Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

DISPERSAL MODELS

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Introduction

We consider in this paper a generalization of the Soboloff potential theorem to the case of "Parabolic potentials"(see definition below). The proof we consider here avoids the use of Marcinkiewicz's general interpolation theorem and, it instead uses a parabolic version of the maximal theorem. These tools are used to address problems of dispersal in sections II and III.

I. Parabolic Potential Operators

We consider the coordinate transformation

$x_1 = \rho^{a_1} \operatorname{coc}\varphi_1 \ldots \cos\varphi_{n-1}$,

$x_2 = \rho^{a_2} \cos\varphi_1 \ldots \cos\varphi_{n-1} \sin\varphi_{n-1}$,

$x_3 = \rho^{a_3} \cos\varphi_1 \ldots \cos\varphi_{n-2}$,

$x_4 = \rho^{a_4} \cos\varphi_1 \ldots \cos\varphi_{n-3} \sin\varphi_{n-2}$,

................................,

$x_n = \rho^{a_n} \sin\varphi_1$.

Where $a_i \geq 1$ and $\sum_1^n \left[\frac{x_i}{\rho^{a_i}}\right]^2 = 1$.

With the parabolic distance $\rho(x,y) = \rho(x-y)$ with root ρ and $\sum_1^n \left[\frac{x_i - y_i}{\rho^{a_i}}\right]^2 = 1$. For details and the fact that ρ is a distance see Fabes and Riviere [6]. We shall be concerned with operators of the form

$$T_\beta(f) = \int_{\mathbb{R}^n} \frac{1}{[\rho(x-y)]^{|a|-\beta}} f(y)\, dy$$

where $|a| = a_1 + a_2 + \dots + a_n$ and $0 < \beta < |a|$.

Theorem A :(Here, we adapt partially the proof in Stein [19])

$$\|T_\beta(f)\|_q < C_p\|f\|_p, \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{\beta}{|a|}.$$

Proof:

Let K be the kernel defined by $K = \frac{1}{\rho^{|a|-\beta}}$.

We now choose $\alpha = \frac{p}{p\beta - |a|}$ and define the functions K_1 and K_2 by

$$K_1 = \begin{cases} K \text{ if } \rho(x) \geq \lambda^\alpha \\ 0 \text{ if } \rho(x) < \lambda^\alpha \end{cases}$$

and

$K_2 = K - K_1$. Where $\lambda > 0$ is a fixed constant.

Then for $\|f\|_p = 1$ and $p^* = \frac{p}{p-1}$, we have

$$|K_1 * f| \leq \|K_1\|_{p*} \|f\|_p \tag{*}$$

and

$$\|K_1\|_{p*} = C\lambda$$

where C is a fixed constant independent of λ or f.

Likewise

$$\|K_2\|_1 = C\int_0^{\lambda^\alpha} \frac{1}{\rho^{|a|-\beta}} \rho^{|a|-1}\, d\rho = C\lambda^{\alpha\beta}.$$

From this we see that

$|\{ K_2 * f > \lambda \}| \leq \frac{1}{\lambda^p} \|K_2\|_1^p.$

If $\|f\|_p = 1$, using the fact that $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{|a|}$ we obtain

$$|\{ K_2 * f > \lambda \}| \leq \frac{C_1}{\lambda^q}.$$

Selecting now $\|f\| = \epsilon_0$ (small) we first see that

$$|K*f| \leq |K_1*f| + |K_2*f|$$

On the other hand $\{|K*f| > \lambda \} \subset \{|K_1*f| > \lambda/2\} \cup \{|K_2*f| > \lambda/2\}$.

Since ϵ_0 is small then $\{|K_1*f| > \lambda/2\} = \emptyset$. In view of what was said before, we get for f such that $\|f\|_p = \epsilon_0$

$$|\{K*f > \lambda| = |\{K_2*f\} > \lambda/2| \leq \frac{C_1}{\lambda^q}.$$

This shows the weak type estimates. Marcinkiewicz's interpolation theorem gives the desired result.

It is , however, interesting to consider the following alternative proof that does not make use of Marcinkiewicz's results. The proof relies on the following simple observation(generalization of Natanson's Lemma [11]).Since K is a decreasing function of the parabolic distance.

$|\int K_2(x-y)f(y)dy| \leq \|K_2\|_1 \cdot M(f)(x)$, where $M(f)(x) = \sup_{R > 0} \frac{1}{R^{|a|}} \int_{\rho(x-y) < R} |f(y)|\,dy$

The above result tells us that

$\{ K*f > \lambda \} \subset \{ cK_2*f > \lambda/2\} \subset \{ c\lambda^{\alpha\beta} M(f)(x) > \lambda/2\}$. Hence

$$D_{K*f}(\lambda) \leq C D_{M(f)}(\tfrac{1}{2}\lambda^{1-\alpha\beta}).$$

On using the value that we have for α and the fact that $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{|a|}$, we see that

$$\int_0^\infty D_{K*f}(\lambda)\lambda^{q-1}d\lambda \leq C_2\int_0^\infty D_{M(f)}(\lambda^{1-\alpha\beta})\lambda^{q-1}d\lambda = C_3\int_0^\infty D_{M(f)}(s)\, s^{p-1}ds \leq C_4\|f\|_p^p.$$

II. **Dispersal:**

Murray [10], Okubo [12], Shigesada [17] and Shigesada et al [18] have considered models of insect dispersal by using a nonlinear diffusion model

$$\frac{\partial u}{\partial t} = c\nabla\bullet[u^m\nabla u]\ ;\ m > 0 \tag{1}$$

where $u(\mathbf{x},t)$ represents the population density of insects; $\mathbf{x}$ could be either a one dimensional or a two dimensional variable. Equation (1) also plays a role in diffusion through porous media (see Murray [10]). In the one dimensional case Okubo [12,page 99] considers the diffusion problem $\frac{\partial u}{\partial t} = D_0\frac{\partial}{\partial x}(\frac{u}{u_0})^m\frac{\partial u}{\partial x}$ and uses Pattle's [14] solution to model the dispersal of a flock of insects that at time $t = 0$ are concentrated on the origin of coordinates. The solution in this case is given by

$$u = u_0(\frac{t_0}{t})^{\frac{1}{m+2}}(1 - \frac{x^2}{x_1^2})_+^{\frac{1}{m}}$$

where $x_1 \equiv r_0(\frac{t}{t_0})^{\frac{1}{m+2}}$, $r_0 \equiv \dfrac{Q\Gamma(1/m + \frac{3}{2})}{\sqrt{\pi}\, u_0\Gamma(1/m + 1)}$, $t_0 \equiv r_0^2\, m/2D_0(m+2)$,

Q being the initial flux of individuals from the origin and Γ is the gamma function. A steady state for the above equation has been studied by Shigesada [17], Teramoto [20], and Shigesada et al [18]. The equivalent plane radially symmetric problem with Q insects released at time $t = 0$ satisfies (see Murray,[10, page 239])

$$\frac{\partial u}{\partial t} = \frac{D_0 \partial}{r \partial r}\left[r\left(\frac{u}{u_0} \right)^m \frac{\partial u}{\partial r} \right].$$

The afore mentioned approaches do not shed some light on the dispersal of a flock of insects whose initial distribution is not concentrated at the origin, nor does it model the long range behavior of the flock when a general initial distribution is known. The purpose of this paper is to model the long range planar behavior of a flock of insects whose initial distribution is known and is not necessary a " δ " measure.

2.Insect Dispersal and Long Range Effects.

If u represents the population density of insects, then the conservation equation

$$\frac{\partial u}{\partial t} + \nabla \bullet J = F(u,x,t) \tag{2.1}$$

is known to form a reasonable basis for studying insect and animal dispersal which have been discussed in detail for the one dimensional case by Okubo in [12] and Shigesada in [17]. In (2.1), J is the flux of material and F, represents the source of material and is a function of u, x, and t. As it is known, one extension of the classical diffusion model which is of particular relevance to insect dispersal is when there is an increase in diffusion due to population pressure. Such models have the flux J, depending on the population density u where

$$J = -D(u)\nabla u, \quad \frac{dD}{du} > 0. \tag{2.2}$$

A typical form of $D(u)$ is $D_0\left(\frac{u}{u_0} \right)^m$, where $m > 0$ and D_0, u_0 are positive constants(see Murray[10], Okubo[12]). It is also known that insects at low population densities frequently tend to aggregate. A model reflecting this fact has a flux(see Murray[10]) given by $J = Uu - D(u)\nabla u$, where U is the transport velocity. An example occurs in the situation where the center of attraction is the origin and the velocity of attraction is constant. Shigesada et al [14] took $U = -U_0 \mathrm{Sgn}(x)$ for the one dimensional case.

In many instances, of insect outbreaks the densities of insect population are not small and a local or short range diffusion flux proportional to the gradient is not sufficiently accurate. Instead of simply taking $J \propto \nabla u$ we now consider

$$J = \underset{r \,\epsilon\, N(x)}{G[} \; \nabla u(\, x + r,t)\,] \tag{2.3}$$

where $N(x)$ is some neighborhood of the point x over which effects are noticed at x, and G is some functional of the gradient. The resulting form for the flux in (2.3) is then

$$J = -D_1(u)\nabla u + \nabla D_2(\nabla^2 u) \;;\; \frac{dD_1(u)}{du} > 0 \tag{2.4}$$

where D_2 is a constant which is a measure of the long range effects and in general is

smaller in magnitude than $D_1(u)$. This approach is due to Othmer[13].

If we now take the flux J as given by (2.1) we have

$$\frac{\partial u}{\partial t} = \nabla \bullet [D_1(u)\nabla u] - \nabla \bullet \nabla (D_2 \nabla^2 u) + F(u,x,t) \tag{2.5}$$

where the first term on the right hand side is the diffusion term and the middle term, the biharmonic term contributes to the long term effects. If we now take $D_1(u) = D_0 (\frac{u}{u_0})^m$, $m \geq 0$, where D_0 is the diffusivity for $u = u_0$(a reference concentration) and D_0, u_0 as usual are positive constants, then we will have as a model describing the long term effects of dispersal of insects density, the equation

$$\frac{\partial u}{\partial t} = -D_2 \nabla^4 u + D_0 \nabla \bullet [\, (\frac{u}{u_0})^m \nabla u\,] + F(u,x,t) \tag{2.6}$$

As $m > 0$, the diffusivity increases with u and in general D_2 is smaller in magnitude than $D_0(\frac{1}{u_0})^m$. Thus the effect of population pressure is incorporated in $D_0(\frac{1}{u_0})^m$. We will also give equation (2.6) an initial data

$$u(x,0) = g(x) \; ; \; x \in \mathbb{R}^2 \tag{2.7}$$

We will also consider the source F to be the logistic term $cu(A - u^s)$ where C is a constant.

Now we shall use the results in Jones [8] and Rosenbloom [15] applied to $\frac{\partial u}{\partial t} - (-1)^{\kappa/2}P(D)u = f \; ; \; u(x,0) = 0. \; ; \; \kappa = 4$ where P is an elliptic polynomial of degree m. There exist a fundamental solution $\Gamma(x,t) = t^{-1/2}\varphi(xt^{-1/4})$, where $\Gamma \in c^{\infty}(\mathbb{R}^2)$. If we now let the symbol $\otimes$ to represent the spatial and time variables convolution and the symbol $*$ to represent the usual spatial variable convolution, then a formal solution to (2.6), (2.7) can be written as

$$u(x,t) = \Gamma \otimes D_0\nabla\bullet[\,(\frac{u}{u_0})^m\nabla u\,] + \Gamma \otimes cu(A - u^s) + \Gamma * g \tag{2.8}$$

u(x,t) is a weak solution of (2.6), (2.7) if the integrals in (2.8) exists in the Lebesgue sense. Under this requirement (2.8) becomes via integration by parts

$$u(x,t) = \kappa_1 K \otimes u^{m+1} + \Gamma \otimes cu(A - u^s) + \Gamma * g \tag{2.9}$$

where $\kappa_1 = D_0 u_0^{-m}(m+1)^{-1}$ and $K = D_x^2\Gamma(x,t) = \sum_{i=1}^{2} \frac{\partial^2\Gamma(x,t)}{\partial x_i^2}$ and where we noted

that $u^m\nabla u = (m+1)^{-1}\nabla u^{m+1}$ so that $\nabla.(u^m\nabla u) = (m+1)^{-1}\nabla^2 u^{m+1}$.

Many techniques can be used to approximate the solution of the integral equation (2.9). However, here we will only indicate a successive approximations method similar to the ones in Calderón [2,3], Calderón and Kwembe [4,5] and Kwembe [9]. We will introduce some weak global solutions as well as local solutions of (2.6), (2.7) in some L^p spaces. Let ϕ be the nonlinear integral operator defined by

$$\phi(u)(x,t) = \kappa_1 K \otimes u^{m+1} + \Gamma \otimes C\,u(A - u^s) + \Gamma * g \tag{2.10}$$

Then by using the method of successive approximations we can show that the iterations

$$u_{k+1} = \phi(u_k) \tag{2.11}$$

converge in the norm defined below to the solution of (2.9) and that ϕ possesses a unique fixed point u satisfying $\|u\| \leq y_0$ for a suitable small y_0. Towards this effort we will consider the following standard estimate of $\Gamma(\mathbf{x},t)$.

$$|\Gamma(\mathbf{x},t)| \leq c(|\mathbf{x}| + t^{1/4})^{-2} \; ; \; t > 0 \tag{2.12}$$

and its spatial derivative estimate of

$$|\partial_{\mathbf{x}}^{\mathbf{n}}\Gamma(\mathbf{x},t)| \leq C(|\mathbf{x}| + t^{1/4})^{-2-n} ; \; t > 0 \tag{2.13}$$

where $|n| = n_1 + n_2$ and where n is the number of derivatives taken with respect to **x**.

DISPERSAL FOR ARBITRARY INITIAL VALUES

We now try to address the problem of obtaining a solution for equation (2.6), subject to the initial condition (2.7). First we consider the case when the source term $F(\mathbf{x},u,t)$ is zero. Then (2.6) reduces to

$$\frac{\partial u}{\partial t} = -D_2\nabla^4 u + D_0\nabla\bullet[(\frac{u}{u_0})^m \nabla u] \tag{2.14}$$

subject to the initial condition (2.7), which is not easy to solve in closed form even when one seeks solutions of the form

$$u(x,t) \propto \exp\{\sigma t + i\boldsymbol{\kappa}\bullet\mathbf{x}\} \tag{2.15}$$

Substituting this into (2.14) we have

$$\sigma = -D_2\kappa^4 - \frac{(m+1)D_0\kappa^2}{u_0^m}\exp\{ m\sigma t + im\boldsymbol{\kappa}\bullet\mathbf{x}\}\ ;\ \kappa = |\boldsymbol{\kappa}|$$

from which it is clear that wave – like solutions are achievable only for $m = 0$ (See Murray [10]).

We shall consider the case $m > 0$ and look at the existence and uniqueness of weak global solutions to (2.14) subject to (2.7). Which is the same as studying the existence and uniqueness of integral equation (2.10) when the source term $Cu(A - u^S) \equiv 0$. Thus after first considering the estimate of the integral operator

$$(\phi u)(x,t) = \kappa_1 K \otimes u^{m+1} + \Gamma * g\ ;\ m > 0. \tag{2.16}$$

We then have the following result:

Theorem 1: Suppose that the initial data $g \in L^m(\mathbb{R}^2)$; $m > 1$. If $\epsilon_0 > 0$ is small and g is such that $||g||_m < \epsilon_0$ .Then, there exists a unique solution u to the problem (2.14), subject to the initial data (2.7), that is global in time. The uniqueness holds in the class of functions u such that $||u||_{3m} < \infty$.

The proof of this theorem is achieved through auxiliary Lemmata. We shall present its proof and those of the connecting Lemmata in III

Remark 1 :

Our approach not only gives a medium range diffusion but allows us to solve insect dispersal for any set of initial values; whereas, Okubo [12] and Murray [10] provides a solution for the case of a "δ" initial value.

Remark 2 :

Following Murray [10, page 249], we take a cell potential energy approach to the long range diffusion. Here we let $u(\mathbf{x},t)$ be the density and μ the potential. Then the flux

J is given by

$$J = -D\nabla\mu(u)$$

where $\mu(u) = \frac{\delta E}{\delta u} = e'(u)$ and

$$E(u) = \int_V e(u)d\mathbf{x}.$$

is the total Energy in a volume V and e(u) is an internal energy per unit volume of an evolving spatial pattern. Thus if $e(u) = \frac{au^2}{2} + \frac{bu^4}{4}$ then the generalized diffusion is obtained to be

$$\frac{\partial u}{\partial t} = D_1\nabla^2 u - D_2\nabla^4 u + D_3\nabla^2 u^3 + F(\mathbf{x},u,t) \qquad (R_2,1)$$

where F is again the source term. If we give $(R_2,1)$ the initial value (2.7) then a formal solution when $F = 0$ is given by

$$u(\mathbf{x},t) = \kappa_1 K \otimes u + \kappa_2 K \otimes u^3 + \Gamma * g$$

$$\text{where } K = D_x^2\Gamma(\mathbf{x},t) = \sum_{i=1}^{2} \frac{\partial^2\Gamma(x,t)}{\partial x_i^2}.$$

using the same approach as above, we define the operator ϕ by

$$(\phi u)(\mathbf{x},t) = \kappa_1 K \otimes u + \kappa_2 K \otimes u^3 + \Gamma * g$$

Then similar results as above are obtainable.

3. Integral Equation Approach(An alternative view)

As noted before the derivations of equations of section 2 were achieved through Othmer's approach which neglected any memory effects. In general though, if $\phi(x)$ is an initial density of a dispersing biotic mass, then if

$$\int_{\mathbb{R}^2} \phi(x)dx > 0 ; \ t > 0,$$

the usual dispersing density $\int_{\mathbb{R}^2} \frac{1}{4\pi t} \exp\{-|x|^2/4t\}\phi(x-y)dx > 0 \ \forall \ y,t > 0$ propagates infinitely fast [15], hence, it is not a realistic view.

We now suppose that $A(t)$ represents the area covered by the dispersing δ biomass at time t. Then around the origin, the area exhibits the following properties,

1) $A(0) = 0$, 2) $A(t) = c_1 t + c_2 t^2 + o(t^2)$.

So that if the area is circular, for example, the radius of the circular region has maximum reach at time t. And so $r \sim Ct^{1/2}$, for $0 < t \leq \epsilon$, where ϵ is small. This estimate holds for any convex bounded region for which the origin is an interior point regardless of the modality of the diffusion.

Next, we suppose that φ is a function expressing the diffusion of a dispersing biomass concentrated at the origin in such a way that φ describes a characteristic function of a convex bounded region with the origin as an interior point. We require, further, that $\int \varphi dx = 1$. If there are no sources or sinks, after an initial distribution $f(x)$, the diffusion in the time span of $[0,t]$ will be given by

$$\int_{\mathbb{R}^2} f(y) \frac{1}{(t)} \varphi\left(\frac{x-y}{\sqrt{t}} \right) dy \qquad (3.1)$$

The function φ has to be chosen in accordance with the particular characteristics of the problem.(for instance the governing equations).

Let $\Psi(x,t) = \frac{1}{t} \varphi\left(\frac{|x|}{\sqrt{t}} \right)$

Then Ψ is the Kernel function that describes the diffusion of a δ biomass.

If $F(x,\tau)$ is the source function at time τ, then within the time interval $(\tau,\tau+d\tau)$, the differential area dy, the dispersing biomass density is given by

$$\Psi(x-y,t-\tau)F(y,\tau)dyd\tau\ ;\ \tau < t \tag{3.2}$$

Combining (3.1) and (3.2) we have the total dispersing biomass described by

$$u(x,t) = \int_{\mathbb{R}^2}\Psi(x-y,t)f(y)dy + \int_0^t\int_{\mathbb{R}^2}\Psi(x-y,t-\tau)F(y,\tau)dyd\tau \tag{3.3}$$

where $u(x,t)$ is the dispersing density located at position x at time t.

If $m(t)$ represent the biomass at time t, then the area $A(t)$ is related to $m(t)$ according to the allometric law

$$\frac{dA(t)}{A(t)} = c\frac{dm}{m}$$

which may also describes the basal metabolic rate of the average biological unit for special values of c. However, in general

$$A(t) \sim Ct^{\alpha}\ ;\ 0 < \alpha < 1$$

The usual assumption is $\alpha = 1/2$. The case in section 2 corresponds to $\alpha = 1/4$ which gives the biharmonic diffusion. If $\varphi(x)$ is radial continuously differentiable and decreasing, then:

$$\frac{d\varphi(r)}{dr} \le 0\ ;\ \varphi(r_m) = 0,\ \varphi(0) = \text{maximum.}\ r_m \text{ being the maximum reach.}$$

Then $\varphi(x) = 1 - c|x|^2 + o(|x|^2)$ or more generally, we may assume

$$\varphi(x) = 1 - C|x|^{\beta} + o(|x|^{\beta})\ ;\ \beta > 0.$$

Looking at this kernel asymptotically, we have

$$\varphi(x) = \exp\{\log[1 - C|x|^{\beta} + o(|x|^{\beta})]\}\ ;\ |x| < \epsilon_0.$$

Since $\dfrac{\log(1 - C|x|^{\beta})}{-C|x|^{\beta}} \to 1$ as $|\mathbf{x}| \to 0$, we have

$$\varphi(x) \sim \exp\{-C|x|^{\beta}\} \text{ for small } |x| \text{ and}$$

$$\exp\{-(c+\epsilon_0)|x|^{\beta}\} \le \varphi(x) \le \exp\{-(c-\epsilon_0)|x|^{\beta}\}\ ;\ |x| < \delta$$

which shows that for small $|x|$ the classical diffusion coincides with this approximation,where $\beta = 2$. Equation (3.3) can be solved by applying similar techniques to the ones studied in section 2.

III. Proof of Theorem 1

Here we shall give the proof of Theorem 1 starting with auxiliary results.

Lemma 1: Let B be a Banach space of Lebesgue measurable functions and let $\phi: B \to B$ be a continuous mapping such that

(1) $\|\phi(f)\| \leq \varphi(\|f\|) + \gamma$, where γ is a real number and φ is a convex non negative function satisfying

(2) $\varphi(0) = 0$ and $0 \leq \varphi'(0) < 1$.

(3) $\frac{\varphi(s)}{s} \to \infty$ as $s \to \infty$

Then there is an ϵ_0 such that whenever $\gamma < \epsilon_0$ we can find a $\delta = \delta(\gamma)$ such that

1) $\phi: B_\delta \to B_\delta$, where B_δ denotes a ball of radius δ

2) $\delta \to 0$ as $\gamma \to 0$.

Proof: consider the curve defined by $y = \varphi(x)$. Since φ is convex and by hypothesis(2) we see that the graph of $y = \varphi(x)$ is underneath the line $y = x$ for $0 < x < x_1$, where x_1 satisfies $x_1 = \varphi(x_1)$. The existence of x_1 is guaranteed by hypothesis (3) Next we consider the modification $y = \varphi(x) + \gamma$ for small γ so that a portion of the graph of $y = \varphi(x) + \gamma$ is underneath $y = x$. Then we can find x_1, x_2 such that $x_1 = \varphi(x_1) + \gamma$ and $x_2 = \varphi(x_2) + \gamma$ where $x_1 < x < x_2$. Thus for $x_1 < x < x_2$ $\varphi(x) + \gamma < x$. Therefore, if $x < x_1$, then $\varphi(x) + \gamma < x_1$. Take $x_1 = \delta(\gamma)$ of the hypothesis and the Lemma follows on noting that $B_\delta = \{ x: \varphi(x) + \gamma < x_1 \}$.

Lemma 2: If $g \in L^m(\mathbb{R}^2)$ and $u \in L^{3m}(\mathbb{R}^2 x \mathbb{R}_+)$; $m > 1/2$, then
$\|(\phi u)\|_{3m} \leq C(\kappa_1, 3m)\|u\|_{3m}^{m+} + C(3m)\|g\|_m$.

Proof: From estimate (2.12), (2.13) and equation (2.16) we have

$$|(\phi u)(x,t)| \leq \kappa_1 C \int_{\mathbb{R}_+}\int_{\mathbb{R}^2} \frac{|u(y,\tau)|^{m+1}\, dy d\tau}{(|x-y| + (t-\tau)^{\frac{1}{4}})^4} + |\Gamma * g|$$

Considering first the first term on the right hand side, we observed from Theorem A, that

$$K \leq \frac{C}{\rho^{|a|-2}}$$

where $a_1 = 1$, $a_2 = 1$ and $a_3 = 4$ and $\beta = 2$. Thus if we assume that

$|u(y,\tau)|^{m+1} \in L^{\frac{p}{m+1}}(\mathbb{R}^2 x\mathbb{R}_+)$ and q such that $\frac{1}{q} = \frac{m+1}{p} - \frac{2}{6}$.

Then Theorem A applied to the above inequality gives

$$\|(\phi\, u)\|_q \leq \kappa_1.C.C_p\|u\|_p^{m+1} + \|\Gamma * g\|_q$$

(A₁)

On taking $p = q$, we have that $p = q = 3m$ and (A₁) becomes

$$\|(\phi\, u)\|_{3m} \leq C(\kappa_1,m)\|u\|_{3m}^{m+1} + \|\Gamma * g\|_{3m}.$$

(A₂)

Next we will show that

$$\|\Gamma * g\|_{3m} \leq C(m)\|g\|_m$$

But this follows directly from the following lemma.

Lemma 3 : Let $F(x,t) = \Gamma * g$. Suppose that $g \in L^m(\mathbb{R}^2)$; $1 < m < \infty$. Then $F: L^{3m,3m}(\mathbb{R}^2 x\mathbb{R}_+) \to L^m(\mathbb{R}^2)$ and $\|F\|_{3m} \leq C(m)\|g\|_m$.

Proof: We shall first indicate a notation explanation. $\|F\|_{3m} \equiv \|F\|_{L^{3m}(\mathbb{R}^2 x\mathbb{R}_+)} \equiv \|F\|_{3m,3m}$.

Now by invoking estimate (2.12) we have

$$|F(x,t)| \le C\int_{\mathbb{R}^2} \frac{|g(x-y)|\,dy}{(|y| + t^{1/4})^2}$$

On taking the $L^{3m}(\mathbb{R}_+)$ norm in the t – variable of both sides we have

$$\Big(\int_{\mathbb{R}_+} |F(x,t)|^{3m} dt\Big)^{\frac{1}{3m}} \le C\Big[\int_{\mathbb{R}_+}\Big[\int_{\mathbb{R}^2} \frac{|g(x-y)|\,dy}{(|y| + t^{1/4})^2}\Big]^{3m} dt\Big]^{\frac{1}{3m}}.$$

Applying Minkowski's integral inequality on the right hand side we have

$$\|F(x,.)\|_{3m} \le C\int_{\mathbb{R}^2}|g(x-y)|\Big[\int_{\mathbb{R}_+} \frac{dt}{(|y|+t^{1/4})^{6m}}\Big]^{\frac{1}{3m}}.$$ From which we have

$$\|F(x,.)\|_{3m} \le C\nu\int_{\mathbb{R}^2} \frac{|g(y)|\,dy}{|x-y|^{2-\frac{4}{3m}}} \tag{A_3}$$

where ν does not exceed $\Big[\int_0^{\infty} \frac{dt}{(1 + t^{\frac{1}{4}})^{6m}}\Big]^{\frac{1}{3m}}$

On taking $a_1 = 1, a_2 = 1$ then $|a| = 2$ and on observing from Theorem A that

$$K \le \frac{C}{\rho^{|a|-\frac{4}{3m}}}$$

followed by the fact that $g \in L^m(\mathbb{R}^2)$, we apply Theorem A to (A_3) to get $\|F\|_{3m,3m} \le C(m)\|g\|_m$.

This concludes the proof of Lemma 2.

We now present the proof of Theorem 1. It suffice to prove that the nonlinear operator

$$(\phi u)(x,t) = \kappa_1 K \otimes u^{m+1} + \Gamma * g \tag{A_4}$$

is a contraction mapping of a ball of radius say r_0 into itself.

Suppose that u_1, u_2 are two functions belonging to this ball. the size of the ball determined by Lemma 1 above. Then

$$|(\phi u_1) - (\phi u_2)| \le \kappa_1 \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} |K(x-y,t-\tau)|\,|u_1^{m+1} - u_2^{m+1}|\,dy d\tau$$

On invoking estimate (2.13), we have

$$|(\phi u_1) - (\phi u_2)| \le C\kappa_2 \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \frac{(|u_1| + |u_2|)^m |u_1 - u_2|\, dy d\tau}{(|x-y| + (t-\tau)^{\frac{1}{4}})^4}$$

where $\kappa_2 = D_0 u_0^{-m}$ and where we have made use of the fact that

$$|u_1^{m+1} - u_2^{m+1}| \le (m+1)|u_1 - u_2|(|u_1| + |u_2|)^m.$$

We note again that $K \le \dfrac{C}{\rho^{|a|-2}}$ where $a_1 = 1$, $a_2 = 1$, $a_3 = 4$ and $|a| = 6$.

If we now let $(|u_1| + |u_2|)^m |u_1 - u_2| \in L^{\frac{p}{m+1}}(\mathbb{R}^2 x \mathbb{R}_+)$, then again Theorem A gives that

$$\|(\phi u_1) - (\phi u_2)\|_{3m} \le C(\kappa_2, m) \|(|u_1| + |u_2|)^m |u_1 - u_2|\|^{m+1}_{\frac{p}{m+1}}$$

where $\dfrac{1}{3m} = \dfrac{m+1}{p} - \dfrac{1}{3}$. This gives $p = 3m$.

Hence

$$\|(\phi u_1) - (\phi u_2)\|_{3m} \le C(\kappa_2, m) \|(|u_1| + |u_2|)^m |u_1 - u_2|\|^{m+1}_{\frac{3m}{m+1}}$$

If we now let $|u_1 - u_2| \in L^{3m}(\mathbb{R}^2 x \mathbb{R}_+)$ and $(|u_1| + |u_2|)^m \in L^3(\mathbb{R}^2 x \mathbb{R}_+)$, we have on the application of Hölder's inequality of exponents $\frac{m+1}{m}$, m+1 respectively

$$\|(\phi u_1) - (\phi u_2)\|_{3m} \leq C(\kappa_2,m)(\|u_1\|_{3m} + \|u_2\|_{3m})^m \|u_1 - u_2\|_{3m} \qquad (A_4)$$

If we now choose from Lemma 2, $\epsilon_0 > 0$ small such that $\|g\|_m < \epsilon_0$. Then the inequality of Lemma 2 satisfies condition (1) of Lemma 1. Hence, there exists a ball of radius say $r_0(\epsilon_0)$ such that $\phi : B_{r_0} \to B_{r_0}$. For $u_1, u_2 \in B_{r_0}$, we have from (A_4) that

$$\|(\phi u_1) - (\phi u_2)\|_{3m} \leq C(\kappa_2,3m)(2r_0)^m \|u_1 - u_2\|_{3m}.$$

On choosing $C(\kappa_2,3m) = 2^{-m}$, we have for r_0 small enough $C(\kappa_2,3m)(2r_0)^m < 1$. Hence ϕ is a contraction mapping a ball of radius $r_0(\epsilon_0)$ into itself. This completes the proof of theorem 1.

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