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# TOWARDS AN ALGEBRAIC APPROACH TO INTEGRABLE SYSTEMS

LUIS A. PIOVAN

§1.A quarter of century ago the discovery of solitary waves (soliton) solutions to Korteweg-de Vries equations opened a path for a reviewing of the phenomena associated with integrability. Nowadays, several aspects of integrability of finite and infinite dimensional systems have grown apart and relate to Lie group Theory, Algebraic Geometry, infinite dimensional Lie algebras, infinite dimensional grasmannians and tau functions. In classical hamiltonian mechanics, the complete integrability over the reals is explained by Arnold [Ar]. If we start with a system of Ordinary Differential Equations:

$$\dot{z} = J(z).gradH, \qquad z \in R^{k+2m},$$

where $J(z)$ = skew-symmetric matrix with polynomial entries in z; and the Poisson bracket:

$$\{f,g\} :=< gradf, J(z).gradg >$$

satisfies the Jacobi identity, then there exist (k+m) independent integrals $H_1, ..., H_{k+m}$ in involution ( i.e. $\{H_i, H_j\} = 0$ ). The first k invariants are called trivial or Casimir invariants since $J(z).gradH_i = 0$. The remaining integrals give rise to m commuting vector fields: $X_i : \dot{z} = J(z).gradH_{i+k}$ constrained to the variety $\bigcap_{i=1}^{k+m} \{H_i = c_i\}$. According to the Liouville-Arnold theorem the compact, connected, invariant manifolds $\bigcap_{i=1}^{k+m} \{z : H_i(z) = c_i\}$ are generically real tori and the varieties $\bigcap_{i=1}^{k} \{H_i = c_i\}$ are generically symplectic leaves. Also, there are canonical coordinates $I, \varphi$ (the action-angle variables) such that the vector field takes the form $\dot{I} = 0, \dot{\varphi} = constant$.

A few examples of these systems are given by the equations of the heavy rigid body motion about a fixed point [Ar] and the 3 body Toda lattice [A-vM,3].

An aspect of integrability in the crosspath of algebraic geometry is algebraic complete integrability. This phenomenon is described in the work of Adler and van Moerbeke [A-vM,1].It requires that the real tori of the Arnold-Liouville picture, when complexified they form the affine parts of compact algebraic groups whose group structure is compatible with the Hamiltonian flows (i.e. complex torus in some projective space or abelian varieties). A compactification of the affine piece can be achieved upon a choice of a divisor at infinity for which the coordinates functions ($z_i$'s) blow up in a meromorphic fashion as functions of the time variable. Usually one picks the minimal divisor for which the coordinate functions blow up (see [AvM, 1,2,3]).

Under certain conditions on the solutions to the differential equations defining the system, it is possible to determine wether a complete integrable system in the sense of Liouville-Arnold is algebraic complete integrable (see [AvM,1]).

However, many of the ingredients of some systems can be described in a totally algebraic way. For instance, by imposing certain conditions on the variety and the divisors at infinity Barth can recover the original affine picture related to the integrable systems [Ba]. The actual question Barth addresses is how to recover an affine part of an abelian variable as a complete intersection.

Therefore,it is better to change the starting point and ask whether an integrable system can be completely described by certain algebraic conditions. In this respect, a missing point was to find a way of describing canonical equations for holomorphic vector fields on abelian varieties. Indeed, by fixing an abelian variety and a polarization one automatically fixes a symplectic structure for the embedding variables and hence can be extended to a symplectic structure over the ring of functions on the abelian variety (supposedly the embedding is projectively normal). A crucial role for describing canonical equations for holomorphic vector fields is played by the Heisenberg group. Indeed, one can define a pair

$$W_X : H^0(A,L)\otimes H^0(A,L) \longrightarrow H^0(A,L^{\otimes 2})$$

$$W_X(s,t) := st' - ts'$$

which is invariant under the action of the Heisenberg group G (see [Pi,1] and [Mu,1,2] for definitions ). This pair can be extended over to the whole homogeneous coordinate ring $R = \bigoplus_{m\geq 0} R_m$ ($R_m$ being the polynomials of degree m in the abelian variety A) as follows

$$[f,g]_X := n.f.X(g) - m.g.X(f) \in R_{m+n}, \quad f \in R_n, \; g \in R_m$$

$$R_k = H^0(A, L^{\otimes k})$$

By picking a section $s$, (our divisor at infinity) say in $R_1$, we can define the derivation produced by the vector X as follows

$$X : R_k \longrightarrow R_{k+1} \qquad X(g) = [s,g]_X = \varphi_s(g)$$

Now, one constructs the affine ring

$$A = \bigoplus_{m\geq 0} R_m/s^m \cong \bigoplus_{m\geq 0} R_m/s.R_{m-1} = \bigoplus_{m\geq 0} A_m$$

( $A_m$ is the degree m affine component ), and define in a unique way $X : A_m \longrightarrow R_{m+1}/s^{m+1} = \bigoplus_{i=0}^{m+1} A_i$ as $X(f/s^m) = [s,f]/s^{m+1}$ ; which coincides with the usual definition of a derivation in affine coordinates.

Thus, the equations defining abelian varieties belong to the kernel of $[s,.] : R_m \longrightarrow R_{m+1}$.

Let $K \subseteq G$ be a subgroup which fixes the linear span of s and also a direct summand of it in $R_1$ . Therefore, K acts on the whole affine ring. Now, the kernel of $\varphi_s$ will consist of functions killed by the vector field X; i.e. integrals of the motion. Thus, once we established canonical equations for vector fields on a polarized abelian variety, we can describe the invariants by requiring them to be killed by the vector field. The degree of the variety $= D.D$ for the polarization divisor $D$ fixes restrictions for an affine part to be a complete intersection , so the possible choice for the degrees of the invariants remain within a finite set. On the other hand we can take advantage of the fact that the kernel of $\varphi_s$ is a K-module.

### §2.Elliptic curves in $P^3$.

The polarization type induced by the embedding in projective space $P^3$ is (4). In an appropriate basis, the action of the Heisenberg group is given by the following table:

| | $Y_0$ | $Y_1$ | $Y_2$ | $Y_3$ |
|---|---|---|---|---|
| $\sigma$ | $Y_2$ | $-Y_3$ | $Y_0$ | $Y_1$ |
| $\chi$ | $Y_1$ | $Y_0$ | $\sqrt{-1}.Y_3$ | $\sqrt{-1}.Y_2$ |
| $\sigma^2$ | 1 | $-1$ | 1 | $-1$ |
| $\chi^2$ | 1 | 1 | $-1$ | $-1$ |
| $\iota$ | 1 | 1 | 1 | $-1$ |

The canonical vector field invariant under the symmetries $\sigma^2, \chi^2, and\ \iota$, is given by the equations

$$\begin{cases} \dot{y}_0 = \gamma y_1 y_2, \\ \dot{y}_1 = \beta y_0 y_2, \\ \dot{y}_2 = \alpha y_0 y_1. \end{cases}$$

Here we have chosen $Y_3^2 = 1$. Moreover, the quadratic integral invariants are of the form $ay_0^2 + by_1^2 + cy_2^2$, with $a\alpha + b\beta + c\gamma = 0$. Thus, this system is precisely the Euler Top equations for a rigid body around a fixed point [Ar] p.143. We can state the following :

**Theorem:** *The only holomorphic vector field on the family of elliptic curves in $P^3$ which is invariant under $\sigma^2, \chi^2$, and $\iota$ and whose divisor at infinity is cut out by an odd section, is given by the Euler-Arnold equations of the rigid body in SO(3).*

### §3. The Toda lattice system and principally polarized abelian surfaces.

This system was studied by many people [Fl],[A-vM] and even interpreted using the grassmanian viewpoint (recent work of Flaschka, Haine, Adler and van Moerbeke [A-vM,5]).

The system I am concerned with here is the 3-body Toda Lattice. In $C^6$ it has the following equations:

$$\begin{cases} \dot{w}_1 = w_1(w_5 - w_6), \\ \dot{w}_2 = w_2(w_6 - w_4), \\ \dot{w}_3 = w_3(w_4 - w_5), \\ \dot{w}_4 = w_3 - w_2, \\ \dot{w}_5 = w_1 - w_3, \\ \dot{w}_6 = w_2 - w_1. \end{cases}$$

Setting each integral of motion to a constant we get an affine surface that is completed into an abelian variety by adding a divisor at infinity formed of three genus 2 curves pairwise touching in a tacnode.

Our point is stated in the following:

**Theorem:** *The only vector field on a principal polarized Jacobian that admits an automorphism of order 3 and the (-1)-reflection, which leaves invariant the affine part, the divisor at infinity and is an algebraic complete integrable system possessing the polarization type (3,3), is the Toda Lattice system.*

The proof of these theorems will appear elsewhere.

## References


[A-vM,1] Adler,M;van Moerbeke,P.:The complex geometry of the Kowalevski-Painlevé analysis. Invent. math.,**97**,3-51 (1989)

[A-vM,2] Adler,M.;van Moerbeke,P.:The algebraic integrability of geodesic flow on so(4). Invent. math., **67**,297-331 (1982)

[A-vM,3] Adler,M.;van Moerbeke,P.:Completely integrable systems - A sistematic approach towards solving integrable systems (preprint 1989) to appear.

[A-vM,4] Adler,M.;van Moerbeke,P.:The Kowalevski and Henon-Heiles motions as Manakov geodesic flows on SO(4). A two dimensional family of Lax pairs. Commun. Math. Phys. **113** (4)(1988)

[A-vM,5] Adler,M.;van Moerbeke,P.:The Toda lattice, Dynkin diagrams, singularities and abelian varieties. Inven. math. **103**,223-278 (1991)

[Ar] Arnold,V.:Mathematical methods of classical Mechanics. Springer 1978

[Ba] Barth,W.:Affine parts of abelian surfaces as complete intersection of four quadrics. Math. Ann.**278** (1-4),117-131 (1987)

[Fl] Flaschka,H.:The Toda lattice, (1) and (2). Phys. Rev. **9**,1924-1925 (1974); Prog. theor. Phys. **51**, 703-706(1974)

[Mu] Munford,D.: On the equations defining abelian varieties I-III. Invent. math.**1**, 287-354(1966) **2** - **3**,75-135,215-244(1967)

[P] Piovan,L.: Algebraically completely integrable systems and Kummer varieties. Math. Ann.**290**, 349-403 (1991).



Instituto de Matemática,
Universidad Nacional del Sur,
8000 Bahía Blanca, Argentina.