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CHARACTERIZATION OF LIPSCHITZ SPACES VIA THE COMMUTATOR OPERATOR OF COIFMAN, ROCHBERG, AND WEISS

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Let $f \in L^1_{\rm loc} \cap \mathcal{S}'$, $0 < \beta < k \leq n$, k – an integer (in particular, $k = [\beta] + 1$), and n – dimension of the ambient space. Let $K(x) = \Omega(x)/|x|^n$ be a Calderón-Zygmund kernel, i.e., $\int_{\Sigma_{n-1}} \Omega = 0$, $\Omega \in C^\infty(\Sigma_{n-1})$, Ω homogeneous of degree 0.

Define:

$$\begin{aligned} C_{f,k}g(x) &= \int_{\mathbf{R}^n} \Delta_h^k f(x) K(h) g(x+h)\, dh, \\ \overline{C}_{f,k}g(x) &= \int_{\mathbf{R}^n} \Delta_{\frac{h}{k}}^k f(x) K(h) g(x+h)\, dh, \end{aligned}$$

where Δ_h^k is the k-th difference operator; i.e., $\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x)$, $\Delta_h^1 f(x) = f(x+h) - f(x)$.

We have the following theorem, which generalizes a theorem by Janson ([j]), dealing with the case $k = 1$.

Theorem 1 *Let $1 < p < q < +\infty$; $1/p - 1/q = \beta/n$. Under above assumptions, the following are equivalent:*

i) $f = f_1 + P$, with $f_1 \in \dot{\Lambda}_\beta$ and P a polynomial of degree $< k$.

ii) $C_{f,k} : L^p \longrightarrow L^q$ is a bounded operator.

iii) $\overline{C}_{f,k} : L^p \longrightarrow L^q$ is a bounded operator.

If, in particular $k = [\beta] + 1$, then i) says that $f \in \dot{\Lambda}_\beta$.

The key ingredient in the proof is the following characterization of Lipschitz spaces, which appears to be new.

Theorem 2 *Suppose $f \in L^1_{\rm loc} \cap \mathcal{S}'$, $0 < \beta < k$, k - integer (in particular, $k = [\beta] + 1$). Then, the following conditions are equivalent:*

i) $f = f_1 + P$ with $f_1 \in \dot{\Lambda}_\beta$ and P a polynomial of degree $< k$.

ii) There exists an open set $U \subset \mathbf{R}^n$ such, that for every $x_0 \in \mathbf{R}^n$ and $t > 0$,

$$\left| \frac{1}{|Q|} \frac{1}{|Q^z|} \int_Q \int_{Q^z} \left[\Delta_{y-x}^k f(x)\right] dy dx \right| \leq c t^\beta, \tag{1}$$

with c independent of $z \in U$, x_0 and t. Here, $Q = Q(x_0, t)$ (cube centered at x_0, sidelength t, sides parallel to the axes), and $Q^z = Q(x_0 + zt, t)$.

iii) The same condition as ii), with (1) replaced by:

$$\left|\frac{1}{|Q|}\frac{1}{|Q^z|}\int_Q\int_{Q^z}\left[\Delta^k_{\frac{y-x}{k}}f(x)\right]\,dydx\right|\le ct^\beta. \tag{2}$$

If any of these conditions hold (thus all hold) then $\|f_1\|_{\dot\Lambda_\beta}$ is comparable with smallest c in ii) and iii).

Proof of Theorem 2: i)⇒ii) & i)⇒iii): This direction follows immediately from the pointwise estimate on the differences of f. ii)⇒i) & iii)⇒i): The key idea is to write

$$\left|\frac{1}{|Q|}\frac{1}{|Q^z|}\int_Q\int_{Q^z}\left[\Delta^k_{y-x}f(x)\right]\,dydx\right| = \mu_t^z * f(x_0), \tag{3}$$

and

$$\left|\frac{1}{|Q|}\frac{1}{|Q^z|}\int_Q\int_{Q^z}\left[\Delta^k_{\frac{y-x}{k}}f(x)\right]\,dydx\right| = \nu_t^z * f(x_0), \tag{4}$$

for appropriate functions μ and ν. Here $\mu_t(x) = 1/t^n\mu(x/t)$; similarly for ν_t. It turns out, that if $z_1,\ldots,z_n$ are linearly independent, then the vector valued functions $(\hat\mu^{z_1},\ldots,\hat\mu^{z_n})$ and $(\hat\nu^{z_1},\ldots,\hat\nu^{z_n})$ satisfy a so-called Tauberian condition; i.e., their absolute value is not identically 0 along any ray (see [j-t]). Thus, $\mu = (\mu^{z_1},\ldots,\mu^{z_n})$ and $\nu = (\nu^{z_1},\ldots,\nu^{z_n})$ are compactly supported, finite, vector valued Borel measures, such, that $\hat\mu$ and $\hat\nu$ satisfy Tauberian condition. We invoke now a version of Calderón's reproducing formula, due to Janson and Taibleson ([j-t]):

Theorem 3 (Janson and Taibleson) *There exist vector valued functions $\eta = (\eta_1,\ldots,\eta_n)$ and $\omega = (\omega_1,\ldots,\omega_n)$ such that $\eta_i,\ \omega_i \in \mathcal{S}$; $i = 1,\ldots,n$ associated with μ and ν respectively, and an integer N (N is the order of f as a tempered distribution), so that:*

$$f = \int_0^\infty f*\mu_t*\eta_t\,\frac{dt}{t}, \qquad \mu_t*\eta_t = \sum_{i=1}^n \mu_t^{z_i}*\eta_{it},$$

$$f = \int_0^\infty f*\nu_t*\omega_t\,\frac{dt}{t}, \qquad \nu_t*\omega_t = \sum_{i=1}^n \nu_t^{z_i}*\omega_{it},$$

both integrals converging in $\mathcal{S}'_N = \mathcal{S}'/P_{N-1}$ where P_{N-1} is the space of polynomials of degree at most $N-1$.

From this, following the argumentation in [j-t], we deduce, that both (3) and (4) (separately) imply that $f = f_0 + P$, with $f_0 \in \dot\Lambda_\beta$ and P a polynomial. We then show, that the polynomial satisfying either (1) or (2) (i.e., P in the place of f) has degree at most $k-1$. To see this, observe, that the double integrals in (1) and (2) are polynomials in t, of degree equal to the degree of P (provided we replace x_0 by tx_0). Thus, coefficients (in t) of order k and higher vanish. Each of these coefficients is a polynomial in z, and therefore is a zero polynomial (in z).

The homogeneous part of each such polynomial of the highest order coincides with the corresponding homogeneous part of P, up to a non-zero constant. Thus P has degree at most $k-1$. This finishes the proof of Theorem 2.

We now prove Theorem 1.

Proof of Theorem 1: i)⇒ii) & i)⇒iii): This part follows immediately from the pointwise estimates on the differences of f, and the boundedness properties of the Riesz potentials.

ii)⇒i) & iii)⇒i): Using the method in [j], we can show, that ii) implies ii) of Theorem 2 and iii) similarly implies iii) of Theorem 2. This finishes the proof of Theorem 1.

References

[c-r-w] R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. **103** (1976), 611–635.

[h] Heideman, Duality and fractional integration in Lipschitz spaces, Studia Math. **50** (1974), 65–85.

[j] S. Janson, Mean oscillation and commutators with singular integral operators, Ark. Math. **16** (1978), 263- 270.

[j-t] S. Janson and M. Taibleson, I teoremi di rappresentazione di Calderón, Rend. Sem. Math. Torino **39** (1981), 27–35.