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SOME PROBLEMS OF SPECTRAL ANALYSIS ON BERGMAN SPACES

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1. Introduction.

The aim of the present note is to review some of the recent advances in multivariable spectral theory on Bergman spaces and to discuss a few open problems in this field. The interest for multivariable Bergman spaces and their operators has increased recently as a consequence of the maturity of the abstract spectral theory for commutative n-tuples of linear operators and also because of the specific problems arising in the function theory of several complex variables. One can already speak of a non-accidental connection between these two fields in the territory of Bergman spaces. However, the research in this area is only at the beginning, a series of interesting open problems being not solved yet. It is our first aim in the sequel to isolate a few of these open problems.

In what is now called abstract spectral theory there are two definitive examples. One is concerned with self-adjoint operators and the other with the unilateral shift, both acting on a separable complext Hilbert space. The latter is a model, although not always explicitly stated, for the spectral theory on any spaces of analytic functions, in particular on Bergman spaces.

Let H be a complex Hilbert space and let $e_0, e_1, \ldots$ denote an orthonormal basis of it. The linear operator $S : H \to H$ which acts as $Se_n = e_{n+1}, n \geq 0$, is bounded and it has the matrix:

$$S = \begin{pmatrix} 0 & & & \\ 1 & 0 & & 0 \\ & 1 & 0 & \\ & & 1 & \\ & & \ddots & \ddots \\ & 0 & & \end{pmatrix}.$$

This is the unilateral shift of multiplicity one. A natural model for S which explains its spectral behaviour is obtained by identifying H with the Hardy space $H^2(\Pi)$ consisting of the closure of complex analytic polynomials in $L^2(\Pi, d\theta), \Pi$ being the unit torus and $d\theta$ denoting the Haar measure on Π. More precisely, $e_k \mapsto z^k,\ k \geq 0$ gives an isometric isomorphism $H \simeq H^2(\Pi)$ in which the operator S becomes the multiplication M_z with the complex variable $z \in \mathbb{C}$.

On the basis of this model, operator theory and the theory of analytic functions in the unit disk interact in quite a few unexpected directions, see [13]. We mention only a well-known theorem of Beurling which asserts that every closed invariant subspace of $S = M_z$ is of the form $fH^2(\Pi)$, where f are the boundary values of an inner function in the unit disk. Moreover, two such invariant subspaces are unitarily equivalent, preserving the action of S, if and only if they coincide.

For an arbitrary bounded analytic function F in the unit disk $\mathbb{D}$ there is a natural operator $F(S)$ acting as the mutiplier M_F on $H^2(\Pi)$, whose spectral picture is completely understood. In particular, $\sigma(F(S)) = \overline{F(\mathbb{D})}$.

With this example in mind, let us now turn to domains in $\mathbb{C}^n$ instead of the unit disk. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain (i.e. a bounded domain of holomorphy) and let $L^2_a(\Omega)$ denote the space of analytic functions in Ω which are square summable with respect to the Lebesque measure on Ω. This is a Hilbert space with reproducing kernel which carries a natural n-tuple of commuting bounded linear operators: $M_\Omega = (M_{z_1}, \ldots, M_{z_n})$. The space above is known as the Bergman space of the domain Ω. It was Stefan Bergman who revealed in the twenties the importance of the reproducing kernel of $L^2_a(\Omega)$ in the study of the geometry of the domain Ω.

The parallel between the spectral behaviour of the unilateral shift $S = M_z$ and that of the Bergman n-tuple M_Ω has guided a series of recent researches in multivariable spectral theory. Of course, there are also some other remarkable examples of commuting n-tuples of operators which invite to a similar parallel, see [6], but the Bergman space setting has the advantage of having a rich function theory behind. A few references which complement our brief presentation are [6], [9], [10], [20], [21].

2. Analytically invariant subspaces.

Let Ω be a bounded pseudoconvex domain of $\mathbb{C}^n$. There are two natural classes of closed analytically invariant subspaces of the Bergman space $L^2_a(\Omega)$ which deserve attention. For an ideal I of the algebra of analytic functions $\mathcal{O}(\Omega)$ one has a subspace as in Beurling's Theorem:

$$I^{(2)} = (I \cdot L^2_a(\Omega))^-$$

On the other hand, for an arbitrary closed subset V of Ω one has the subspace:

$$S(V) = \{f \in L^2_a(\Omega);\ f|_V = 0\} .$$

The relationship between these two classes of analytic ideals gives rise to a series of natural but rather difficult problems. To begin with, let us consider the reduced ideal $I(V) = \{f \in \mathcal{O}(\Omega);\ f|_V = 0\}$ attached to the set V above. Then obviously $I(V)^{(2)} \subseteq S(V)$ and the question is whether this inclusion is an equality or not. The answer is yes in a few very particular cases treated in [17] and [18]:

Theorem 1. *Let Ω be a bounded, strongly pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary. Then the following assertions hold:*

a) For an ideal $I \subset \mathcal{O}(\bar{\Omega})$ with $V(I) \cap \Omega = \phi$ one has $I^{(2)} = L^2_a(\Omega)$;

b) If V is a complex analytic manifold which interests $\partial\Omega$ transversally, then $I(V \cap \bar{\Omega}) \cdot L^2_a(\Omega) = S(V \cap \Omega)$.

c) Every finite codimensional analytic subspace S of $L^2_a(\Omega)$ is of the form $S = I \cdot L^2_a(\Omega)$ where the set of common zeros of I is finite.

It is interesting to remark that the subspaces of the form $I \cdot L^2_a(\Omega)$ appearing in b) and c) are closed in $L^2_a(\Omega)$. Actually the proof of Theorem 1 shows more, namely that in these nice cases I and $L^2_a(\Omega)$ are analytically transversal, that is $\mathrm{Tor}_q^{\mathcal{O}(\bar{\Omega})}(I, L^2_a(\Omega)) = 0$ for $q \geq 1$, see [18]. The occurence of homological algebra at this stage is not unexpected if one takes into account the similar Nullstellensatz problem for $\mathcal{O}(\Omega)$ instead of the Bergman space. Moreover, the known results for polydomains contained in [8] and [14] strongly suggest that a criterion for the validity of the equality $I(V)^{(2)} = S(V)$, V being of course an analytic subset of Ω, must involve either the geometry of the set V and its relative position with respect to $\partial\Omega$, or the homological algebra of the ideal $I(V)$.

To conclude this discussion, let us isolate the main question.

Problem 1. a) *Let V be a closed analytic subset of the bounded pseudoconvex domain Ω. Find geometric criteria on V and $\partial\Omega$ for the inclusion $I(V)^{(2)} \subset S(V)$ to be an equality.*

b) (conjecture) *If $I \subset \mathcal{O}(\bar{\Omega})$ is an ideal, then* $\mathrm{Tor}_q^{\mathcal{O}(\bar{\Omega})}(I, L^2_a(\Omega)) = 0$ *for $q \geq 1$ whenever $I \cdot L^2_a(\Omega)$ is a closed subspace of $L^2_a(\Omega)$.*

A few particular cases which support conjecture b) are presented in [18].

With the notations above, let $I, \mathcal{J}$ be ideals of $\mathcal{O}(\bar{\Omega})$ and consider their corresponding analytic subspaces $I^{(2)}, \mathcal{J}^{(2)}$. An important question is to decide when these two subspaces are unitarily and analytically equivalent, that is when there exists a unitary operator $U : I^{(2)} \to \mathcal{J}^{(2)}$ with $Uf_\varphi = fU_\varphi$ for $f \in \mathcal{O}(\bar{\Omega})$ and $\varphi \in I^{(2)}$.

A substantial progress in this direction is summarized in the following theorem.

Theorem 2. (Douglas-Paulsen-Yan, [10]) *Let B denote the unit ball of $\mathbb{C}^n$ and let I and $\mathcal{J}$ be polynomial ideals supported by sets of codimension greater or equal than 2(in Ω).*

Then $I^{(2)}$ and $\mathcal{J}^{(2)}$ are unitarily-analytically equivalent if and only if $I = \mathcal{J}$.

A geometric approach which extends the hypotheses of this result is currently developed by K. Yan [24]. It is expected that this rigidity phenomenon holds in a very general context.

Problem 2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $I, \mathcal{J}$ be ideals of $\mathcal{O}(\bar{\Omega})$. Find necessawry and sufficient conditions for the equality $I^{(2)} = \mathcal{J}^{(2)}$ to hold.*

As we have noted in Theorem 1. a), it is not always true that $I^{(2)} = \mathcal{J}^{(2)}$ implies $I = \mathcal{J}$. So the problem is to find the most natural conditions under which this implication holds.

A colateral question which turns out to be related to the latter open problem is to identify the quotient space $L^2_a(\Omega)/I^{(2)}$ with a natural function space carried by the support V of the ideal I. Having in mind the case of a polydisk and a linear subspace of it, we raise the following problem.

Problem 3. *Let Ω be a bounded pseudoconvex domain of $\mathbb{C}^n$ and let V be a complex submanifold of a neighborhood of $\bar{\Omega}$, endowed with the induced Hermitian metric.*

Is the restriction map $L^2_a(\Omega) \to L^2_a(V \cap \Omega)$ well defined?

If the answer to this problem were yes, then there is some evidence that the quotient $L^2_a(\Omega)/\mathcal{J}(V)^{(2)}$ would be isometrically isomorphic with $L^2_a(V \cap \Omega)$, at least when V intersects $\partial\Omega$ transversally, see [18].

3. The spectral picture of analytic Toeplitz tuples.

Let Ω be a bounded pseudoconvex domain of $\mathbb{C}^n$ and let $f_1, \dots, f_n \in H^\infty(\Omega)$, that is a system of bounded holomorphic functions on Ω. Our next aim to to discuss the spectral properties of the analytic Toeplitz n-tuple $M_f = (M_{f_1}, \dots, M_{f_m})$, where M_{f_i} is the multiplication operator with f_i on the Bergman space $L^2_a(\Omega), i = 1, \dots, m$. The most recent and comprehensive result was obtained by adapting the L^2-estimates for the $\bar{\partial}$-operator of Hörmander [12] and Skoda [22] to this setting.

Theorem 3. ([11]). *Let Ω be a bounded pseudoconvex domain of $\mathbb{C}^n$ and let $f = (f_1, \dots, f_m) \in H^\infty(\Omega)^m$. Then:*

a). $\sigma(M_f) = \overline{f(\Omega)}$;

b). $\sigma_e(M_f) \subset \cap(\overline{f(U \cap \Omega)};\ U \supset \sigma_e(M_z)$ open).

Above σ denotes Taylor's joint spectrum [23] while σ_e stands for the essential joint spectrum. The symbol M_z denotes the n-tuple of multiplication operators with the complex coordinates of $\mathbb{C}^n$.

It is well known that $\sigma_e(M_z)$ is contained in the topological boundary of Ω, [15], but this inclusion may be strict even for $n = 1$, see [5]. The case $n = m = 1$ of the theorem is treated in [4] and [5].

The spectral picture of the Bergman n-tuple M_Z is by far the best understood, cf. [6], [15], [16], [20], [21]. The key technical tool in studying the Bergman n-tuple is a sheaf model which localizes the Bergman space [15], [16].

It is known that the inclusion b) in Theorem 3 is actually an equality when the domain Ω is strictly pseudoconvex with C^2-boundary, see [11]. So the next problem arises naturally.

Problem 4. (Conjecture) *For any bounded pseudoconvex domain Ω, the inclusion* b) *in Theorem 3 is an equality.*

A more intriguing question whose solution seems to be completely different from the present proof of Theorem 3 is the following.

Problem 5. *Let Ω be a bounded pseudoconvex domain and let $p \in [1, \infty]$. Is Theorem 3 still valid for $L^p_a(\Omega)$ instead of the Bergman space and the same m-tuple $f \in H^\infty(\Omega)^m$?*

We notice that for $p = \infty$ the above problem is equivalent to the Corona Problem for the domain Ω, the answer to which is unknown even for the unit ball in $\mathbb{C}^n$.

In the process of localizing the Bergman space in order to prove for instance Theorem 3 one needs to control the homology of the m-tuple T_f on various function spaces. In particular, the following result is expected to hold.

Problem 6. (Conjecture) *Let Ω be a bounded pseudoconvex domain of $\mathbb{C}^n$ and let $f \in H^\infty(\Omega)^m$. Then Koszul's complex $K.(w - f; C^\infty(\mathbb{C}^n)\hat{\otimes}L^2_a(\Omega))$ is exact in positive degree and has separated homology in degree zero.*

The above condition is a variation of Bishop's property (β), cf. [11], [16]. It is known that the conjecture is true when the partial derivatives of the entries of f are uniformly bounded on Ω, see [16]. A second indication towards a positive solution to Problem 6 is the equality $\sigma(T_f) = \sigma_r(T_f)$, whereby the latter denotes the right (joint) spectrum.

To complete the spectral picture of an m-tuple T_f as above one has to take into account the Fredholm index data, too. This is essentially done in [16] and [21], while the remaining open problems are a bit too technical for this report.

Finally we have to mention that even for $n = m = 1$ not all of the above problems are completely elucidated, see [4].

Although it is difficult to find the origins and to track the history of these problems, it is certain that they go beyond the spectral analysis motivation exposed before. Two different perspectives on some similar questions appear in [19] and [22].

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