\cup_{-\infty}^{\infty} G_n = G$ and $\cap_{-\infty}^{\infty} G_n = \{0\}$,

(iv) $\sup\{$ order $G_n/G_{n+1} : n \in \mathbf{Z}\} < \infty$.

The dual group or character group of G will be denoted by Γ; thus $\gamma \in \Gamma$ if $\gamma : G \to \mathbf{T}$ is continuous and $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $x, y \in G$. If Γ is topologized with the so-called compact-open topology and if we set

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \quad \text{for all} \quad x \in G_n\},$$

then

(i)* each Γ_n is a compact open subgroup of Γ,

(ii)* $\Gamma_n \subsetneq \Gamma_{n+1}$ for all $n \in \mathbf{Z}$,

(iii)* $\cup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\cap_{-\infty}^{\infty} \Gamma_n = \{\gamma_0\}$, where $\gamma_0(x) = 1$ for all $x \in G$,

(iv)* order $(\Gamma_{n+1}/\Gamma_n) =$ order (G_n/G_{n+1}) for all $n \in \mathbf{Z}$.

We choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $(\mu(G_n))^{-1} = \lambda(\Gamma_n)$ for each $n \in \mathbf{Z}$; we define m_n by $m_n := (\mu(G_n))^{-1}$. Once a Haar measure has been introduced on G, we can define the Lebesgue spaces $L^p(G)$, $0 < p \leq \infty$, and the Fourier transform: for $f \in L^1(G)$ and $\gamma \in \Gamma$ we set $\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}d\mu(x)$. The inverse Fourier transform will be denoted by $^\vee$. It is easy to see (denoting the characteristic function of a set A by χ_A) that if $\Delta_n := m_n \chi_{G_n}$ then $(\Delta_n)^\wedge = \chi_{\Gamma_n}$.

Before giving the definition of the Hardy spaces on such groups G we first give an example of a LC Vilenkin group and some general references. Let $D = \{x : x = (x_i)_{-\infty}^{\infty}, x_i \in \{0,1\}$ for all $i \in \mathbf{Z}$ and $x_i = 0$ for $i < N_x\}$. Addition in D is defined coordinate wise modulo 2 and we introduce a topology on D by defining a metric on D as follows: $d(x,y) = 2^{-n}$ if $(x-y)_n = 1$ and $(x-y)_i = 0$ for $i < n$. Then D is a LCA topological group and if we define the subgroups D_n of D by

$$D_n = \{x \in D : x_i = 0 \quad \text{for} \quad i < n\}$$

then it is a simple exercise to show that D and the sequence $(D_n)_{-\infty}^{\infty}$ satisfy conditions (i), (ii), (iii) and (iv). Furthermore, the elements of the character group Γ_D of D can be described as follows: for any $y = (y_i)_{-\infty}^{\infty} \in D$ define $\gamma_y \in \Gamma_D$ by

$$\gamma_y(x) = \exp(\pi i \sum_{k=-\infty}^{\infty} x_k y_{-k})$$

whenever $x = (x_i)_{-\infty}^{\infty} \in D$. Then $\Gamma_D = \{\gamma_y : y \in D\}$.

Other examples of LC Vilenkin groups are the additive group of a p-adic field or a p-series field or, more generally, of any local field. A detailed exposition of harmonic analysis on local fields was given by M. H. Taibleson in [9]. The first mentioning of LCA topological groups G containing a doubly infinite sequence of sub-groups $(G_n)_{-\infty}^{\infty}$ with properties (i) through (iv) was made in a series of papers by R. Spector, see [8]. A further reference, containing additional examples of LC Vilenkin groups, is [3].

We now turn to the definitions of the (homogeneous) Herz spaces $K(\alpha, p, q)$ and the Hardy spaces $H^p(G)$ on a LC Vilenkin group G.

Definition 1. *Let $\alpha \in \mathbf{R}$ and $0 < p, q \leq \infty$. A measurable function $f : G \to \mathbf{C}$ belongs to the Herz space $K(\alpha, p, q)$ if*

$$||f||_{K(\alpha,p,q)} := \Big(\sum_{\ell=-\infty}^{\infty} \Big((m_\ell)^{-\alpha} ||f\chi_{G_\ell \setminus G_{\ell+1}}||_p\Big)^q\Big)^{1/q} < \infty,$$

with the usual modification if $q = \infty$.

Clearly, $K(0, p, p) = L^p(G)$ and it is easy to show that the following (continuous) inclusion relations hold.

(a) If $0 < q_1 \leq q_2 \leq \infty$ then $K(\alpha, p, q_1) \hookrightarrow K(\alpha, p, q_2)$.

(b) If $0 < p_1 \leq p_2 \leq \infty$ then $K(\alpha - 1/p_2, p_2, q) \hookrightarrow K(\alpha - 1/p_1, p_1, q)$.

In [5] the definitions and some of the basic properties for the spaces of tests functions $\mathcal{S}(G)$ and distributions $\mathcal{S}'(G)$ are given. The definitions and results are immediate generalizations of corresponding results described in [9] for local fields and they will not be repeated here.

Definition 2. *Let $0 < p \leq 1$. A distribution $f \in \mathcal{S}'(G)$ belongs to the Hardy space $H^p(G)$ if the function $f^* : G \to \mathbf{C}$ defined by $f^*(x) = \sup_\ell |\Delta_\ell * f(x)|$ belongs to $L^p(G)$; we set $||f||_{H^p} := ||f^*||_p$.*

Definition 3. *A function* $\phi \in L^\infty(\Gamma)$ *is a (Fourier) multiplier of* $H^p(G)$, $\phi \in \mathcal{M}(H^p)$, *if there exists a constant* $C > 0$ *so that for all* $f \in H^p(G)$ *we have* $||(\phi\hat{f})^\vee||_{H^p} \leq C||f||_{H^p}$.

In joint work with T.S. Quek the following theorems were proved. Here we use as notation: if $\phi \in L^\infty(\Gamma)$ and $k \in \mathbf{Z}$ then $\phi^k := \phi\chi_{\Gamma_{k+1}\backslash\Gamma_k}$.

Theorem 1. *Let* $0 < p < 1$ *and* $p < r < \infty$. *If* $\phi \in L^\infty(\Gamma)$ *satisfies*

$$\sup_k \, (m_k)^{1/p-1}||(\phi^k)^\vee||_{K(1/p-1/r,r,p)} < \infty \, ,$$

then $\phi \in \mathcal{M}(H^p)$.

Theorem 2. *If* $0 < p \leq 1$ *and* $\phi \in L^\infty(\Gamma)$ *satisfies*

$$\sum_{k=-\infty}^{\infty} ((m_k)^{1/p-1}||(\phi^k)^\vee||_p)^{2p/(2-p)} < \infty \, ,$$

then $\phi \in \mathcal{M}(H^p)$.

A proof of these theorems will appear in [6]. The same paper also contains the proofs of several results concerning the sharpness of the preceding theorems. We mention here two such results.

Theorem 3. *Let* $0 < p < 1$ *and* $p < r < \infty$. *For every* $q > p$ *there exists a* $\phi \in L^\infty(\Gamma)$ *such that*

$$\sup_k \, (m_k)^{1/p-1}||(\phi^k)^\vee||_{K(1/p-1/r,r,q)} < \infty$$

and $\phi \notin \mathcal{M}(H^p)$.

Theorem 4. *Let* $0 < p \leq 1$. *For every* q *with* $2p/(2-p) < q \leq \infty$ *there exists a function* $\phi \in L^\infty(\Gamma)$ *such that*

$$(*) \qquad \sum_{k=-\infty}^{\infty} ((m_k)^{1/p-1}||(\phi^k)^\vee||_p)^q < \infty$$

and $\phi \notin \mathcal{M}(H^p)$.

In view of the fact that condition $(*)$ in Theorem 4 is not sufficient for a function $\phi \in L^\infty(\Gamma)$ to be an $H^p(G)$-multiplier, it is of some interest to find additional conditions to obtain $H^p(G)$-multipliers. The next theorem provides one such result. We state it here only for the case $q = \infty$ in $(*)$; the general case is considered in [6].

Theorem 5. *Let $0 < p \leq 1$ and let $\phi \in L^\infty(\Gamma)$ satisfy*

$$\sup_k \, (m_k)^{1/p-1} \|(\phi^k)^\vee\|_p < \infty .$$

If $(\alpha_k) \in \ell^{2p/(2-p)}(\mathbf{Z})$ and if ψ is defined by $\psi(\gamma) = \sum_{-\infty}^{\infty} \alpha_k \phi^k(\gamma)$ then $\psi \in \mathcal{M}(H^p)$.

Thus far we have only been able to prove the sharpness of Theorem 5 for the case $p = 1$; in [6] a proof of the following result can be found.

Theorem 6. *Let $(\alpha_k) \in \ell^\infty(\mathbf{Z}) \setminus \ell^2(\mathbf{Z})$. Then there exists a $\phi \in L^\infty(\Gamma)$ so that $\sup_k \|(\phi^k)^\vee\|_1 < \infty$ and ψ, defined by $\psi(\gamma) = \sum_{-\infty}^{\infty} \alpha_k \phi^k(\gamma)$, does not belong to $\mathcal{M}(H^1)$.*

We now briefly discuss how the multiplier theorems stated here for Hardy spaces defined on LC Vilenkin groups, compare with similar results for multipliers on Lebesgue or Hardy spaces defined on $\mathbf{R}^n$. While doing so we shall also mention some open problems. In [1], A. Baernstein and E. T. Sawyer proved various multiplier theorems for $H^p(\mathbf{R}^n)$-spaces. Before stating their results we first introduce some notation.

For $0 \leq \alpha < \infty$ and $0 < p, q \leq \infty$ we define the homogeneous Herz spaces on $\mathbf{R}^n$ by

$$\dot{K}_p^{\alpha,q} = \{f \in L^1_{loc}(\mathbf{R}^n) : \|f\|_{\dot{K}_p^{\alpha,q}} := \Big(\sum_{k=-\infty}^{\infty} \Big(2^{k\alpha} \|f \chi_{A(k)}\|_p\Big)^q\Big)^{1/q} < \infty\} ,$$

with the usual modification if $q = \infty$; here $A(k) = \{x \in \mathbf{R}^n : 2^k \leq |x| \leq 2^{k+1}\}$.

The inhomogeneous Herz spaces $K_p^{\alpha,q}$ are defined by $K_p^{\alpha,q} = \dot{K}_p^{\alpha,q} \cap L^p$, and $\|f\|_{K_p^{\alpha,q}} = \|f\|_p + \|f\|_{\dot{K}_p^{\alpha,q}}$.

Next, fix a function $\eta \in C^\infty(\mathbf{R}^n)$ such that $0 \leq \eta(\xi) \leq 1$ for all $\xi \in \mathbf{R}^n$, $\eta(\xi) = 1$ for $\frac{1}{2} \leq |\xi| \leq 2$ and $supp(\eta) \subset \{\frac{1}{4} \leq |\xi| \leq 4\}$. For $m \in L^\infty(\mathbf{R}^n)$ and $\delta > 0$ define m_δ by $m_\delta(\xi) = m(\delta\xi)\eta(\xi)$. In [1, Theorem 3a] Baernstein and Sawyer proved the following.

Theorem B&S. *Let $0 < p < 1$ and assume $m \in L^\infty(\mathbf{R}^n)$ satisfies*

$$\sup\{\|(m_\delta)^\wedge\|_{K_1^{n(1/p-1),p}} : \delta > 0\} < \infty .$$

Then $m \in \mathcal{M}(H^p(\mathbf{R}^n))$.

Baernstein and Sawyer also proved the sharpness of their result. There is an obvious similarity between Theorem B&S and the case $r = 1$ of Theorem 1 and this raises the question whether or not the spaces $K_1^{n(1/p-1),p}$ in Theorem B&S can be replaced by the spaces $K_r^{n(1/p-1/r),p}$, or even the spaces $\dot{K}_r^{n(1/p-1/r),p}$, with $r > p$.

In [1] the authors point out that Theorem B&S does not extend to $p = 1$. In their paper they modify the definition of the Herz spaces to obtain a result for $H^1(\mathbf{R}^n)$-multipliers. It would be of interest to determine if the case $p = 1$ of Theorem 2 can be modified so as to obtain a similar multiplier theorem for $H^1(\mathbf{R}^n)$ and if such a modification would extend to $0 < p \leq 1$.

As for a comparison of Theorem 5 with corresponding known results for $H^p(\mathbf{R}^n)$-multipliers we mention here that in [2] M. Cowling, G. Fendler and J. J. F. Fournier proved the following.

Theorem CFF. *Let $1 < p < \infty$. Suppose that for each $n \in \mathbf{Z}$ the function $m_n \in \mathcal{M}(L^p(\mathbf{R}^n))$ and that $\sup_n |||m_n|||_p \leq 1$ where $|||m_n|||_p$ denotes the multiplier norm of $m_n \in \mathcal{M}(L^p(\mathbf{R}^n))$. If $(\alpha_n) \in \ell^s(\mathbf{Z})$ for $s = |2p/(2-p)|$ and if m is defined by $m = \sum_{-\infty}^{\infty} \alpha_n m_n \chi_{A(n)}$, then $m \in \mathcal{M}(L^p(\mathbf{R}^n))$.*

A comparison of Theorem CFF and Theorem 5 immediately raises two questions. (1) Can Theorem CFF be extended to multipliers for $H^p(\mathbf{R}^n)$-spaces with $0 < p \leq 1$? (2) Can Theorem 5 be extended to multipliers for $L^p(G)$-spaces, $p > 1$? Moreover, there are also obvious questions about the sharpness of such results.

As a final remark we mention that under the assumptions of Theorem 1 the functions ϕ are, in fact, multipliers for power-weighted Hardy spaces $H^p_\alpha(G)$ for α satisfying $-1 + p/r < \alpha \leq 0$, see [5] for the definition of such spaces. This raises the question whether some of the other theorems stated in this paper can be extended to power-weighted Hardy spaces and/or whether such theorems also hold for other classes of weights besides the power-weights.

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C. W. Onneweer
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
USA