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SINGULAR INTEGRAL EQUATIONS, SPACES OF HOMOGENEOUS TYPE AND BOUNDARY ELEMENTS IN NON-SMOOTH DOMAINS

VILHELM ADOLFSSON*, BJORN JAWERTH† and RODOLFO TORRES‡

1 Introduction

The method of layer potentials is a well known, classical method for solving boundary value problems. In recent years much progress has been made concerning the use of this method for studying both elliptic and parabolic partial differential equations in non–smooth domains; see for example [18], [10], [11], [14], [3], and [4]. Solving boundary value problems in domains with very little regularity is of fundamental importance in applications, but it is also of great interest from a mathematical point of view. In fact, when the domain is non–smooth, the applicability of the method of layer potentials relies on the use of deep and powerful techniques from harmonic analysis such as singular integrals, maximal functions, Hardy spaces, etc. The purpose of this article is to briefly illustrate how some of these techniques can be taken one step further and be used, in a discrete setting, to do numerical analysis for boundary value problems in domains which are merely Lipschitz.

The approach we will describe was initiated by B. Dahlberg and G. Verchota in [12], where they constucted a Galerkin method for the Dirichlet and Neumann problems for the Laplace equation in Lipschitz domains. Their procedure was then improved and extended to other elliptic problems in [1]. The parabolic case will be analyzed in detail in a forthcoming paper [2]. Here, we shall only attempt to describe some of the ideas involved by presenting a more abstract version, which unifies the elliptic and parabolic cases. In fact, based on the

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works just mentioned, we will discuss a general scheme to approximate solutions of integral equations of the second kind in the context of spaces of homogeneous type. We will then indicate how this leads to a boundary element method in non–smooth domains. Finally, we will mention the type of error estimates that can be obtained. Of course, error estimates are crucial if one wants to apply the method for numerical computations (our original goal). For the sake of brevity, we will usually only state the results and make some comments about the proofs. More details can be found in the references given.

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2 Spaces of Homogeneous Type

It is well known that many problems in mathematics and the applied sciences can be reduced to the problem of solving an integral equation of the form

$$g(x) + \int K(x,y)g(y)dy = f(x).$$

Here, the integral could be in R^n or in some more general measure space Y, and one may try to solve such an equation in different function spaces, e.g. $C(Y)$ or $L^p(Y)$. We can write the above equation as

$$(I+T)g = f,$$

where I is the identity operator and

$$Tg(x) = \int K(x,y)g(y)dy.$$

The possibility of solving the equation is connected with the properties of the operator T, but in this general formulation it is of course not possible to characterize, in a simple manner, which operators or which properties of their kernels yield a solvable equation. In certain cases the kernel is not a locally integrable function, and the operator T has to be viewed as a principal value singular integral. The singularity of the kernel is usually related to some geometrical feature and to some appropriate metric associated with the problem in question. Certainly, this is the type of singularity Calderón–Zygmund operators possess, and, indeed, a large class of important integral equations is captured by considering the case when T is a Calderón–Zygmund operator. Hence, it makes sense to consider the above type of integral equations in the context of spaces of homogeneous type (in the sense of Coifman and Weiss [8]). This is the most general setting in which Calderón–Zygmund theory, and other related results in harmonic analysis, can be developed. As we will see, even if one is only interested in integral equations for boundary value problems, one is forced to consider, if not a general space of homogeneous type, at least R^n with a different metric; this is the case when dealing with parabolic equations.

A space of homogeneous type (Y, ρ, μ) is a set Y together with a quasimetric ρ and a doubling measure μ. More specifically, $\rho : Y \times Y \to [0, \infty)$ is a function satisfying

i) $\rho(x, y) = 0$ if and only if $x = y$
ii) $\rho(x, y) = \rho(y, x)$ for all x, y
iii) $\rho(x, y) \leq C_1(\rho(x, z) + \rho(z, y))$ for all x, y, z and some $C_1 > 0$,

and μ is a positive measure with the property that for all $x \in Y$ and $r > 0$, there exists a fixed constant $C_2 > 0$, such that

$$\mu(B(x, 2r)) \leq C_2\mu(B(x, r)),$$

where,

$$B(x, r) = \{y \in Y : \rho(x, y) < r\}.$$

As we already mentioned, most of the usual techniques for dealing with integral operators in R^n extend to this more abstract setting. In particular, we will use the fact that the Hardy-Littlewood Maximal Operator

$$Mf(x) = \sup_{r>0} \left\{ \mu(B(x, r))^{-1} \int_{B(x,r)} |f(y)| d\mu(y) \right\},$$

is bounded on $L^p(Y, \mu)$ for $1 < p < \infty$. We will also need the following simple lemma, the proof of which is similar to the proof of the corresponding result in the euclidean case.

Lemma 1

i) For $\alpha > 1$,

$$\sup_{r>0} \left\{ \mu(B(x, r))^{\alpha-1} \int_{Y \setminus B(x,r)} \frac{|f(y)|}{\mu(B(x, \rho(x, y)))^{\alpha}} d\mu(y) \right\} \leq C_\alpha Mf(x).$$

ii) For $0 < \alpha < 1$,

$$\sup_{r>0} \left\{ \mu(B(x, r))^{\alpha-1} \int_{B(x,r)} \frac{|f(y)|}{\mu(B(x, \rho(x, y)))^{\alpha}} d\mu(y) \right\} \leq C_\alpha Mf(x).$$

Remark. Actually, for the lemma to be true as stated it is necessary to assume that for some $C_3 < 1$, $\mu(B(x, r)) \leq C_3\mu(B(x, 2r))$. This may fail for r large if, for example, diam $Y = \sup_{x,y \in Y} \rho(x, y) < \infty$. However, we shall only use the estimates in the lemma in situations involving small r, $r \to 0$.

3 Approximation of Linear Operators in a Normed Space

Let us consider the integral equation $(I + T)g = f$ again. Even when the equation can be solved, it is not always possible to compute $(I+T)^{-1}$ explicitly (or it is computationally too expensive to do so). Instead, we may try to find approximate solutions that can be calculated more easily. That is, we want to find functions g_n which are easy to calculate and so that $(I + T)g_n \to f$ in some sense. More generally, we may try to construct approximating equations $(I + T_n)g_n = f_n$ (perhaps in some finite dimentional space), so that $g_n \to g$ when $f_n \to f$. There are many ways to formalize this approximation procedure in the context of Banach spaces; see for example [17]. We will describe one suitable for our purposes.

Our approach involves two main steps. First we obtain some convergence result when the operator T is approximated by operators T_n in a suitable way. In applications this will allow us to replace a singular integral operator by a weakly singular one, which is easier to handle. The second step concerns the discretization of the problem and the reduction to a finite dimensional situation.

Let V be a normed space. A generalized approximation of V is a family of triples $\{V_h, p_h, r_h\}_h$, where V_h is a normed space and

$$\begin{aligned} p_h &: V_h \to V \quad \text{(prolongation operator)} \\ r_h &: V \to V_h \quad \text{(restriction operator)} \end{aligned}$$

are linear and continuous operators. This approximation is called stable if

$$(1) \qquad ||p_h|| \leq C, \quad ||r_h|| \leq C, \quad \text{for all } h$$

and convergent if

$$(2) \qquad ||u - p_h r_h u|| \to 0 \quad \text{as } h \to 0, \quad \text{for each } u \in V.$$

Assume now that V and V_h are Hilbert spaces with inner products $< \, , >$ and $< \, , >_h$, respectively. Assume further that p_h is an isometry and that

$$(3) \qquad < p_h u, v > = < u, r_h v >_h \quad \text{for all } u \in V \text{ and } v \in V_h$$

For a linear and continuous operator A acting on a Hilbert space V, we let A^* denote its formal transpose. Let $\{V_h, p_h, r_h\}_h$ be a stable and convergent generalized approximation of V as described above. We say that the approximation is well adapted to a given operator A, if there exists a family of linear operators $A_h : V_h \to V_h$ such that

$$(4) \qquad ||A_h|| \leq C \quad \text{for all } h$$

and

$$(5) \qquad ||Au - p_h A_h r_h u|| \quad \to 0 \qquad \text{as } h \to 0 \ \text{ for all } u \in V$$
$$(6) \qquad ||A^* u - p_h A_h^* r_h u|| \quad \to 0 \qquad \text{as } h \to 0 \ \text{ for all } u \in V.$$

Given an operator A and an approximation well adapted to this operator, we want to approximate the solution g of the equation

$$(7) \qquad Ag = f, \quad f \in V$$

by solutions g_h of the equations

$$(8) \qquad A_h g_h = r_h f.$$

We have the following result.

Lemma 2 *(First Approximation)*

Let $\{V_h, p_h, r_h, A_h\}_h$ be a an approximation of a Hilbert space V well adapted to an operator A. Assume that the operators A_h are invertible and that for some constant $C > 0$ they satisfy

$$(9) \qquad ||u|| \leq C||A_h u|| \quad \textit{for all } u \in V_h \textit{ and all } h > 0.$$

Then, given $f \in V$, the equation (7) always has a unique solution $g \in V$. Moreover, if g_h is the unique solution in V_h of the equation (8), then $p_h g_h \to g$ (strong convergence).

The proof of this lemma is not difficult. The sequence $\{p_h g_h\}$ is easily seen to be uniformly bounded and, hence, one can extract a subsequence from it which converges weakly to some element $g \in V$. Using (6) one can then show that g is a solution of (7). Finally, using (5) one can prove that if g is any solution of (7) and g_h is a solution of (8), then the (whole) sequence $p_h g_h$ converges strongly to g.

The hypothesis of uniform invertibility of the operators A_h in the above lemma is, of course, the crucial point. To verify this hypothesis in a specific application is what requires most of the work. In the examples below, concerning boundary value problems and layer potentials, the corresponding estimates were obtained in [18] and [3], where a method essentially equivalent to the lemma is used. A more abstract formulation of that method, also in the form of a Hilbert space result, appears in [14].

For simplicity, we shall often state results in the Hilbert space context, or even just for L^2 on some measure space. Nevertheless, with obvious modifications the above lemma extends to the case of a reflexive Banach space. In particular, we could include L^p spaces, $1 < p < \infty$, as well. (The reflexivity is used in the proof to extract a weakly convergent subsequence from a bounded one.)

For theoretical purposes the above lemma is usually sufficient, since it shows that (7) can be solved. As we will see, in certain cases one can obtain the operators A_h by some regularization procedure and they act on some V_h's which are also (infinite dimensional) L^2 spaces. For numerical computations, however, we would like each V_h to be finite dimensional. This can be achieved by a second approximation.

Let $\{V_h, p_h, r_h\}_h$ be an approximation of a Hilbert space V. A second approximation of V is given by a family $\{X_h, \tilde{p}_h, \tilde{r}_h\}_h$, where $X_h \subset V_h$ is a closed subspace, $\tilde{p}_h$ is the restriction of p_h to X_h, and $\tilde{r}_h = \Pi_h r_h$ where Π_h is the projection operator from V_h onto X_h. We shall also assume that

$$(10) \qquad ||\tilde{r}_h u - r_h u|| \to 0 \quad \text{as } h \to 0 \quad \text{for every } \; u \in V.$$

This last condition says that the space X_h looks more and more like the space V_h as h approaches zero. If we think of the spaces X_h as finite dimensional, this condition is very natural and implies that the dimension of the space X_h must increase as h decreases. Condition (10) also implies that $\{X_h, \tilde{p}_h, \tilde{r}_h\}_h$ is itself a convergent and stable approximation of V. Moreover, we have the following fact.

Lemma 3 *(Second approximation)*

Let V be a Hilbert space and let $A : V \to V$ be a linear and continuous operator. Let $\{V_h, p_h, r_h, A_h\}_h$ be an approximation of V well adapted to A and satisfying (9). Let $\{X_h, \tilde{p}_h, \tilde{r}_h\}_h$ be a second approximation of V. Assume further that the operators $\Pi_h A_h : X_h \to X_h$ are uniformly invertible. Given $f \in V$, let $\tilde{g}_h \in X_h$ be the unique solution of the equation

$$\Pi_h A_h \tilde{g}_h = \tilde{r}_h f.$$

Then, $\{\tilde{p}_h \tilde{g}_h\}_h$ converges strongly to the unique solution of the equation $Ag = f$.

The result follows from Lemma 2 after showing that $\{X_h, \tilde{p}_h, \tilde{r}_h, \Pi_h A_h\}$ is also well adapted to A.

Again, simple modifications yield a similar result for reflexive Banach spaces. The important hypothesis is now the uniform invertibility of the operators $\Pi_h A_h$. Even though the operators A_h are assumed to be uniformly invertible, it is not a priori clear that they can be inverted within, say, some finite dimensional subspaces. In the following section we shall describe a way to construct the (finite element) spaces X_h in such a way that the above lemma can be applied.

4 Localized Spaces

Given an $h > 0$ and that $A_h : V_h \longrightarrow V_h$ is invertible, our goal is to show that $\Pi_h A_h : X_h \longrightarrow X_h$ is also invertible if we choose the spaces X_h in the right way. Since h will be fixed, we shall drop this subscript in this section. In addition, we will assume that $V_h = L^2(Y)$ for some space of homogeneous type (Y, ρ, μ). We will heavily rely on some of the ideas in [12] and [1].

A finite dimensional space $X \subset L^\infty(Y)$ is called a localized space with variable scale if the following properties hold.

i) There are pairwise disjoint sets $E_j \subset Y$ whose union equals Y, points $P_j \in E_j$, and positive numbers ρ_j, K, c_0, and ϵ_0, such that each point in Y lies in at most K of the sets $B_j = B(P_j, \rho_j)$, $\rho_j > \epsilon_0$, $E_j \subset B_j$, and

$$||f||_{L^\infty(E_j)} \le \frac{c_0}{\mu(B_j)} \sup \left| \int_Y fw \, d\mu \right|, \tag{11}$$

where the supremum is taken over all $w \in X$ with $||w||_\infty \le 1$ and w supported in B_j.
ii) All constant vectors belong to X.

We say that the covering $\{B_j\}$ has the the finite intersecton property and we will refer to (11) as the localization property. Since we are assuming X to be of finite dimension, $X^2 = X \cap L^2(Y)$ is automatically a closed subspace of $L^2(Y)$. Hence, the projection of $V = L^2(Y)$ onto X^2 is well defined. Again, the assumption concerning the finite dimensionality of X is made only to simplify the presentation somewhat. In the case that Y is the boundary of a Lipschitz domain, it is shown in [12] and [1] that the localization property implies that $X^p = X \cap L^p$ is actually a closed subspace of L^p and that there is always a bounded projection onto X^p. The arguments therein extend mutatis mutandis to the case of a space of homogeneous type. Condition *ii*) is a technical one and its use will become clear later on.

In applications, the space X will play the role of a boundary element space. It is not hard to check that if, for example, X is a space consisting of piecewise linear functions over a partition of some hypersurface in R^n, then the localization property is satisfied.

We shall now restrict our attention to the case of operators of the form $A = I + T$ and we shall state a condition on T which will guarantee the invertibility of ΠA.

Let X be a localized space with variable scale. An operator T is said to satisfy the local approximation property with constant δ if there are vectors $\phi_j \in X$ such that

$$\sum_j \int_{B_j} |Tf - \phi_j|^2 \, d\mu \le \delta ||f||_{L^2}^2, \quad \text{if } f \in L^2(Y). \tag{12}$$

Notice that because of the boundedness of T and the finite intersection property, there always exists some δ for which (12) is satisfied. We can simply take $\phi_j \equiv 0$. The problem is to make δ small.

Condition (12), though technical, is not unusual in dealing with singular integrals. In fact, quite often one is lead to substract some average or the value of a function at a particular point to exploit some cancellaton or decay of the kernel of an operator. It turns out that (12) is also related to compactness. We refer to [1] for more details about this. In any case, being able to make δ small is all that we need, as the next result shows.

Theorem 1

Suppose that X is a localized space of variable scale in a space Y of homogeneous type. Then, there exists $\delta_0 > 0$ with the following property: if the operator T satisfies the local approximation property with constant $\delta < \delta_0$, then

$$(13) \qquad ||\Pi A \Pi f||_{L^2} \geq C||\Pi f||_{L^2} \quad \text{for all } f \in L^2(Y).$$

Here C and δ_0 only depend on the operator norm of A^{-1}, the parameters in the definition of the space of homogenous type, and the parameters K and c_0 in the definition of X.

We shall sketch the proof of this theorem to illustrate the natural use of the Hardy–Littlewood maximal operator in this problem.

We may assume $f \in X$ and show that $||\Pi A f||_{L^2} \geq C||f||_{L^2}$. Since A is invertible, it is enough to show that

$$||Af||_{L^2} \leq C||\Pi A f||_{L^2} + \delta ||f||_{L^2}$$

for a sufficiently small δ. By writing

$$Af = Af - Tf + \phi_j + Tf - \phi_j,$$

and using (12) we are then reduced to estimating $||Af - Tf + \phi_j||$. But,

$$Af - Tf + \phi_j = f + \phi_j \in X,$$

so, by (11), for $x \in E_j$,

$$\begin{aligned} |(Af - Tf + \phi_j)(x)| &\leq \frac{c_0}{\mu(B_j)} \sup_w \left| \int (Af - Tf + \phi_j) w \, d\mu \right| \\ &\leq M(\Pi A f)(x) + C \left(\int_{B_j} \frac{|Tf - \phi_j|^2}{\mu(B_j)} d\mu \right)^{1/2}. \end{aligned}$$

Hence,

$$\int_{E_j} |Af - Tf + \phi_j|^2 \, d\mu \leq C \left(\int_{E_j} (M(\Pi Af)(x))^2 + \int_{B_j} |Tf - \phi_j|^2 \right).$$

By the finite intersection propety, (12), and the boundedness of the Hardy–Littlewood maximal operator we obtain

$$\begin{aligned} ||f||_{L^2}^2 &\leq C||Af||_{L^2}^2 \\ &\leq C(||M(\Pi Af)||_{L^2}^2 + \delta||f||_{L^2}^2) \\ &\leq C(||\Pi Af||_{L^2}^2 + \delta||f||_{L^2}^2), \end{aligned}$$

which finishes the proof if δ is sufficiently small. Notice that since we are assuming that the space X is of finite dimension, the theorem actually gives the invertibility of ΠA on X^2.

Next we shall show that when T is a weakly singular integral operator, then the approximation property is indeed satisfied for δ small if the balls B_j are picked sufficiently small.

Let $T : L^2(Y) \to L^2(Y)$ be a linear and continuous operator. We say that T is a weakly singular integral operator if

$$Tf(x) = \int_Y K(x,y) f(y) d\mu(y),$$

and, for some $\kappa > 0$, $L > 0$, $0 < \alpha < 1$ and $0 < \epsilon \leq 1$, the kernel K satisfies

(14) $$|K(x,y)| \leq \kappa\mu(B(x,\rho(x,y)))^{\alpha} \quad \text{for all } x \neq y$$

(15) $$|K(x,y) - K(z,y)| \leq Lm(x,z)^{\epsilon}\mu(B(x,\rho(x,y)))^{-1-\epsilon}$$

for $\rho(x,z) \leq \frac{1}{2}\rho(x,y)$. Here $m(x,z) = \inf_{x,z\in B} \mu(B)$.

We have the following pointwise estimate.

Lemma 4

Let T be a weakly singular integral operator and X a localized space. Then, for each $\gamma > 2$ and a given $f \in L^2(Y)$, there exist $\phi_j \in X$ such that

(16) $$\begin{aligned} |(Tf - \phi_j)(x)| \leq\; & C\,\kappa\mu(B(p_j,(\gamma+1)\rho_j))^{1-\alpha}|Mf(x)| \\ & + C\,L\left(\frac{\mu(B(p_j,\rho_j))}{\mu(B(p_j,(\gamma-2)\rho_j))}\right)^{\epsilon}|Mf(x)| \end{aligned}$$

for all $x \in B_j$. Here the constant C only depends on the parameters in the definition of the space of homogeneous type Y.

To prove the lemma, we fix γ and choose $\phi_j \equiv T(f\chi_{Y\setminus\tilde{B}_j})(P_j)$, where $\tilde{B}_j = B(P_j, \gamma\rho_j)$. Here we use the fact that constants are in X. We write

$$\begin{aligned} |(Tf-\phi_j)(x)| &= |T(f\chi_{\tilde{B}_j})(x) + T(f\chi_{Y\setminus\tilde{B}_j})(x) + \phi_j| \\ &\leq |T(f\chi_{\tilde{B}_j})(x)| + |T(f\chi_{Y\setminus\tilde{B}_j})(x) - T(f\chi_{Y\setminus\tilde{B}_j})(P_j)| \\ &\leq I + II. \end{aligned}$$

Using part (i) of Lemma 1 and (14), we get that I is bounded by the first term on the right hand side of (16). Similarly, to estimate II, we exploit the extra decay of $K(x,y) - K(z,y)$ "at infinity" and use part (ii) of Lemma 1 to get the second term in (16).

The above lemma yields the main result of this section.

Theorem 2

Let $A = I + T$ be invertible in $V = L^2(Y)$, where T is a weakly singular integral operator. Let X be a localized space with variable scale given by a covering $\{B_j\}$. Then, there exists $\rho_0 > 0$ with the property that if $\sup_j \rho_j \leq \rho_0$, then $\Pi A : X^2 \longrightarrow X^2$ is also invertible. Moreover, ρ_0 only depends on the operator norm of A^{-1}, the parameters in the definition of the space Y of homogenous type, the parameters K and c_0 in the definition of X, and the constants κ and L associated with T.

We only need to show that T satisfies the approximation property with constant $\delta < \delta_0$ where δ_0 is given by Theorem 1. This will follow from the finite intersection property and the boundedness of the Hardy–Littlewood maximal operator if we can prove that

$$|(Tf - \phi_j)(x)| \leq \delta_1 |Mf(x)|$$

for $x \in B_j$ and a sufficiently small constant δ_1. To accomplish this we first chose γ sufficiently large so that the second term on the right hand side of (16) is bounded by $\delta_1|Mf(x)|$. Finally, with that γ fixed, we can take care of the other term in (16) by imposing $\sup_j \rho_j \leq \rho_0$ for a sufficiently small ρ_0.

Recall that in our aproximation procedure, we will have to use Theorem 1 and Theorem 2 at each stage h. Now, in the applications, all the parameters which δ_0 depends on will be uniformly bounded in h, and, hence, $\delta_h = \delta_0$ will work at every stage. This yields the uniform invertibility of the operators ΠA_h. However, to obtain the approximation property with constant $\delta < \delta_0$ at stage h, will force us to take the number ρ_h in Theorem 2 smaller and smaller as h goes to zero. The reason for this is that the constants κ_h associated with the oerators T_h will blow up (all the other parameters which ρ_h depends on will remain uniformly bounded in h). From the numerical point of view, the behaviour of ρ_h as $h \longrightarrow 0$ determines how the "grid", or the support of the finite elements used to approximate the solution of the original equation, has to be refined. In this sense, we want to remark that the dependence of ρ_0 on κ in Theorem 2 is local. That is, the kernel of T may satisfy (14) with different constants κ_j on appropriate balls $\tilde{B}_j$. We see from the discussion about the proofs of Lemma 4 and Theorem 2 that is only the blow up of the constants κ_{jh} at level h that must be compensated for with a smaller size of the corresponding ρ_{jh} (cf. [1]).

We conclude this section by pointing out that all of the above results remain valid if we replace the "balls" by another family of sets for which the corresponding Hardy–Littlewood maximal function is bounded on L^p.

5 Boundary Value Problems

Within the general context of the previous sections, we shall next discuss a Galerkin method for elliptic equations, along the lines of [12] (cf. also [1]), as well as a version for parabolic equations.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. For a function f defined on the boundary of Ω, let us consider the Dirichelet problem for the Laplace equation:

$$(D) \quad \begin{cases} \Delta u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega. \end{cases}$$

In addition, we let $S_T = \partial\Omega \times (0,T)$, and consider, for a function f defined on S_T, the initial–Dirichlet problem for the heat equation:

$$(I-D) \quad \begin{cases} \Delta u &= \partial_t u \text{ in } \Omega \times (0,T) \\ u &= f \text{ on } S_T \\ u(x,0) &= 0 \text{ in } \Omega. \end{cases}$$

Both problems can be solved by employing layer potentials. For a function g defined on $\partial\Omega$ and $x \in \Omega$, the electrostatic double layer potential is defined by

$$\mathcal{D}g(x) = c_n \int_{\partial\Omega} \frac{< x - Q, \nu(Q) >}{|x-Q|^n} g(Q)d\sigma(Q),$$

while for a g defined on S_T and $(x,t) \in \Omega \times [0,T)$ the caloric double layer potential is given by

$$\mathcal{D}g(x,t) = k_n \int_0^t \int_{\partial\Omega} \frac{< x - Q, \nu(Q) >}{(t-s)^{\frac{n}{2}+1}} \exp\left(-\frac{|x-Q|^2}{4(t-s)}\right) g(Q,s)d\sigma(Q)ds.$$

Here c_n and k_n are two normalizing constants that depend on the dimension, ν is the unit normal to the boundary of Ω, and the integrals over $\partial\Omega$ are with respect to surface measure σ.

These layer potentials satisfy the partial differential equations in (D) and $(I-D)$, respectively, but they have boundary values given by $Ag(P) = (\frac{1}{2}I + T)g(P)$, $P \in \partial\Omega$ and $Ag(P,t) = (\frac{1}{2}I+T)g(P,t)$, $(P,t) \in S_T$. Here, in both cases, T is an operator whose kernel is obtained from the kernel of the corresponding $\mathcal{D}$ by formally replacing x by P. Unlike the classical case of a smooth domain, these kernels are no longer locally integrable (even if the domain is C^1), and the operators T have to be interpreted in the principal value sense.

In order to solve (D) or $(I-D)$, we may then take $u = \mathcal{D}(A^{-1}f)$. In the process of showing that this works, the fact that A is bounded on $L^p(\partial\Omega)$ or $L^p(S_T)$ is needed. This, in turn, is a consequence of some deep results from harmonic analysis about the boundedness of the Cauchy integral operator due to Calderón [5] and Coifman-McIntosh-Meyer [6]. The result in [5] was used in [13] and [15] to study the case of C^1 domains. It was also shown in those works that the methods from functional analysis used in the smooth case still apply when the domain is C^1, and, as a consequence, the operator A is invertible in L^p for $1 < p < \infty$. Nevertheless, these "soft" methods do not give the best estimates of the operator norm of A^{-1}, thus preventing the use of some approximation procedure to consider the Lipschitz case.

The result in [6] yields the boundedness of A in the Lipschitz case, but different techniques have to be used to prove invertibility. The invertibility of A in L^2 now follows from a priori energy estimates derived from certain Rellich type identities, see [18] and [3], and it follows that the operator norm of A^{-1} only depends on the Lipschitz character of the domain. Moreover, a more detailed analysis shows that A is also invertible in L^p for a certain, optimal range of p. This was done in [10] and [4].

Now, we want to approximate the integral equation $A = \frac{1}{2}I + T$, following the general procedure in the previous sections. (Notice that the presence of the factor $\frac{1}{2}$ is not important, since we can always take $\tilde{A} = 2A = I+2T = I+\tilde{T}$.) To obtain a first approximation, we want to regularize T in order to pass to weakly singular operators. In the present situation this can be done in a very simple manner. The idea is to approximate the domain Ω by smooth domains Ω_h and the cylinder $\Omega \times (0,T)$ by smooth cylinders $\Omega_h \times (0,T)$ for which the singularity of the kernels become locally integrable. The precise technical conditions on the approximating domains Ω_h are given by the following lemma from [12].

Lemma 5

Given a bounded Lispchitz domain in Ω, there exists a family of smooth domains $\{\Omega_h\}$, $0 < h < h_0$ with the following properties.
i) There is a finite covering $\{U\}$ of $\partial\Omega$ with open sets such that for every U there is an orthonormal coordinate system (x,y) and a smooth function $\varphi_h : R^{n-1} \to R$ with the property that $U \cap \Omega_h = U \cap \{(x,y) : \ y > \varphi_h(x)\}$ and $U \cap R^n \setminus \overline{\Omega_h} = U \cap \{(x,y) : \ y < \varphi_h(x)\}$. Furthermore, to each U there is a Lipschitz function φ such that $U \cap \Omega = U \cap \{(x,y) : \ y > \varphi(x)\}$ and $U \cap R^n \setminus \overline{\Omega} = U \cap \{(x,y) : \ y < \varphi(x)\}$.

ii) There is a positive constant L such that the φ_h's satisfy $\|\varphi - \varphi_h\|_\infty \leq Lh$, $\|\nabla\varphi_h\|_\infty \leq L$ and $\nabla\varphi_h \to \nabla\varphi$ a.e. as $h \to 0$. Furthermore, there is a Lipschitz homeomorphism $F_h : \partial\Omega \to \partial\Omega_h$ such that locally $F_h(\xi, \varphi(\xi)) = (\xi, \varphi_h(\xi))$. The Jacobians of F_h converge almost everywhere to 1 and the Lipschitz constants of F_h and their inverses are uniformly bounded in h.
iii) The Hessian of φ_h satisfies the bound $\|\nabla_2\varphi_h\|_\infty \leq L/h$.

We shall call such a family Ω_h, $h > 0$, a smooth approximation of Ω. For the parabolic case, we let $S = S_T$ and $S_h = S_{T,h} = \partial\Omega \times (0, T)$, and we have Lipschitz homomorphisms $G_h : S \to S_h$ defined by $G_h(P, t) = (F_h(P), t)$, so that locally $G_h((\xi, \varphi(\xi)), t) = ((\xi, \varphi_h(\xi)), t)$, with properties analogous to those of F_h.

Notice that $\partial\Omega$, with the restriction of the ecludian metric of R^n and with the surface measure, is a space of homogeneous type. Similarly, S, with the restriction of the parabolic metric in $\mathrm{R}^n \times \mathrm{R}$, given by

$$\rho((P,t),(Q,s)) = \sum_{i=1}^{n} |P_i - Q_i|^2 + |t - s|,$$

and the measure given by $d\sigma \times dt$, is also a space of homogeneous type. In both cases we shall just denote them by Y. At each level h, associated with the approximating domains or cylinders, we have in the analogous way spaces of homogeneous type Y_h.

We now let $V = L^2(Y)$ and $V_h = L^2(Y_h)$. In the elliptic case, we define the prolongation and restriction operators by $p_h f = f \circ F_h$ for $f \in V_h$ and $r_h f = f \circ F_h^{-1}$ for $f \in V$. In the parabolic case, we define $p_h f = f \circ G_h$ for $f \in V_h$ and $r_h f = f \circ G_h^{-1}$ for $f \in V$. It is easy to check that in both cases $\{V_h, p_h, r_h\}_h$ is a stable and convergent generalized approximation of the Hilbert space V. Moreover, if we let $A_h = I + T_h$ be the boundary value operators of the double layer potentials corresponding to the approximating domains or cylinders, then $\{V_h, p_h, r_h, A_h\}_h$ is well adapted to A. To proof of this last assertion is a rather straightforward exercise about singular integrals. Using the properties of $\partial\Omega$, one is led to show that for every $f \in V$, $Tf - \tilde{T}_h f$ tends to zero, where $\tilde{T}_h$ is a version in the original domain or cylinder of the operator T_h, obtained by a change of variables. This can be achieved in a "standard" way by truncating the integrals and by analyzing the parts "close to" and "far away from" the singularity of the kernel. More details can be read off from the arguments in [18], p. 586-587 and [3], p. 374-377.

By the results in [18] and [3], the operators A_h are uniformly invertible. Hence, the first approximation lemma applies, and we have accomplished our first goal of reducing the problem from the singular integral equation to a weakly singular one. We now want to proceed with the second approximation in order to discretize our problem.

At each stage h we select localized spaces X_h as in Section 4. In the elliptic case the balls are just the intersection of R^n balls with the boundary of the domain, while in the parabolic case, for convenience, we chose the "balls" to be of the form

$$B((P_0,t_0),r) = \{\{P : |p - P_0| < r\} \times \{t : |t - t_0| < r^2\}\} \cap S_h,$$

for some $(P_0, t_0) \in S_h$.

By Theorem 1 and Theorem 2, we will have finished the second approximation procedure once we have shown how small the radii of the balls at each stage h have to be. Now, in local coordinates, the factor $< P - Q, \nu(Q) >$ appearing in the kernels of the operators T_h can be written as

$$(\varphi_h(x) - \varphi_h(y) - \nabla\varphi_h(y)(x-y))(1 + |\nabla\varphi_h(y)|^2)^{-1/2},$$

and by the properties of the approximating domains we have the estimate

$$|\varphi_h(x) - \varphi_h(y) - \nabla\varphi_h(y)(x-y)| \leq \frac{L}{h}|x-y|^2.$$

Using this it is easy to see that the estimate in Lemma 4 now takes the form

$$|(T_h f - \phi_h)(x)| \leq CL(\frac{\gamma\rho_j}{h} + \frac{1}{\gamma})|Mf(x)|$$

where C is independent of h. (Actually, in the parabolic case a modification of the proof of Lemma 4 is required since the operators T_h do not satisfy (14), although they do satisfy (15). We shall omit the details). Moreover, following the remarks at the end of Section 4, we can also localize the above estimate to

$$|(T_h f - \phi_{j,h})(x)| \leq C(\gamma\rho_{j,h}\kappa_{j,h} + \frac{L}{\gamma})|Mf(x)|,$$

where $\kappa_{j,h} = ||\nabla_2\varphi_{j,h}||_\infty$. Finally to get the second approximation we only need to take γ large (independent of h) and

$$\lim_{h\to 0}\sup_j \kappa_{j,h}\rho_{j,h} = 0.$$

Notice that $\kappa_{j,h}$ is a measure of the local curvature of the domains $\partial\Omega_h$. Thus, at the "corners" of the domain Ω, and because of the properties of the approximating domains, the $\kappa_{j,h}$ blow up like $1/h$. However, where the original domain is "flat", or smooth, this behavior can be improved. As a consequence, the refinement of the mesh size (given by the $\rho_{j,h}$'s) can be carried out in an adaptive way, taking into consideration the local regularity of the original domain Ω.

6 An Error Estimate

In order to be able to use the above Galerkin method for the solution of boundary value problems numerically, it is crucial to have some error estimate. We have shown that if $\tilde{g}_h \in X_h^2$ is the unique solution of $\Pi_h A_h \tilde{g}_h = \tilde{r}_h f$, where $\tilde{r}_h = \Pi_h r_h$, then $\tilde{p}_h \tilde{g}_h = p_h \tilde{g}_h$ converges strongly to g. The aim here is to show a certain rate of this convergence in the case that g has some additional regularity. In the elliptic case error estimates were derived in [1]. Here we will briefly indicate how to obtain an estimate in the parabolic case, which is technically more involved. Again, we will only consider the L^2 result. Complete proofs will appear elsewhere.

The smoothness we require of g can be described by the following space.

Let $H^{1/2}(-\infty, T)$ denote the collection of functions in $L^2(-\infty, T)$ for which the norm

$$||u||^2_{H^{1/2}(-\infty,T)} \equiv \int_{-\infty}^{T} u^2(s)ds + \int_{-\infty}^{T}\int_{-\infty}^{T} \frac{|u(t)-u(s)|^2}{|t-s|^2} dsdt$$

is finite. We shall denote the second term by $||u||^2_{1/2;T}$. Furthermore, we let $H^{1,1/2}(S_T)$ be the closure of $\{v : v = u|_{S_T} \text{ with } u \in C_0^\infty(\mathrm{R}^n \times (0,\infty))\}$ with respect to the norm

$$||v||^2_{H^{1,1/2}(S_T)} \equiv \iint_{S_T} |\nabla_{tan} v|^2 + v^2(P,t)dPdt + \int_{\partial D} ||v(P,\cdot)||^2_{1/2;T} dP.$$

If $v_i \longrightarrow v$ in $H^{1,1/2}(S_T)$ and v_i are smooth in a neighborhood of S_T we let $\nabla_{tan} v$ denote the limit of $\nabla_{tan} v_i$. Here, and in the sequel, ∇ refers to the space variables only.

As before, we let $S_{T,h} = \partial\Omega_h \times (0,T)$, where $\partial\Omega_h$ is the boundary of the smooth approximating domain. We assume that on each $S_{T,h}$ there is a function space X_h with all the properties discussed in the previous sections. Recall that we also have $G_h : S_T \longrightarrow S_{T,h}$ defined by $G_h(P,t) = (F_h(P),t)$, and $p_h f = f \circ G_h$ and $r_h f = f \circ G_h^{-1}$. Suppose that $f \in L^2(S_T)$ and that $g \in L^2(S_T)$ is the unique solution of $Ag = f$, where A is the limit of the double layer potential. We have that $Ag(P,t) = \lim_{x \to P \in \partial\Omega} \mathcal{D}g(x,t) = (\frac{1}{2} + T)f(P,t)$, where we let $Tg(P,t) = \lim_{\varepsilon \to 0^+} T_\varepsilon f(P,t)$ with

$$T_\varepsilon f(P,t) = k_n \int_0^{t-\varepsilon} \int_{\partial\Omega} \frac{< P-Q, \nu(Q) >}{(t-s)^{\frac{n}{2}+1}} \exp(-\frac{|P-Q|^2}{4(t-s)}) g(Q,s) dQds.$$

For notational convenience, we have set $k_n = \frac{1}{2}(4\pi)^{-n/2}$. Further, we let T_h and $T_{h,\varepsilon}$ denote the corresponding entities but relative to $S_{T,h}$ instead. The following theorem is our main result.

Theorem 3

Suppose $\rho_h = \max_j \rho_{j,h} = \mathcal{O}(h)$ *and that* $g \in H^{1,1/2}(S_T)$. *Then*

$$\begin{aligned}||g - p_h\tilde{g}_h||_{L^2(S_T)} \leq\ & Ch\left(\iint_{S_T} |\nabla_{\tan} g|^2 dPdt\right)^{1/2} \\ & + Ch\sqrt{|\log h|}\left(\int_{\partial D} ||g(P,\cdot)||^2_{1/2;T} dP\right)^{1/2}.\end{aligned}$$

The basic idea is simple, and it is the same as the one employed in [1], for the corresponding elliptic situation. For the parabolic problem considered here, the use of the Fourier transform in the time variable reduces the analysis to an elliptic like problem. This type of approach was used in [15] for C^1 domains and in [3] for Lipschitz domains. We will make use of some of their arguments.

The main obstacle in obtaining an estimate of the rate of convergence discussed above is that $\nabla\varphi_h \to \nabla\varphi$ a.e., but not uniformly. However, $\varphi_h \to \varphi$ uniformly, with a rate h, so, exploiting the smoothness of the function g we can use Green's formula to move the derivatives from φ and φ_h to g and g_h, or rather to the remaining parts of the integrands in the layer potentials for $\partial\Omega$ and $\partial\Omega_h$, respectively.

The following lemma is straightforward.

Lemma 6

We have

$$\begin{aligned}||g - p_h\tilde{g}_h||_{L^2(S_T)} \leq\ & (1 + ||\tilde{A}_h^{-1}||\, ||T_h||)||r_h g - \Pi_h r_h g||_{L^2(S_{T,h})} \\ & + ||\tilde{A}_h^{-1}||\, ||T_h r_h g - r_h T g||_{L^2(S_{T,h})},\end{aligned}$$

where $\tilde{A}_h = \Pi_h A_h \Pi_h$.

The first term here is estimated by using the next lemma.

Lemma 7

Suppose $\rho_h = \mathcal{O}(h)$. *Then*

$$||r_h g - \Pi_h r_h g||_{L^2(S_{T,h})} \leq Ch(||\nabla_{\tan} g||_{L^2(S_T)} + (\int_{\partial D} ||g(p,\cdot)||^2_{1/2;T} dP)^{1/2}).$$

To estimate $||T_h r_h g - r_h T g||_{L^2(S_{T,h})} \leq ||Tg - p_h T_h r_h g||_{L^2(S_T)}$, we note that $T(1) = T_h(1) = \frac{1}{2}$ on the boundaries. Hence, we have

$$\begin{aligned}
&Tg(P,t) - p_h T_h r_h g(P,t) = \\
&\quad = T(g(\cdot,\cdot) - g(P,t))(P,t) - (p_h T_h r_h g(P,t) - \frac{1}{2} p_h r_h g(P,t)) \\
&\quad = T(g(\cdot,\cdot) - g(P,t))(P,t) - T_h(r_h g(\cdot,\cdot) - r_h g(G_h(P,t)))(G_h(P,t)) \\
&\quad = \lim_{\varepsilon\to 0+} [T_\varepsilon(g(\cdot,\cdot) - g(P,t))(P,t) - T_{h,\varepsilon}(r_h g(\cdot,\cdot) - r_h g(G_h(P,t)))(G_h(P,t))].
\end{aligned}$$

This is essentially a sum of integrals with kernels

$$K(P,t;Q,s) = \frac{< P-Q, \nu(Q) >}{(t-s)^{\frac{n}{2}+1}} \exp(-\frac{|P-Q|^2}{4(t-s)})$$

and

$$K_h(P,t;Q,s) = \frac{< F_h(P)-Q, \nu_h(Q) >}{(t-s)^{\frac{n}{2}+1}} \exp(-\frac{|F_h(P)-Q|^2}{4(t-s)}).$$

It is now enough to give an estimate independent of ε of the term ocurring inside the limit. We briefly sketch the basic ideas. Using a partition of unity, we can, via local coordinates, pull back the integrals in question to integrals over R^{n-1}. Using Green's formula we can move the derivatives away from the functions φ and φ_h, coming from the normals ν and ν_h in K and K_h above. Due to the localization, we can assume that the boundary terms vanish when using Green's formula, and that the integrals exist on R^n_+. We are essentially left with a sum of integrals of which the following two are typical:

$$\begin{aligned} I_1(y,t) &= -nk_n \int_0^{t-\varepsilon} \int_{\mathrm{R}^{n-1}} \frac{\Delta_{x,y}}{(t-s)^{\frac{n}{2}+1}} \exp(-\frac{L_{x,y}}{4(t-s)}) \\ &\quad \times (G(x,s)-G(y,s))dxds. \end{aligned}$$

and

$$I_2(y,t) = -nk_n \int_0^{t-\varepsilon} \int_{\mathrm{R}^{n-1}} \frac{\Delta_{x,y}}{(t-s)^{\frac{n}{2}+1}} \exp(-\frac{L_{x,y}}{4(t-s)})(G(y,s)-G(y,t))dxds$$

Here $\Delta_{x,y} = \varphi(x) - \varphi_h(x) - (\varphi(y) - \varphi_h(y))$, $L_{x,y} = |y-x|^2 + (\varphi(y) - \varphi(y))^2$, and G is obtained from g by a local change of variables.

To estimate I_2 we note that

$$\begin{aligned} |I_2(y,t)| &\leq (\int_0^{t-\varepsilon} (\frac{1}{(t-s)^{\frac{n}{2}}} \int_{\mathrm{R}^{n-1}} \Delta_{x,y} \exp(-\frac{|x-y|}{4(t-s)}) \Psi_0(x) dx)^2 ds)^{1/2} \\ &\quad \times (\int_0^{t-\varepsilon} \frac{|G(y,s)-G(y,t)|^2}{(t-s)^2} ds)^{1/2}, \end{aligned}$$

where Ψ_0 is an appropriate smooth "cut off" function. We separate the two cases $|x-y| < h$ and $|x-y| > h$, so that

$$\int_{\mathrm{R}^{n-1}} \Delta_{x,y} \exp(-\frac{|x-y|}{4(t-s)}) \Psi_0(x) dx = \int_{|x-y|<h} \cdots + \int_{|x-y|>h} \cdots$$

$$
\begin{aligned}
&\leq C(t-s)^{\frac{n}{2}} \int_{|u|<h/2(t-s)^{1/2}} |u| \exp(-|u|^2)du \\
&\quad + Ch(t-s)^{\frac{n-1}{2}} \int_{|u|>h/2(t-s)^{1/2}} \exp(-|u|^2)du \\
&= I + II.
\end{aligned}
$$

The term I_2 can thus be estimated by

$$
\begin{aligned}
\int_0^{t-\varepsilon} \left(\frac{1}{(t-s)^{\frac{n}{2}}} I\right)^2 ds &\leq C \int_0^{t-\varepsilon} \left(\int_0^{h/2(t-s)^{1/2}} r^{n-1} \exp(-r^2)dr\right)^2 ds \\
&= C \int_0^{t-h^2/4} \left(\int_0^{h/2(t-s)^{1/2}} r^{n-1} \exp(-r^2)dr\right)^2 ds \\
&\quad + C \int_{t-h^2/4}^{t-\varepsilon} \left(\int_0^{h/2(t-s)^{1/2}} r^{n-1} \exp(-r^2)dr\right)^2 ds \\
&\leq C \int_0^{t-h^2/4} \left(\frac{h}{2(t-s)^{1/2}}\right)^2 ds + C \int_{t-h^2/4}^{t-\varepsilon} ds \\
&\leq Ch^2 |\log h|.
\end{aligned}
$$

With similar estimates for the term involving II, we get

$$
|I_2(y,t)| \leq C(T) h \sqrt{|\log h|} \left(\int_0^{t-\varepsilon} \frac{|G(y,s) - G(y,t)|^2}{(t-s)^2} ds\right)^{1/2}.
$$

To estimate I_1 we follow [15] and use the Fourier transform. Then, remembering that the functions have support only for $t > 0$, we get

$$
\widehat{I}_1(y,\tau) = -nk_n \int_{\mathrm{R}^{n-1}} \frac{\Delta_{x,y}}{L_{x,y}^{n/2}} H^1(\varepsilon/L_{x,y}, \tau L_{x,y})(\widehat{G}(x,\tau) - \widehat{G}(y,\tau)dx,
$$

where we have used the notation $H^1(\varepsilon,\tau) = \int_\varepsilon^\infty \exp(-\frac{1}{4s} + i\tau s)ds/s^{n/2+1}$. This integral is essentially bounded by $ChM(\widehat{\nabla G}(\cdot,\tau))(y)$ plus two terms of the form

$$
C|\varphi_h(y) - \varphi(y)| \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{(\widehat{G}(x,\tau) - \widehat{G}(y,\tau))(\Psi_0(x))}{(|y-x|^2 + (\varphi(y) - \varphi(x))^2)^{n/2}} dx \right|
$$

and

$$
C \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{(\varphi_h(x) - \varphi(x))(\widehat{G}(x,\tau) - \widehat{G}(y,\tau))(\Psi_0(x))}{(|y-x|^2 + (\varphi(y) - \varphi(x))^2)^{n/2}} dx \right|.
$$

From the following lemma ([7] and [16]) we obtain the desired estimate.

Lemma 8
Let

$$T_\varepsilon(y) = \int_{|x-y|>\varepsilon} \frac{(f(x)-f(y))g(x)\prod_k(\Psi_i(x)-\Psi_i(y))}{(|x-y|^2+(\Phi(x)-\Phi(y))^2)^{\frac{n+k}{2}}}\,dx.$$

Suppose that $||\nabla\Phi||_\infty \leq M$*, and that* $k \geq 2$ *is an even integer. Then*

$$||\sup_{\varepsilon>0}|T_\varepsilon(y)|||_{L^r(R^{n-1})} \leq C(M)\prod_k ||\nabla\Psi_i||_{L^\infty(R^{n-1})}||\nabla f||_{L^p(R^{n-1})}||g||_{L^q(R^{n-1})}$$

where $\frac{1}{r} = \frac{1}{q}+\frac{1}{p}, 1 < p \leq \infty, 1 < q \leq \infty, 1 < r < \infty.$

Hence,

$$||I_1(\cdot,\tau)||_{L^2(\mathrm{R}^{n-1})} \leq Ch||\widehat{\nabla G}(\cdot,\tau)||_{L^2(\mathrm{R}^{n-1})}$$

and using the boundedness of the maximal function and Plancharel's theorem, we have

$$||I_1||_{L^2(\mathrm{R}^{n-1}\times\mathrm{R})} \leq Ch||\nabla G||_{L^2(\mathrm{R}^{n-1}\times\mathrm{R})}.$$

The theorem follows from these, (and similar), estimates.

To conclude, we mention that in the paper [9] Costabel has developed a different method to study numerically boundary value problems for parabolic equations using a combination of single and double layer potentials. However, for a general Lipschitz domain, his approach yields only an error estimate in the "energy" space $H^{-1/2,-1/4}(S_T)$ (see [9] for the precise definition of this Sobolev space of negative order and further details). This estimate is weaker than our L^2 result. We also want to point out that Costabel posed the problem of whether it was possible to develop a method using the classically preferred integral equations of the second kind. We have successfully solved this problem.

References

[1] V. Adolfsson, M. Goldberg, B. Jawerth and H. Lennerstad, *Localized Galerkin estimates for boundary integral equations on Lipschitz domains*, preprint.

[2] V. Adolfsson, B. Jawerth, and R. Torres, *Boundary elements for parabolic equations in non–smooth cylinders*, preprint.

[3] R. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math. 111 (1989), 339–37.

[4] R. Brown, *The initial–Neumann problem for the heat equation in Lipschitz cylinders*, Trans. Amer. Math. Soc. 320 (1990) 1–52.

[5] A. Calderón, *Cauchy integral on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 1324–1327.

[6] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitiziennes*, Ann. of Math. 116 (1982), 361–387.

[7] R. Coifman and Y. Meyer, *Non–linar harmonic analysis, operator theory and P.D.E.*, Beijing Lectures Notes in harmonic Analysis, Ann. of Math. Studies 112, (1986) 3–46.

[8] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645

[9] M. Costabel, *Boundary integral operators for the heat equation*, Integral Equations Oper. Theory 13 (1990), 498–552.

[10] B. Dahlberg and C. Kenig, *Hardy spaces and the L^p–Neumann problem for Laplace's equation in a Lipschitz domain*, Ann. of Math. 125(1987), 437–465.

[11] B. Dahlberg, C. Kenig, and G. Verchota, *Boundary Value Problems for the Systems of Elastostatics in Lipschitz Domains*, Duke Math. J. 57 (1988), 795–818.

[12] B. Dahlberg and G. Verchota, *Galerkin methods for the boundary integral equations of elliptic equations in non–smooth domains*, Contemporary Mathematics 107 (1990), 39–60.

[13] E. Fabes, M. Jodeit, and N. Rivière,*Potential techniques for boundary value problems on C^1 domains*, Acta Math. 141 (1978), 165–186.

[14] E. Fabes, C. Kenig, and G. Verchota, *The Dirichlet Problem for the Stokes System on Lipschitz Domains*, Duke Math. J. 5 (1988), 7–793.

[15] E. Fabes and N. Rivière, *Dirichlet and Neumann problems for the heat equation in C^1–cylinders*, Proc. Symp. Pure Math. 35 (1979), 179–198.

[16] N. Lin, *Galerkin method for the boundary integral equation of the Dirichlet problem of Laplace equations in Lipschitz domains*, Ph.D. Thesis, Washington University, 1990.

[17] R. Temam, *Numerical Analysis*, D. Reidel Publishing Company, Dordrecht-Holland, 1973.

[18] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal. 59 (1984), 572–611.

Vilhelm Adolfsson, Department of Mathematics, Box 1917, Brown University, Providence, RI 02912.
Björn Jawerth, Department of Mathematics, University of South Carolina, Columbia, SC 29208.
Rodolfo Torres, Courant Institute of Mathematical Sciences, New York University, New York, NY 10012.

Vilhelm Adolfsson
Brown University

Björn Jawerth
University of South Carolina

Rodolfo Torres
Courant Institute