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SOME REMARKS ON LIFTING AND INTERPOLATION THEOREMS

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ABSTRACT. The aim of this note is to clarify the relations between the Sarason-Nagy-Foias lifting theorem and the Cotlar-Sadosky lifting theorem. By exploiting the method of unitary extensions of an isometry, combined with an extension of Adamjan, Arov and Krein scattering approach, a generalization of Sarason's interpolation theorem is given, from which the above mentioned lifting theorems follow directly as well as formulas for all the solutions of related interpolation problems.

I. A GENERALIZATION OF SARASON'S INTERPOLATION THEOREM.

A fundamental operatorial approach to interpolation problems was started by the following theorem, due to Sarason's [S.1]:

(I.1) THEOREM. *Let* S *be the shift in* $L^2 \equiv L^2(\mathbf{T})$ *with respect to Lebesgue measure and* $K \subset H^2$ *a closed subspace such that its orthogonal complement with respect to* H^2 *is invariant under* S. *If* $T = P_K S_{|K}$ *and* $A' \in L(K)$ *commutes with* T *then there exists* $h \in H^\infty$ *such that* $A'g = P_K(hg)$, $\forall\, g \in K$, *and* $\|A'\| = \|h\|_\infty$.

As usual, $\mathbf{T}$ denotes the unit circle on the complex plane C and S is given by $(Sf)(z) \equiv zf(z)$. For any $p \geq 1$, $H^p = \{f \in L^p \equiv L^p(\mathbf{T}) : \hat{f}(n) = 0 \text{ if } n < 0\}$, where $\hat{f}$ is the Fourier transform of f. If G,H are Hilbert spaces, $L(G,H)$ is the set of bounded linear operators from G to H, $L(G) = L(G,G)$; if K is a closed subspace of G, P_K denotes the orthogonal projection of G onto K, i_K the injection of K in G and $G \ominus K$ the orthogonal complement of K in G. Also, $\vee$ means "closed linear span of".

In applications, the function h of the above statement solves several interpolation problems. It appears because the operator A can be "lifted" to an operator that commutes with the shift, which is consequently given by the multiplication by a fixed function. Thus, the description of all the "liftings" of A gives a description of all the solutions to the related interpolation problems. These remarks motivate the following

(I.2) THEOREM. *Let* $U_1 \in L(G_1)$ *and* $U_2 \in L(G_2)$ *be unitary operators in Hilbert spaces,* $B_1 \subset G_1$ *and* $B_2 \subset G_2$ *closed subspaces such that* $U_1 B_1 \subset B_1$, $U_2^{-1} B_2 \subset B_2$, $\vee\{U_1^n B_1 : n \leqslant 0\} = G_1$ *and* $\vee\{U_2^n B_2 : n \geqslant 0\} = G_2$. *If* $A \in L(B_1, B_2)$ *is such that*

$$A\, U_1\, i_{B_1} = P_{B_2}\, U_2\, A$$

set $\mathcal{A} = \{\tilde{A} \in L(G_1, G_2) : \tilde{A}U_1 = \tilde{A}U_2,\ A = P_{B_2} \tilde{A}\, i_{B_1},\ \|A\| = \|\tilde{A}\|\}$.

Then:

a) $\mathcal{A}$ *is non void.*

b) *Assume moreover that* $\|A\| = 1$. *Let* H *be a Hilbert space and* $r_1 : B_1 \to H$, $r_2 : B_2 \to H$ *isometries such that* $H = (r_1 B_1) \vee (r_2 B_2)$ *and* $A = r_2^* r_1$. *An isometry* W *acting in* H, *with domain* $(r_1 B_1) \vee (r_2 U_2^{-1} B_2)$ and range $(r_1 U_1 B_1) \vee (r_2 B_2)$ *is defined by* $W(r_1 b_1 + r_2 U_2^{-1} b_2) = r_1 U_1 b_1 + r_2 b_2$, $\forall\ b_1 \in B_1$, $b_2 \in B_2$.

Let $\mathcal{U}$ *be the family of the equivalence classes of minimal unitary extensions of* W. *There exists a bijection* α *from* $\mathcal{U}$ *to* $\mathcal{A}$ *that can be obtained as follows. If* $U \in L(G)$ *is such a unitary extension, let* $\tilde{r}_1 : G_1 \to G$ *and* $\tilde{r}_2 : G_2 \to G$ *be the isometries determined by* $\tilde{r}_1 U_1^n{}_{|B_1} = U^n r_1$ *and* $\tilde{r}_2 U_2^n{}_{|B_2} = U^n r_2$, $\forall\ n \in \mathbb{Z}$; *then*

$$\alpha(U,G) := \tilde{r}_2^*\tilde{r}_1 \in A.$$

A proof can be obtained from the following remarks, which we only sketch because they are essentially contained in the proof of the Nagy-Foias commutant lifting theorem given in [A.1] and in Sarason's presentation [S.2] of it.

(I.3) REMARKS

(a) If $X \in L(F_1,F_2)$ is a contraction between Hilbert spaces, $\exists$ F, a Hilbert space, and $\sigma_1: F_1 \to F$, $\sigma_2: F_2 \to F$, isometries, such that $F = (\sigma_1 F_1) \vee (\sigma_2 F_2)$ and $X = \sigma_2^*\sigma_1$. F can be obtained as the Hilbert space generated by $F_1 \times F_2$ and the scalar product $<(f_1,f_2),(f_1',f_2')> \equiv <f_1,f_1'>_{F_1} + <Af_1,f_2'>_{F_2} + <f_2,Af_1'>_{F_2} + $ $+ <f_2,f_2'>_{F_2}$ while σ_1,σ_2 are defined by the correspondences $f_1 \to (f_1,0)$, $f_2 \to (0,f_2)$, respectively.

(b) If, moreover, $V_1 \in L(F_1)$ and $V_2 \in L(F_2)$ are isometries such $X V_1 = V_2^* X$ then $V(\sigma_1 f_1 + \sigma_2 V_2 f_2) \equiv \sigma_1 V_1 f_1 + \sigma_2 f_2$ defines an isometry with domain $(\sigma_1 F_1) \vee (\sigma_2 V_2 F_2)$ and range $(\sigma_1 V_1 F_1) \vee (\sigma_2 F_2)$ such that $V\sigma_1 = \sigma_1 V_1$, $V^{-1}\sigma_2 = \sigma_2 V_2$. We say that V is the *isometric coupling* of V_1 and V_2 generated by X. If V_1 and V_2 are unitary operators, $V \in L(F)$ is also a unitary operator.

(c) Let D,R be closed subspaces of a Hilbert space F and V an isometry from D onto R. We say that $(V',F') \in U \equiv U(V,F)$ if $V' \in L(F')$ is a unitary extension of V to the Hilbert space $F' = \vee\{V'^n F: n \in \mathbf{Z}\}$; we say that $(V',F') \approx (V'',F'')$ in U if $\exists$ $\rho \in L(F',F'')$, a unitary operator such that $\rho_{|F} = I_F$ (the identity in F) and $\rho V' = V''\rho$. $U \equiv U(V,F)$ is the *set of equivalence classes of minimal unitary extensions of* V *to Hilbert spaces that contain* V. It is not empty. [and $(V',F') \approx (V'',F'')$ iff $P_F V'^n{}_{|F} = P_F V''^n{}_{|F}$, $\forall\, n \in \mathbf{Z}$].

(d) Assume the conditions of (a) and (b). For $j = 1,2$ let $V'_j \in L(F'_j)$ be a unitary extension of V_j such that the minimality condition $F'_j = \vee\{V'^n_j F_j : n \in \mathbf{Z}\}$ holds. Then:

1. If $(V',F') \in U$, isometric extensions $\sigma'_1 : F'_1 \to F'$ of σ_1 and $\sigma'_2 : F'_2 \to F'$ of σ_2 are determined by $\sigma'_1 V'^n_1|_{F_1} = V'^n \sigma_1$ and $\sigma'_2 V'^{-n}_2|_{F_2} = V'^n \sigma_2$, $\forall\ n \in \mathbf{Z}$, respectively. It follows that $\sigma'_1 V'_1 = V'\sigma'_1, \sigma'_2 V'_2 = V'^{-1}\sigma'_2$, V' is the coupling of V'_1 and V'_2 generated by $\sigma'^*_2\sigma'_1$ and $X = P_{F_2}\sigma'^*_2\sigma'_1|_{F_1}$. [If $\|X\| = 1$, $\|X\| = \|\sigma'^*_2\sigma'_1\|$].

2. $(V',F') \approx (V'',F'')$ in U iff, with obvious notation, $\sigma'^*_2\sigma'_1 = \sigma''^*_2\sigma''_1$.

3. Let $Y \in L(F'_1,F'_2)$ be a contraction such that $YV'_1 = V'^*_2 Y$ and $P_{F_2}Y|_{F_1} = X$. If $V' \in L(F')$ is the coupling of V'_1 and V'_2 generated by Y, then $(V',F') \in U$.

A proof of theorem (I.2) follows from the above remarks, with $X = A$ (assuming $\|A\| = 1$), $F_1 = B_1$, $F_2 = B_2$, $V_1 = U_1|_{B_1}$, $V_2 = U_2^{-1}|_{B_2}$, $F'_1 = G_1$, $F'_2 = G_2$, $V'_1 = U_1$, $V'_2 = U_2^{-1}$.

II. LIFTING THEOREMS

If $T \in L(H)$ is a contraction in a Hilbert space Nagy's dilation theorem shows that $\exists\ U \in L(G)$, unique up to unitary isomorphisms, such that U is unitary, $H \subset G = \vee\{U^n H : n \in \mathbf{Z}\}$ and $T^n = P_H U^n|_H$, $\forall\ n \geqslant 0$. U is called the minimal unitary dilation of T. An intertwining between two contractions can be lifted to an intertwining between their unitary dilations. That is the content of Nagy-Foias well known generalization of Sarason's interpolation theorem, which can be stated as follows.

(II.1) THEOREM. *For* $j = 1,2$ *let* $T_j \in L(H_j)$ *be a contraction in a Hilbert space and* $U_j \in L(G_j)$ *its minimal unitary dilation. If* $X \in L(H_1,H_2)$ *and* $XT_1 = T_2X$, *then* $\exists\, Y \in L(G_1,G_2)$ *such that:*

$$YU_1 = U_2Y \ , \ P_{H_2}Y_{|H_1} = X, \ \|Y\| = \|X\| .$$

Theorem (I.2) is a particular case of Nagy-Foias theorem, which is obtained by setting, in (II.1), $T_1 = U_{1|B_1}$ and $T_2 = P_{B_2}U_{2|B_2}$. Conversely, the commutant lifting theorem (II.1) follows from (I.2) if we set

$$B_1 = \vee\{U_1^nH_1 : n \geqslant 0\} \ , \ B_2 = \vee\{U_2^{-n}H_2 : n \geqslant 0\} \ , \ A = XP^{B_1}_{H_1} .$$

Concerning dilation and commutant lifting theorems basic references are [N-F] and [F-F].

In order to state another lifting type theorem we shall use the following notation: $e_n(t) = e^{int}$, $n \in \mathbf{Z}$ and $t \in \mathbf{R}$; P is the space of trigonometric polynomials, i.e., of finite sums $\Sigma a_n e_n$, with $n \in \mathbf{Z}$ and $a_n \in \mathbf{C}$, $P_+ = \{\Sigma a_n e_n \in P : a_n = 0 \text{ if } n < 0\}$, $P_- = \{\Sigma a_n e_n \in P : a_n = 0 \text{ if } n \geqslant 0\}$; $C(\mathbf{T})$ is the Banach space of complex continuous functions on $\mathbf{T}$ and $M(\mathbf{T})$ its dual, i.e., the space of complex Radon measures on $\mathbf{T}$. If $\mu = \{\mu_{jk}\}_{j,k=1,2}$ is a matrix with entries in $M(\mathbf{T})$ and $f = (f_1,f_2) \in C(\mathbf{T}) \times C(\mathbf{T})$, we set

$$\langle \mu f,f\rangle = \Sigma\{\int_{\mathbf{T}} f_j \bar{f}_k \, d\mu_{jk} : j,k=1,2\}.$$

Then the Cotlar-Sadosky theorem [C-S] can be stated as follows.

(II.2) THEOREM. *If the matrix measure* $\mu = \{\mu_{jk}\}_{j,k=1,2}$ *is such that* $\langle \mu f,f\rangle \geqslant 0$, $\forall\, f = (f_1,f_2) \in P_+ \times P_-$, *there exists a positive matrix measure* $\sigma = \{\sigma_{jk}\}_{j,k=1,2}$ *such that* $\langle \sigma f,f\rangle = \langle \mu f,f\rangle$, $\forall\, f \in P_+ \times P_-$.

The above statement implies that $\langle \sigma f,f\rangle \geqslant 0$, $\forall\, f = (f_1,f_2) \in C(\mathbf{T}) \times C(\mathbf{T})$, i.e., $\{\sigma_{jk}(\Delta)\}$ is a positive matrix for any Borel set $\Delta \subset \mathbf{T}$, and

$$\sigma_{11} = \mu_{11} \quad , \quad \sigma_{22} = \mu_{22} \quad , \sigma_{12} = \overline{\sigma}_{21} = \mu_{12} + h\, dt,$$

with dt Lebesgue measure in $\mathbf{T}$ and $h \in H^1$.

Theorem (II.2) can be obtained by setting, in (I.2), $G_1 = L^2(\mu_{11})$, $G_2 = L^2(\mu_{22})$, U_1 and U_2 the corresponding shifts, $B_1(B_2)$ the closure of P_+ (P_-) in $G_1(G_2)$ and defining $A \in L(B_1, B_2)$ by

$$<Af_1, f_2> = \int_{\mathbf{T}} f_1 \overline{f}_2 \, d\mu_{12} \quad , \quad \forall\ (f_1, f_2) \in P_+ \times P_-.$$

Then $\widetilde{A}$ as in (I.2) is given by the multiplication by a function $u = \widetilde{A}e_0$ so $<\widetilde{A}f_1, f_2> = \int_{\mathbf{T}} f_1 \overline{f}_2\, u\, d\mu_{22}$, $\forall\ (f_1, f_2) \in P \times P$. Since $\|\widetilde{A}\| = \|A\| \leqslant 1$, the matrix measure σ given by $\sigma_{11} = \mu_{11}$, $\sigma_{22} = \mu_{22}$, $\sigma_{12} = \overline{\sigma}_{21} = u\, d\mu_{22}$ is as stated.

III. A FUNCTIONAL VERSION OF THE INTERPOLATION THEOREM

All the functions that solve several interpolation problems are given by the following consequence of theorem (I.2). In its statement $L^2(E)$ denotes the space of measurable functions $f: \mathbf{T} \to E$, a separable Hilbert space, such that $\|f\|^2 := \int_{\mathbf{T}} \|f(t)\|_E^{\,2} dt < \infty$, while $L^\infty(\mathbf{T}; E_1, E_2)$ is the space of essentially bounded measurable functions $\theta: \mathbf{T} \to L(E_1, E_2)$ and M_θ is the operator from $L^2(E_1)$ to $L^2(E_2)$ given by $M_\theta f(t) = \theta(t) f(t)$. Each such θ can be obtained as the boundary values of its Poisson transform, which we also call θ and is a bounded function such that $\theta(\rho e^{it}) = \Sigma\{\rho^{|n|} e^{int} \hat{\theta}(n) : n \in \mathbf{Z}\}$, $\rho \in [0,1)$, $\hat{\theta}(n) \in L(E_1, E_2)$ and $\sup\{\|\theta(z)\| : z \in \mathbf{D}\} < \infty$. In fact $\hat{\theta}$ can be obtained by $\hat{\theta}(n) = P_{E_2} S_2^{\,-n} M_\theta|_{E_1}$.

(III.1) THEOREM. *For* $j = 1,2$ *let* E_j *be a separable Hilbert space,* S_j *the shift in* $L^2(E_j)$ *and* B_j *a closed subspace of* $L^2(E_j)$ *such that:* $S_1B_1 \subset B_1$, $S_2^{-1}B_2 \subset B_2$, $\vee\{S_1^nB_1: n \leqslant 0\} = L^2(E_1)$, $\vee\{S_2^nB_2: n \geqslant 0\} = L^2(E_2)$. *If* $A \in L(B_1,B_2)$ *is such that*

$$A\, S_1|_{B_1} = P_{B_2} S_2\, A$$

set $F_A = \{\theta \in L^\infty(\mathbf{T};E_1,E_2): A = P_{B_2}M_\theta|_{B_1}, \ \|\theta\|_\infty = \|A\|\}$. *Then:*

a) F_A *is non void.*

b) Assume moreover that $\|A\| = 1$. *Let* H *be a Hilbert space and* $r_1: B_1 \to H$, $r_2: B_2 \to H$ *isometries such that* $H = (r_1B_1) \vee (r_2B_2)$ *and* $A = r_2^*r_1$. *An isometry* W *acting in* H, *with domain* $(r_1B_1) \vee (r_2S_2^{-1}B_2)$ *and range* $(r_1S_1B_1) \vee (r_2B_2)$ *is defined by* $W(r_1b_1 + r_2S_2^{-1}b_2) = r_1S_1b_1 + r_2b_2$, $\forall\, b_1 \in B_1$, $b_2 \in B_2$.

A bijection from the family U *of the equivalence classes of minimal unitary extensions of* W *and* F_A *is obtained by associating to each* $(U,G) \in U$ *the function*

$$\theta(z) = P_{E_2}\tilde{r}_2^*[U(U-zI)^{-1} + \bar{z}(U^*-\bar{z}I)^{-1}]\tilde{r}_1 i_{E_1} , \quad |z| < 1 ,$$

with the isometry $\tilde{r}_j: L^2(E_j) \to G$ *determined by* $\tilde{r}_jS_j^n|_{B_j} = U^nr_j$, $\forall n \in \mathbf{Z}$, $j = 1,2$.

This result follows from (I.2) because each $\tilde{A} \in A$ is such that $\tilde{A}S_1 = \tilde{A}S_2$, so it is given by the multiplication by a function θ, (i.e., $\tilde{A} = M_\theta$ and $\|\tilde{A}\| = \|\theta\|_\infty$) which can be obtained by the formula

$$\theta(\rho e^{it}) = \Sigma\{\rho^{|n|}e^{int}P_{E_2}S_2^{-n}\tilde{A}i_{E_1}: n \in \mathbf{Z}\} , \ \rho \in [0,1),$$

where i_{E_1} is the injection of E_1 into $L^2(E_1)$ and P_{E_2} the projection of $L^2(E_2)$ onto E_2, while $\tilde{A} = \tilde{r}_2^*\tilde{r}_1$ and $S_2^{-n}\tilde{r}_2^* = \tilde{r}_2^*U^{-n}$.

In applications, the additional conditions $E_1 \subset B_1$ and $E_2 \subset S_2B_2$ usually hold. Then, with the notation of (III.1), $\tilde{r}_1 i_{E_1} = $ $= i_H r_1 i_{E_1}$. Also, $\tilde{r}_2 S_2 i_{(S_2^{-1}E_2)} = U i_H r_2 i_{(S_2^{-1}E_2)}$, so $P_{E_2}\tilde{r}_2{}^* = $ $= P_{E_2} S_2 r_2{}^* P_H U^*$. Thus, since $\theta(z) = P_{E_2}\tilde{r}_2{}^*[U(U-zI)^{-1} + $ $+ \bar{z}(U^*-\bar{z}I)^{-1}]\tilde{r}_1 i_{E_1}$, we see that in this case

$\theta(z) = P_{E_2} S_2 r_2{}^*\{P_H[(U-zI)^{-1} + \bar{z}(I-\bar{z}U)^{-1}] i_H\} r_1 i_{E_1}$.

If W is, as above, an isometry acting in a Hilbert space H and U is a unitary extension of W to a space $G \supset H$, the function $\psi_U: D \to L(H)$ given by $\psi_U(z) = P_H[(I-zU)^{-1}] i_H$ is called a generalized resolvent of W. Let N and M be the orthogonal complements in H of the domain and the range of W, respectively; if $H^\infty(D;N,M)$ is the space of analytic functions $\phi: D \to L(N,M)$ such that $\|\phi\|_\infty := \sup\{\|\phi(z)\|: z \in D\} < \infty$, a bijection from the unit ball of this space, $\{\phi \in H^\infty(D;N,M): \|\phi\|_\infty \leqslant 1\}$ onto the set $\{\psi_U: U \in U(W,H)\}$ of all generalized resolvents of W is given by Chumakin's formula [Ch]:

$$\psi_U(z) = \{I-z[WP_D+\phi(z)P_N]\}^{-1}$$

Consequently:

(III.2) COROLLARY. *In the same conditions and notation of theorem* (III.1.b) *assume also that* $E_1 \subset B_1$ *and* $E_2 \subset S_2B_2$. *Let* N *and* M *be the orthogonal complements in* H *of the domain and the range of* W, *respectively, and set* $B(D;N,M) = \{\phi \in H^\infty(D;N,M): \|\phi\|_\infty \leqslant 1\}$. *Then a bijection between* $B(D;N,M)$ *and* F_A *is obtained by associating to each function* $\phi \in B(D;N,M)$ *the function* $\theta \in F_A$ *given by*

$$\theta(z) = P_{E_2} S_2 r_2{}^*\{z^{-1}[\psi(\bar{z})^*-I] + \bar{z}\psi(\bar{z})\} r_1 i_{E_1}$$

with $\psi(z) = \{I-z[WP_D + \phi(z)P_N]\}^{-1}$.

IV. SOME REMARKS ON APPLICATIONS

We obtain a proof of Sarason's theorem (I.1) and a description of all the functions h as in its statement if, in (III.1) and (III.2), we set $E_1 = E_2 = \mathbb{C}$, $B_1 = H^2$, $B_2 = H^2_- \oplus K$ (where H^2_- denotes the orthogonal complement in L^2 of H^2), $A = A'P^{H^2}K$. Sarason's theorem gives elegant proofs of the solutions of the classical interpolation problems of Nevanlinna-Pick and Carathéodory-Féjer; by the same method we can apply (III.1) and (III.2) to those problems.

The scope of Sarason's interpolation method has been extended by the following result, due to Rosenblum and Rovnyak.

(IV.1) THEOREM. *Let a linear operator* ρ *in a vector space* X *and* $b, c \in X$ *be given. Let* F *a linear subspace of the dual space* X' *such that* $\rho' F \subset F$ *and* $\sum_{j=0}^{\infty} |(\rho^j c, x')|^2 < \infty$, $\forall\ x' \in F$. *The following are equivalent:*

(i) $\exists\ g \in H^{\infty}$ *such that* $\|g\|_{\infty} \leqslant 1$ *and* $(b, x') = \sum_{j=0}^{\infty} \hat{g}(j)(\rho^j c, x')$, $\forall\ x' \in F$.

(ii) *For all* $x' \in F$, $\sum_{j=0}^{\infty} |(\rho^j b, x')|^2 \leqslant \sum_{j=0}^{\infty} |(\rho^j c, x')|^2$.

In [R-R] (IV.1) is proved as a consequence of the commutant lifting theorem. The same ideas show that (III.1) and (III.2) can be applied, as we now sketch. Let K be the closure in H^2 of $\{\sum_{j=0}^{\infty} (\rho^j c, x') e_j;\ x' \in F\}$ and set in (III.1) $E_1 = E_2 = \mathbb{C}$, $B_1 = H^2$, $B_2 = H^2_- \oplus K$; if condition (ii) in (IV.1) holds there exists a contraction $X \in L(K, H^2)$ such that $X[\sum_{j=0}^{\infty} (\rho^j c, x') e_j] = \sum_{j=0}^{\infty} (\rho^j b, x') e_j$, $\forall\ x' \in F$. Setting $A = i_{B_2} X^*$, (III.1) and (III.2) give the functions θ such that each $g(z) := [\theta(\bar{z})]^-$ is as stated.

In a similar way we can prove the following vectorial extension of Nehari's theorem, due to Page [See N]. Recall that, if E is a separable Hilbert space, then $L^2(E) = \oplus\{S^jE: j \in \mathbf{Z}\}$, so for each $f \in L^2(E)$ $\exists\, \hat{f}: \mathbf{Z} \to E$ such that $f = \Sigma\{e_j\hat{f}(j): j \in \mathbf{Z}\}$; in fact, $\hat{f}(j) = P_E S^{-j} f$. Then $H^2(E) = \{f \in L^2(E): \hat{f}(j) = 0 \text{ if } j < 0\}$ and $H^2_-(E)$ is the orthogonal complement in $L^2(E)$ of $H^2(E)$. Let E_1 and E_2 be separable Hilbert spaces, S_1 and S_2 the corresponding shifts in $L^2(E_1)$ and $L^2(E_2)$, respectively. Set $H^\infty(\mathbf{T};E_1,E_2) = \{\theta \in L^\infty(\mathbf{T};E_1,E_2): \hat{\theta}(n) = 0 \text{ if } n < 0\}$. A linear operator Γ from the space H_0 of linear combinations of vectors e_jv, $j \geqslant 0$ and $v \in E_1$ to $H^2_-(E_2)$ is a Hankel operator if $P_{H^2_-(E_2)} S_2\Gamma = \Gamma S_{1|H_0}$. If Γ is bounded, it defines an operator from $H^2(E_1)$ to $H^2_-(E_2)$ that we also call Γ. Then:

(IV.2) THEOREM. a) *A Hankel operator* Γ *is bounded iff* $\exists\ \theta \in L^\infty(\mathbf{T};E_1,E_2)$ *such that*

$$(*) \qquad \Gamma = P_{H^2_-(E_2)} M_{\theta|H^2(E_1)}$$

in which case $\|\Gamma\| = \text{dist}\{\theta, H^\infty(\mathbf{T};E_1,E_2)\}$.

b) *If* Γ *is a bounded Hankel operator,* $\exists\ \theta \in L^\infty(\mathbf{T};E_1,E_2)$ *such that* (*) *holds and* $\|\Gamma\| = \|\theta\|_\infty$.

In fact, let $\theta \in L^\infty(\mathbf{T};E_1,E_2)$ and set $\Gamma_\theta = P_{H^2_-(E_2)} M_{\theta|H^2(E_1)}$; clearly, Γ_θ is a Hankel operator and $\|\Gamma_\theta\| \leqslant \|\theta\|_\infty$. If $\theta_1 - \theta \in L^\infty(\mathbf{T};E_1,E_2)$ then $\Gamma_{\theta_1} = \Gamma_\theta$ so $\|\Gamma_\theta\| \leqslant \text{dist}\{\theta, H^\infty(\mathbf{T};E_1,E_2)\}$. If Γ is a bounded Hankel operator, we assume that $\|\Gamma\| = 1$ and apply (III.1) with $B_1 = H^2(E_1)$, $B_2 = H^2_-(E_2)$, $A = \Gamma$; thus F_A is non void. If $\theta \in F_A$ then $\Gamma = \Gamma_\theta$ and $\|\Gamma\| = \|\theta\|_\infty$; thus,

$\|\Gamma\| = \|\Gamma_\theta\| \leq \text{dist}\{\theta, H^\infty(\mathbf{T};E_1,E_2)\} \leq \|\theta\|_\infty = \|\Gamma\|$. Consequently: $\Gamma = P_{H^2_-(E_2)} M_\theta|_{H^2(E_1)}$ and $\|\Gamma\| = \|\theta\|_\infty = \text{dist}\{\theta, H^\infty(\mathbf{T};E_1,E_2)\}$ iff $\theta \in F_A$. All such functions are described by (III.2).

Let us still remark that a Schur type analysis of the family of minimal unitary extensions of an isometry can be developed in a natural way [A.2], and thus applied to interpolation problems of the type we have considered in this note; when it is applied to the Carathéodory-Fejér problem it gives the classical Schur parameters [A.3].

REFERENCES

[A-A] V.M.ADAMJAN and D.Z.AROV, *On unitary couplings of semi-unitary operators*, Amer.Math.Soc.Transl.Ser.2,95 (1970), pp.75-129.

[A-A-K] V.M.ADAMJAN, D.Z.AROV and M.G.KREIN, *Infinite Hankel matrices and generalized problems of Carathéodory-Fejér and I.Schur*, Funkcional.Anal.Prilozhen.. 2,4 (1968), pp.1-17.

[A.1] R.AROCENA, *Unitary extensions of isometries and contractive intertwining dilations*, Op.Theory: Adv.and Appl.41 vol.II (1989), pp.13-23.

[A-2] R.AROCENA, *Schur Analysis of a class of translation Invariant Forms*, in Analysis and Partial Differential Equations. A Collection of Papers dedicated to Mischa Cotlar (C.Sadosky ed.) Marcel Dekker, New York and Basel, 1990.

[A-3] R.AROCENA, *On a geometric interpolation of Schur parameters*, Revista de la Unión Matemática Argentina 34 (1988), pp.150-165.

[C-S] M.COTLAR and C.SADOSKY, *On the Helson-Szegö theorem and a related class of modified Toeplitz kernels*, Proc. Symp. Pure Math.Amer.Math.Soc.25.I (1979),pp.383-407.

[Ch] M.E.CHUMAKIN, *Generalized resolvents of isometric operators*, Sibirskii Matem.Zhurnal Vol.8,N°4 (1967), pp. 876-892.

[F-F] C.FOIAS and A.E.FRAZHO, *The Commutant Lifting Approach to Interpolation Problems*, Birkhäuser Verlag, Basel-Boston-Berlin, 1990.

[N-F] B.Sz.NAGY and C.FOIAS, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdan, 1970.

[N] N.K.NIKOLSKII, *Treatise on the Shift Operator*, Springer-Verlag, New York, 1986.

[R-R] M.ROSENBLUM and J.ROVNYAK, *Hardy Classes and Operator Theory*, Oxford University Press, New York, 1985.

[S.1] D.SARASON, *Generalized interpolation in* H^∞, Trans.Am. Math.Soc. 127 (1967), pp.179-203.

[S.2] D.SARASON, *New Hilbert Spaces from Old*, in Paul Halmos. Celebrating 50 Years of Mathematics. (J.Ewing & F.W. Gehring eds.), Springer-Verlag, 1991, pp.195-204.

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