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MEAN VALUE AND HARNACK INEQUALITIES FOR A CERTAIN CLASS OF DEGENERATE PARABOLIC EQUATIONS

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1 Introduction

In this paper we study the behavior of solutions of degenerate parabolic equations of the form

(1.1) $$v(x)u_t(x,t) = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x,t)D_{x_j}u(x,t)),$$

where the coefficients are measurable functions, and the coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

(1.2) $$w_1(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq w_2(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2$$

for $\xi = (\xi_1, ..., \xi_n) \in R^n$ and $(x,t) \in \Omega \times (a,b)$, Ω a bounded open set in R^n.

We are going to assume some conditions on the weights (non-negative functions that are locally integrable) v, w_1, w_2 and on the functions λ_j, $j = 1, ..., n$, in order to be able to derive mean value and Harnack inequalities for solutions of (1.1). The assumptions on λ_j, which we list below, are the ones stated in [FL2].

(1.3) $\lambda_1 \equiv 1, \lambda_j(x) = \lambda_j(x_1, ..., x_{j-1}), j = 2, ..., n, \forall x \in R^n$.

(1.4) Let $\prod = \{x \in R^n : \prod x_k = 0\}$. Then $\lambda_j \in C(R^n) \cap C^1(R^n \setminus \prod)$ and $0 < \lambda_j(x) \leq \Lambda$, $\forall x \in R^n \setminus \prod$, $j = 1, ..., n$.

(1.5) $\lambda_j(x_1, ..., x_i, ..., x_{j-1}) = \lambda_j(x_1, ..., -x_i, ..., x_{j-1})$, for $j = 2, ..., n$ and $i = 1, ..., j-1$.

(1.6) There is a family of $n(n-1)/2$ non-negative numbers $\rho_{j,i}$ such that $0 \leq x_i(D_{x_i}\lambda_j)(x) \leq \rho_{j,i}\lambda_j(x)$, for $2 \leq j \leq n$, $1 \leq i \leq j-1$ and $\forall x \in R^n \setminus \prod$.

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Denote $\Gamma = \Omega \times (a,b)$ and define $H = H(\Gamma)$ to be the closure of $Lip(\Gamma)$ under the norm

$$\text{(1.7)} \qquad \|u\|^2 = \iint_\Gamma u^2(x,t)(v(x)+w_2(x))dxdt + \iint_\Gamma |\nabla_\lambda u(x,t)|^2 w_2(x)dxdt + \iint_\Gamma u_t^2(x,t)v(x)dxdt,$$

where $\nabla_\lambda u = (\lambda_1 D_{x_1}u, \dots, \lambda_n D_{x_n}u)$. Thus, $H(\Gamma)$ is the collection of all $(n+2)$-triples (u,β,B) such that there exists $u_k \in Lip(\Gamma)$ with $u_k \longrightarrow u$, $\nabla_\lambda u_k \longrightarrow \beta$, $(u_k)_t \longrightarrow B$, the convergence being in the appropriate L^2 space. We need to verify that β is uniquely determined and for this it is enough to show that for every $F \in C_0^\infty(\Gamma)$, $\int_\Gamma u\nabla_\lambda F = -\int_\Gamma \beta F$. In order to prove this, note that since $u \in H$, there exists $\{u_k\} \subset Lip(\Gamma)$ such that $u_k \longrightarrow u$ in H. Then, by (1.3),

$$\int_\Gamma u_k\lambda_i \frac{\partial F}{\partial x_i} = -\int_\Gamma \frac{\partial}{\partial x_i}(u_k\lambda_i)F = -\int_\Gamma \lambda_i \frac{\partial u_k}{\partial x_i}F.$$

Therefore,

$$\int_\Gamma u_k \nabla_\lambda F = -\int_\Gamma (\nabla_\lambda u_k)F.$$

By Schwarz's inequality and assuming that $w_2^{-1} \in L^1_{loc}$,

$$\left|\int_\Gamma u_k\nabla_\lambda F - \int_\Gamma u\nabla_\lambda F\right| \le \int_\Gamma |u_k-u|w_2^{1/2}|\nabla_\lambda F|w_2^{-1/2}$$
$$\le \|u_k-u\|_{L^2_{w_2}}\left(\int_\Gamma |\nabla_\lambda F|^2 w_2^{-1}\right)^{1/2} \le c\|u_k-u\|_{L^2_{w_2}}.$$

Hence, $\int_\Gamma u_k\nabla_\lambda F \longrightarrow \int_\Gamma u\nabla_\lambda F$ and similarly we can show $\int_\Gamma(\nabla_\lambda u_k)F \longrightarrow \int_\Omega \beta F$. In the same way we prove B is uniquely determined, if $v^{-1} \in L^1_{loc}$. We also define $H_o(\Gamma)$ to be the closure of $Lip_o(\Gamma)$, Lipschitz functions with compact support in Γ, under the norm defined in (1.7). It is easy to see that the bilinear form b on $Lip(\Gamma) \cap H(\Gamma)$ defined by

$$b(u,\phi) = \iint_\Gamma \{u_t\phi v + \langle A\nabla u, \nabla\phi\rangle\}dxdt$$

can be continued to all of $H(\Gamma)$ (here and in the rest of the paper the vector ∇u is understood to be the vector $(\frac{1}{\lambda_1}\beta_1, \dots, \frac{1}{\lambda_n}\beta_n)$ where $\nabla_\lambda u = (\beta_1,\dots,\beta_n)$). We say $u \in H(\Gamma)$ is a solution of (1.1) if $b(u,\phi) = 0$ for any $\phi \in H_o$; $u \in H(\Gamma)$ is a subsolution if $b(u,\phi) \le 0$ for any $\phi \in H_o(\Gamma)$, ϕ positive in the H-sense, i.e., ϕ can be approximated in $H(\Gamma)$ by positive functions with compact support in Γ; $u \in H(\Gamma)$ is a supersolution if $b(u,\phi) \ge 0$ for any $\phi \in H_o$, ϕ positive in the H-sense.

We also define $\tilde H = \tilde H(\Omega)$ to be the closure of $Lip(\Omega)$ under the norm

$$|||u|||^2 = \int_\Omega u^2(x)(v(x)+w_2(x))dx + \int_\Omega |\nabla_\lambda u(x)|^2 w_2(x)dx,$$

and $\tilde{H}_o(\Omega)$ to be the closure of $Lip_o(\Omega)$ under the norm defined above.

Next we will define a natural distance (associated with the functions $\lambda_j, j = 1, ..., n$) and state some of its properties. This metric was first introduced by [FL1].

A vector $v \in R^n$ is called a λ-subunit vector at a point x if $\langle v, \xi\rangle^2 \le \sum \lambda_j^2(x)\xi_j^2$, $\forall \xi \in R^n$. An absolutely continuous curve $\gamma : [0,T] \longrightarrow R^n$ is called a λ-subunit curve if $\dot\gamma(t)$ is a λ-subunit vector at $\gamma(t)$ for a.e. $t \in [0,T]$.

For any $x, y \in R^n$ we define $d : R^n \times R^n \longrightarrow R^+$ by

$$d(x,y) = \inf\{T \in R_+ : \text{there exists a } \lambda\text{-subunit curve } \gamma : [0,T] \longrightarrow R^n \text{ with } \gamma(0) = x, \gamma(T) = y\}.$$

One can check that this is a well-defined metric. There is a quasi-metric δ (a function $\delta : R^n \times R^n \longrightarrow R^+$ is called a quasi-metric if there exists $M \ge 1$ such that $\delta(x,y) \le M\{\delta(x,z) + \delta(z,y)\}$ for all x, y, $z \in R^n$) equivalent to d, and sometimes easier to work with than d (see [FL2]). If $x \in R^n$ and $t \in R$ put $H_o(x,t) = x$ and $H_{k+1}(x,t) = H_k(x,t) + t\lambda_{k+1}(H_k(x,t))e_{k+1}$ for $k = 0, ..., n-1$, where $\{e_k\}$ is the standard basis in R^n. Define $\varphi_j(x^\star, .) = (F_j(x^\star, .))^{-1}$, the inverse function of $F_j(x^\star, .)$, where $F_j(x,s) = s\lambda_j(H_{j-1}(x,s))$, for $j = 1, ..., n$ and $x^\star = (|x_1|, ..., |x_n|)$.

We define $\delta : R^n \times R^n \longrightarrow R^+$ as

$$\delta(x,y) = Max_{j=1,...,n}\varphi_j(x^\star, |x_j - y_j|).$$

Note that

(1.8) $\delta(x,y) < s$ is equivalent to $|x_j - y_j| < F_j(x^\star, s)$, $1 \le j \le n$.

In (1.9), (1.10), (1.11) below we state some basic facts concerning δ and d (see also [FL2]).

(1.9) There exists $a \ge 1$ such that for any $x, y \in R^n$,

$$a^{-1} \le \frac{d(x,y)}{\delta(x,y)} \le a.$$

Consequently, δ is a quasi-metric with $\delta(x,y) \le a^2[\delta(x,y)+\delta(z,y)]$ and $\delta(x,y) \le a^2\delta(y,x)$.

(1.10) For any $x \in R^n$, $s > 0$ and $\theta \in]0,1[$

$$\theta^{G_j} \le \frac{F_j(x^\star, \theta s)}{F_j(x^\star, s)} \le \theta$$

where $G_1 = 1$ and $G_j = 1 + \sum_{l=1}^{j-1} G_l \rho_{j,l}$, for $j = 2, ..., n$.

(1.11) We denote $S(x,r) = \{y \in R^n : d(x,y) < r\}$ and $Q(x,r) = \{y \in R^n : \delta(x,y) < r\}$ and we will call $S(x,r)$ a d-ball and $Q(x,r)$ a δ-ball. Note that there is a constant $A > 1$ such that $|S(x,2r)| \leq A|S(x,r)|$ and $|Q(x,2r)| \leq A|Q(x,r)|$, where $|\ |$ denotes Lebesgue measure. Also, by (1.8), $|Q(x,r)| = \prod_{j=1}^{n} F_j(x^*, r)$. If $Q = Q(x,r)$, we write $r = r(Q)$.

In general we say that a non-negative and locally integrable function $w(x)$ is a doubling weight ($w \in D$) if there exists a constant $A > 1$ such that $w(2Q) \leq Aw(Q)$ for any δ-ball Q, where $2Q = Q(x, 2r)$, if $Q = Q(x,r)$ and $w(Q) = \int_Q w(x)dx$.

(1.12) If $w \in D$ then there exists $\alpha > 0$ such that, $\forall r > 0$, $\forall \theta \in]0,1]$, $\forall x \in R^n$, $w(Q(x,\theta r)) \geq \theta^\alpha w(Q(x,r))$.

Given $1 < p < \infty$, we say $w \in A_p$ if there is a constant $c > 0$ such that for all δ-balls Q in R^n,

(1.13) $$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/p-1}dx\right)^{p-1} \leq c.$$

Note that if we have the A_p condition with respect to δ, we have the same condition holding for the metric d, i.e. (1.13) holds with Q replaced by S (using doubling and the equivalence between d an δ). If v is a weight, $w \in A_p(v)$ means an analogous inequality holds with dx and $|Q|$ replaced by $v(x)dx$ and $v(Q)$, respectively. We use the notation $A_\infty(v) = \bigcup_{p>1} A_p(v)$. The theory of weights in homogeneous spaces was studied by A.P. Calderon in [C] and frequently we refer to this paper.

If x, $y \in R^n$, we shall denote by $H(t,x,y) = (H_1(t,x,y), ..., H_n(t,x,y))$ the solution at time t of the Cauchy problem $\dot{H}_j(.,x,y) = y_j\lambda_j(H(.,x,y))$, $H_j(0,x,y) = x_j$, $j = 1, ..., n$.

Given $\alpha = (\alpha_1, ..., \alpha_n)$, $\epsilon = (\epsilon_1, ..., \epsilon_n)$ with $0 < \epsilon_j < \alpha_j$, $j = 1, ..., n$, we denote $\Delta_\epsilon^\alpha = \{y \in R^n : \epsilon_j \leq y_j \leq \alpha_j, j = 1, ..., n\}$. If $\sigma \in \{-1,1\}^n$, we put $T_\sigma y = (\sigma_1 y_1, ..., \sigma_n y_n)$, $Q^\sigma(x,r) = \{y \in Q(x,r) : \sigma_j(y_j - x_j) \geq 0, j = 1, ..., n\}$ and $\Delta_\epsilon^\alpha(\sigma) = T_\sigma(\Delta_\epsilon^\alpha)$.

Now we can state two results proved in [FS].

Let $\gamma \in]0,1[$ and $\sigma \in \{-1,1\}^n$ be fixed. Then there exists ϵ, $\alpha \in R^n$ as above such that, $\forall r > 0$ and $\forall x \in R^n$,

(1.14) $$|H(r,x,\Delta_\epsilon^\alpha(\sigma)) \cap Q^\sigma(x,r)| \geq (1-\gamma)|Q^\sigma(x,r)|,$$

where $H(r,x,\Delta_\epsilon^\alpha(\sigma)) = \{H(x,r,y) : y \in \Delta_\epsilon^\alpha(\sigma)\}$.

Also, there are positive constants c_1, c_2 depending only on ϵ, α and $\rho_{j,i}$ such that

$$(1.15)\qquad c_1|S(x,r)| \le \prod \int_0^r \lambda_j(H(t,x,y))dt \le c_2|S(x,r)|$$

for each $x \in R^n$, $r > 0$ and $y \in \Delta_\epsilon^\alpha(\sigma)$.

If $q \ge 2$, we say that **Sobolev** inequality holds for w_1, w_2 if for any $u \in \tilde{H}_o(Q)$, Q a δ-ball in R^n,

$$(1.16)\qquad (\frac{1}{w_2(Q)}\int_Q |u|^q w_2 dx)^{1/q} \le cr(Q)(\frac{1}{w_1(Q)}\int_Q |\nabla_\lambda u|^2 w_1 dx)^{1/2}.$$

Given $q \ge 2$, we say the **Poincaré** inequality holds for w_1, w_2 and μ if there are constants $c > 0$ and $a > 0$ (see (1.9)) such that for any δ ball Q and every $u \in \tilde{H}(a^2Q)$ we have

$$(1.17)\qquad (\frac{1}{w_2(Q)}\int_Q |u - av_{\mu,Q}u|^q w_2 dx)^{1/q} \le cr(Q)(\frac{1}{w_1(Q)}\int_{a^2Q} |\nabla_\lambda u|^2 w_1 dx)^{1/2},$$

where $av_{\mu,Q} = \frac{1}{\mu(Q)}\int_Q u d\mu$ and $a^2Q = Q(x, a^2r)$ if $Q = Q(x,r)$.

The reason that we have a^2Q on the right side of (1.17) is that we do not have a Kohn type argument (see also [J]) for the quasi-metric δ. In the d-metric, we can state (1.17) with equal balls on both sides. For the metric δ, however, we have convenient cut-off functions (see[FL1]) that are important in order to get Caccioppoli estimates for solutions of (1.1) (see (C.1), (C.2) and (C.3)). This explains the reason that we work with δ instead of d.

We can now state our main results.

THEOREM A (**Harnack's inequality**) Suppose that:

(i) v, w_1, $w_2 \in A_2$

(ii) the Poincaré inequality holds for w_1, w_2 and w_1, v with $\mu = 1$ and some $q > 2$

(iii) $w_2 v^{-1} \in A_\infty(v)$.

If u is a non-negative solution of (1.1) in the cylinder $R = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta)$, then

$$ess\ sup_{R^-} u \le c_1 exp\{c_2[\alpha^{-2}\beta\Lambda(Q(x_0,\alpha)) + \alpha^2\beta^{-1}(\lambda(Q(x_0,\alpha)))^{-1}]\} ess\ inf_{R^+} u,$$

where $R^- = Q(x_0, \alpha/2) \times (t_0 - 3\beta/4, t_0 - \beta/4)$, $R^+ = Q(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta)$, $\Lambda(Q) = w_2(Q)/v(Q)$, $\lambda(Q) = w_1(Q)/v(Q)$, for a δ-ball Q. Here the constants c_1, c_2 depend only on the constants which arise in (i), (ii), (iii).

We write

$$⨍⨍_R f(x,t)m(x,t)dxdt = \iint_R f(x,t)m(x,t)dxdt / \iint_R m(x,t)dxdt.$$

THEOREM B (**Mean value inequality**) Assume that hypotheses (i),(ii),(iii) of Theorem A hold. Let $0 < p < \infty$, α, $\beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$ and let $Q(x_0, \alpha) = Q$, $Q(x_0, \alpha') = Q'$ and $R = Q \times (t_0 - \beta, t_0 + \beta)$, $R'_+ = Q' \times (t_0 - \beta', t_0 + \beta)$. If u is a solution of (1.1) in R, then u is bounded in R'_+ and

$$ess\ sup_{R'_+} |u|^p$$

$$\leq \ D(\alpha^2\beta^{-1}\lambda(Q)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q) + 1)^{h/(h-1)} \iint_R |u|^p(\alpha^{-2}\beta w_2 + v)\,dx\,dt,$$

where $D \leq C^{\frac{1}{h-1}}$ if $p \geq 2$, and $D \leq c^{\log(\frac{3}{p})}C^c$ if $0 < p < 2$, and $C = c\frac{\alpha^{2+b}\beta}{(\alpha-\alpha')^{2+b}(\beta-\beta')}$. Here $h > 1$, $c > 0$ and $b > 0$ are constants which are independent of u, p, α, α', β, β'.

The organization of the paper is as follows. In section 2 we prove the following **Sobolev interpolation inequality**:

THEOREM D: Let w_1, w_2 be doubling weights, $v \in A_2$ and suppose (1.17) holds with w_1, w_2, $\mu = 1$ and some $q > 2$. If $w_2 v^{-1} \in A_\infty(v)$ then there exists $h > 1$ and constants $c > 0$, $b > 0$ such that for every ϵ satisfying $0 < \epsilon \leq 1$,

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2\,dx$$

$$\leq c\epsilon^{-b}\Big(\frac{1}{v(Q)}\int_{(1+\epsilon)Q} u^2 v\,dx\Big)^{h-1}\Big(\frac{r(Q)^2}{w_1(Q)}\int_{(1+\epsilon)Q} |\nabla_\lambda u|^2 w_1\,dx + \frac{1}{v(Q)}\int_{(1+\epsilon)Q} u^2 v\,dx\Big)$$

for all $u \in \tilde{H}((1+\epsilon)Q)$.

In section 3 we prove Theorem B. First we show, for $p \geq 2$, the following mean value inequality for subsolutions of (1.1):

$$(\star)\ ess\ sup_{R'_+} u_+^p$$

$$\leq \ (p^2C)^{\frac{h}{h-1}}(\alpha^2\beta^{-1}\lambda(Q)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q) + 1)^{h/(h-1)} \iint_R u_+^p(\alpha^{-2}\beta w_2 + v)\,dx\,dt,$$

where C is as in Theorem B and $u_+ = max(u, 0)$. This inequality is less precise than the one we will eventually obtain because of the presence of the factor p^2 on the right. In order to prove the above inequality we apply Theorem D to the function $H_M(u(., \tau))$ where

$$H_M(s) = \begin{cases} s^{p/2} \text{ if } s \in [0, M] \\ M^{p/2} + \frac{p}{2} M^{(p-2)/2}(s - M) \text{ if } s \geq M \\ 0 \text{ if } s < 0, \end{cases}$$

and therefore $H_M(u(.,\tau))$ is an element of $\tilde{H}(Q(x_\circ, \alpha)$ for a.e. $\tau \in (t_\circ - \beta', t_\circ + \beta)$. The first idea would be to apply Theorem D to the function $u_+^{p/2}(.,\tau)$ but at this point we do not know if $u_+^{p/2}(.,\tau)$ belongs to $\tilde{H}(Q(x_\circ, \alpha)$. Hence we have to work with $H_M(u)$, and in order to proceed with the proof of $(\star)$ we show the following Caccioppolli inequality for $H_M(u)$.

(C.1) Let $2 \leq p < \infty$ and u be a subsolution of (1.1) in R. Let $w_2 \in A_2$ and α, α', β, β' satisfy $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$. Then

$$\begin{aligned} &ess\ sup_{\tau \in (t_\circ - \beta', t_\circ + \beta)} \int_Q H_M(u(x,\tau))^2 v(x)dx + \iint_{R'_+} |\nabla_\lambda (H_M(u))|^2 w_1(x)dxdt \\ \leq\ & c \iint_R u^2 H_M'(u)^2 \left(\frac{w_2}{(\alpha - \alpha')^2} + \frac{v}{\beta - \beta'}\right)dxdt, \end{aligned}$$

with c independent of all parameters.

The next step is to apply $(\star)$ for $p = 2$ to deduce that u_+ is locally bounded. This fact allow us to apply Theorem D to the function $u_+^{p/2}(.,\tau)$ for a.e. $\tau \in (t_\circ - \beta', t_\circ + \beta)$. The Caccioppoli inequality we can deduce from (C.1) for the function $u_+^{p/2}$ is not precise enough since it will have a factor p^2 in the right hand side (note that $uH_M'(u) \leq \frac{p}{2} u_+^{p/2}$) and this is the term we want to eliminate from $(\star)$. But with a different test function from the one used in the proof of (C.1), namely, $\phi(x,t) = \eta^2 g(u)\chi(t, \tau_1, \tau_2)$ where

$$g(s) = \begin{cases} s^{p-1} \text{ if } s \in [0, M] \\ M^{p-2}s \text{ if } s \geq M \\ 0 \text{ if } s < 0, \end{cases}$$

and η is a convenient C^∞ function with compact support, we can deduce the following Caccioppolli inequality for subsolutions of (1.1):

(C.2) Let $2 \leq p < \infty$ and u be a subsolution of (1.1) in R. Let $w_2 \in A_2$ and α, α', β, β' satisfy $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$. Then

$$\begin{aligned} &ess\ sup_{\tau \in (t_\circ - \beta', t_\circ + \beta)} \int_Q u_+(x,\tau)^p v(x)dx + \iint_{R'_+} |\nabla_\lambda u_+^{p/2}|^2 w_1(x)dxdt \\ \leq\ & c \iint_R u_+^p \left(\frac{w_2}{(\alpha - \alpha')^2} + \frac{v}{\beta - \beta'}\right)dxdt, \end{aligned}$$

with c independent of all parameters.

Now following the steps of the proof of $(\star)$ using (C.2) instead of (C.1) we can prove that for $p \geq 2$

$$(\star\star)\ ess\ sup_{R'_+} u_+^p$$
$$\leq\ (C)^{\frac{h}{h-1}}(\alpha^2\beta^{-1}\lambda(Q)^{-1}+1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q)+1)^{h/(h-1)}\iint_R u_+^p(\alpha^{-2}\beta w_2+v)dxdt,$$

and Theorem B will follow from $(\star\star)$ and an iteration argument like the one given in lemma (3.4) of [GW2]. Finally we conclude section 3 by making some comments about the proof of mean value inequalities for u^p, when $p<0$, where u is a positive solution of (1.1). These inequalities will be necessary in the proof of Theorem A and in order to show them we need the following generalization of (C.2):

(C.3) Let $-\infty<p<+\infty$, $p\neq 0,1$, u satisfy $0<m<u(x,t)<M<\infty$ in R, $w_2\in A_2$. Then if $p>1$ and u is a subsolution in R, or if $p<0$ and u is a supersolution in R,

$$ess\ sup_{\tau\in(t_o-\beta',t_o+\beta)}\int_{Q'}u(x,\tau)^p v(x)dx+\frac{p-1}{p}\iint_{R'_+}|\nabla_\lambda u^{p/2}|^2 w_1(x)dxdt$$
$$\leq\ c\iint_R u^p\left(\frac{p}{p-1}\frac{w_2(x)}{(\alpha-\alpha')^2}+\frac{v(x)}{\beta-\beta'}\right)dxdt.$$

Moreover, if $0<p<1$ and u is a supersolution in R, then

$$ess\ sup_{\tau\in(t_o-\beta,t_o+\beta')}\int_{Q'}u(x,\tau)^p v(x)dx+\left|\frac{p-1}{p}\right|\iint_{R'_-}|\nabla_\lambda u^{p/2}|^2 w_1dxdt$$
$$\leq\ c\iint_R u^p\left(\left|\frac{p}{p-1}\right|\frac{w_2}{(\alpha-\alpha')^2}+\frac{v}{\beta-\beta'}\right)dxdt.$$

In this paper we do not present the proofs of (C.2) and (C.3) since their proofs are similar to the ones given in section 2 of [GW2].

In section 4, we prove

THEOREM E: Let v and w_1 be weights such that there exists $s>1$ with

$$(1.18)\qquad \left(\frac{r(I)}{r(B)}\right)^2\left(\frac{1}{|I|}\int_I\left(\frac{v}{v(B)}\right)^s dx\right)^{1/s}\left(\frac{1}{|I|}\int_I\left(\frac{w_1}{w_1(B)}\right)^{-s}dx\right)^{1/s}\leq c$$

for all δ-balls I, B with $I\subset 2a^2B$ (a as in (1.9)), where c is a constant independent of the balls. Let $Q=Q(\xi,r)$ and φ be a C^1 function such that $\varphi\equiv 1$ in $Q(\xi,kr)$, $1/2\leq k<1$, $0\leq\varphi\leq 1$, $supp\varphi\subset Q$ and

$$\varphi(x)\varphi(H(t_o,x,y))\leq\varphi(H(t,x,y))$$

for all $x, y, t, t_\circ$ with $0 \leq t \leq t_\circ$. Then, if $u \in Lip(Q)$,

$$\int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx \leq c \frac{v(Q)}{w_1(Q)} r(Q)^2 \int_Q |\nabla_\lambda u(x)|^2 \varphi(x) w_1(x) dx,$$

where $A_Q = \frac{1}{\varphi(Q)} \int_Q u(x)\varphi(x)dx$.

Finally, in section 5, we prove Theorem A. This theorem follows as an application of Bombieri's lemma ([GW2]). In order to verify the hypotheses of Bombieri's lemma we need Theorem B and Theorem F, which we state next. We write $(v \otimes 1)(A) = \int\int_A v(x)dxdt$, where $v = v(x), x \in R^n$, and $A \subset R^{n+1} = \{(x,t) : x \in R^n, t \in R\}$.

THEOREM F: Suppose v is a doubling weight, $w_2 \in A_2$, (1.18) holds and $w_2 v^{-1} \in A_\infty(v)$. Let Q_R be a δ-ball of radius R, $t_\circ \in (a,b)$ and $\tilde{w}_2 = w_2/w_2(Q_R)$ and $\tilde{v} = v/v(Q_R)$. If u is a solution of (1.1) in $Q_{3R/2} \times (a,b)$ which is bounded below by a positive constant, then there are constants c_1, M_2, κ and V such that if for $s > 0$ we define

$$E^+ = \{(x,t) \in Q_R \times (t_\circ, b) : logu < -s - M_2(b - t_\circ) - V\}$$

$$E^- = \{(x,t) \in Q_R \times (a, t_\circ) : logu > s - M_2(a - t_\circ) - V\},$$

then

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E^+) \leq c_1 (\frac{1}{s} \frac{v(Q_R)}{w_1(Q_R)} \frac{R^2}{b - t_\circ})^\kappa (b - t_\circ)$$

and

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E^-) \leq c_1 (\frac{1}{s} \frac{v(Q_R)}{w_1(Q_R)} \frac{R^2}{t_\circ - a})^\kappa (t_\circ - a).$$

Here c_1 and κ depend only on the constants in the conditions on v and w_2, $M_2 \approx \frac{w_2(Q_R)}{R^2 v(Q_R)}$, and V is a constant which depends on u.

In order to prove this theorem, if we follow the steps of Lemma (4.9) of [GW2], we just have to verify that a certain test function (see [FL1]) satisfies the conditions of Theorem E. This will be done in Lemma 5.4.

2 Interpolation Inequality

In this section we prove Theorem D . We start with

Theorem 2.1 *Let w_1, w_2, and μ be doubling weights and suppose (1.17) holds for w_1,*

w_2 with any μ, and for some $q > 2$. If $Q = Q(\xi, r)$ and $w_2 v^{-1} \in A_\infty(v)$ then there exist $h > 1$ and a constant $c > 0$, independent of Q and u, such that

$$\begin{aligned}&\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx \\ &\leq c\Big(\frac{1}{v(Q)}\int_Q u^2 v dx\Big)^{h-1}\Big(\frac{r^2}{w_1(Q)}\int_{Q(\xi,a^2r)} |\nabla_\lambda u|^2 w_1 dx + (av_{\mu,Q}|u|)^2\Big)\end{aligned}$$

for all $u \in \tilde{H}(a^2Q)$ (a as in (1.9)). Also if (1.17) is replaced by (1.16), then

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx \leq c\Big(\frac{1}{v(Q)}\int_Q u^2 v dx\Big)^{h-1}\Big(\frac{r^2}{w_1(Q)}\int_Q |\nabla_\lambda u|^2 w_1 dx\Big)$$

for all $u \in \tilde{H}_o(Q)$.

Proof: The proof follows as in [GW1], theorem 3; the only differences are that we obtain $Q(\xi, a^2 r)$ in the second integral on the right when we apply Poincare's inequality and in the end we use the results of Calderon for weights in homogeneous spaces (see [C]) .

Corollary 2.2 *Let w_1, w_2 be doubling weights and suppose (1.17) holds with w_1, w_2, $\mu = 1$ and some $q > 2$. If $w_2 v^{-1} \in A_\infty(v)$, then there exists $h > 1$ and a constant $c > 0$ such that*

$$\begin{aligned}&\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx \\ &\leq c\Big(\frac{1}{v(Q)}\int_Q u^2 v dx\Big)^{h-1}\Big(\frac{r^2}{w_1(Q)}\int_{a^2Q} |\nabla_\lambda u|^2 w_1 dx + \frac{1}{v(Q)}\int_Q u^2 v dx\Big)\end{aligned}$$

for all $u \in \tilde{H}(a^2Q)$, $Q = Q(\xi, r)$.

Proof: The conclusion of Theorem (2.1) holds for $\mu = 1$. But, by Schwarz's inequality,

$$\begin{aligned}av_Q|u| &= \frac{1}{|Q|}\int_Q |u| dx = \frac{1}{|Q|}\int_Q u v^{1/2} v^{-1/2} dx \\ &\leq \frac{1}{|Q|}\Big(\int_Q u^2 v dx\Big)^{1/2}\Big(\int_Q \frac{1}{v} dx\Big)^{1/2} \leq \Big(\frac{1}{v(Q)}\int_Q u^2 v dx\Big)^{1/2},\end{aligned}$$

where in the last inequality we used the fact that $v \in A_2$.

In the next section we prove mean value inequalities. In order to be able to iterate a certain inequality as was done in [GW2] we need a refinement of the above corollary This refinement is Theorem D and to prove it we need the following lemmas.

Lemma 2.3 *Given $Q = Q(\xi, s)$ and $0 < r < s$, there exists $x_1, ..., x_{m(r,s)}$ in Q, and $k \geq 1$ independent of ξ, r, s, such that*

$$(i)\ Q(x_j, r/k) \cap Q(x_h, r/k) = \emptyset, h \neq j$$

$$(ii)\ Q(\xi, s) \subset \cup_{j=1}^{m(r,s)} Q(x_j, r).$$

Moreover, $m(r,s) \leq c(\frac{s}{r})^\nu$ for some constant ν depending only on the dimension.

Proof: If we apply theorem (1.2), page 69, of [CoW] to the open covering of Q given by $(S(x, \frac{r}{4a}))_{x \in Q}$, there exist $x_1, ..., x_{m(r,s)}$ in Q such that: $S(x_h, \frac{r}{4a}) \cap S(x_j, \frac{r}{4a}) = \emptyset$ if $j \neq h$ and $Q(\xi, s) \subset \cup_{j=1}^{m(r,s)} S(x_j, \frac{r}{a})$. By (1.9), $S(x_j, \frac{r}{4a}) \supset Q(x_j, \frac{r}{4a^2})$ and $S(x_j, \frac{r}{a}) \subset Q(x_j, r)$. Therefore, if we choose $k = 4a^2$, (i) and (ii) follow. It remains to find an upper bound for m(r,s). First, we note that $Q(x_j, \frac{r}{k}) \subset Q(\xi, a^2 \frac{k+1}{k} s)$. But, $\frac{r}{k} = \frac{2a^4(k+1)s}{k} \frac{r}{2a^4(k+1)s}$, and so by (1.10), there exists $\nu > 0$, such that

$$|Q(x_j, \frac{r}{k})| \geq (\frac{r}{2a^4(k+1)s})^\nu |Q(x_j, \frac{2a^4(k+1)s}{k})|,$$

and since the $Q(x_j, \frac{r}{k})$ are disjoint,

$$|Q(\xi, \frac{a^2(k+1)s}{k})| \geq \sum_j |Q(x_j, \frac{r}{k})| \geq c(\frac{r}{s})^\nu \sum_j |Q(x_j, \frac{2a^4(k+1)s}{k})|.$$

But, $Q(x_j, \frac{2a^4(k+1)s}{k}) \supset Q(\xi, \frac{a^2(k+1)s}{k})$ and so $|Q(\xi, \frac{a^2(k+1)s}{k})| \geq c(\frac{r}{s})^\nu m(r,s) |Q(\xi, \frac{a^2(k+1)s}{k})|$. Therefore, $m(r,s) \leq c(\frac{s}{r})^\nu$.

Lemma 2.4 *If $\delta(y,z) < s$ then $F_j(z^*, s) \leq (2a^2)^{G_j} F_j(y^*, s)$. G_j as in (1.10).*

Proof: Since $Q(z,s) \subset Q(y, 2a^2 s)$, $F_j(z^*, s) \leq F_j(y^*, 2a^2 s)$. By (1.10), it follows that

$$F_j(z^*, s) \leq F_j(y^*, 2a^2 s) \leq (2a^2)^{G_j} F_j(y^*, s).$$

Lemma 2.5 *If $0 < \epsilon < 1$ and $\eta \in Q = Q(\xi, s)$, then $Q(\eta, \epsilon s/(2a^2)^\zeta) \subset Q(\xi, (1+\epsilon)s)$, where $\zeta = \max_{j=1,...,n} G_j$.*

Proof: If $y \in Q(\eta, \epsilon s/(2a^2)^\zeta)$ then by (1.8), $|y_j - \eta_j| \leq F_j(\eta^*, \epsilon s/(2a^2)^\zeta)$ and by (1.10) and Lemma (2.4)

$$F_j(\eta^*, \frac{\epsilon s}{(2a^2)^\zeta}) \leq \frac{\epsilon}{(2a^2)^\zeta} F_j(\eta^*, s) \leq \epsilon F_j(\xi^*, s).$$

Therefore,

$$|y_j - \xi_j| \leq |y_j - \eta_j| + |\eta_j - \xi_j| \leq \epsilon F_j(\xi^*, s) + F_j(\xi^*, s) = (1+\epsilon) F_j(\xi^*, s) \leq F_j(\xi^*, (1+\epsilon)s),$$

where in the last inequality we used (1.10).

Proof of Theorem D.

Let $Q = Q(\xi, s)$. By Lemma (2.5), $\delta(Q, \partial(1+\epsilon)Q) \geq \frac{\epsilon s}{(2a^2)^\zeta}$. Apply Lemma (2.3) to $r = \frac{\epsilon s}{(2a^2)^\zeta a^2}$ to find $x_1, ..., x_{m(r,s)} \in Q$ such that: $Q(x_j, r/k) \cap Q(x_h, r/k) = \emptyset$ if $j \neq h$, $Q(\xi, s) \subset \cup_{j=1}^{m(r,s)} Q(x_j, r)$ and $m(r,s) \leq c(s/r)^\nu$.

Note that, by (2.5), $Q(x_j, a^2 r) = Q(x_j, \frac{\epsilon s}{(2a^2)^\zeta}) \subset Q(\xi, (1+\epsilon)s) = (1+\epsilon)Q$. Then using Corollary (2.2), doubling for w_2, doubling for v and w_1 and the fact that $Q(x_j, 2a^2 s) \supset Q(\xi, s)$ and $Q(\xi, 2a^2 s) \supset Q(x_j, s)$,

$$\int_Q |u|^{2h} w_2 dx \leq \sum_{j=1}^{m(r,s)} \int_{Q(x_j,r)} |u|^{2h} w_2 dx$$

$$\leq c \sum_{j=1}^{m(r,s)} w_2(Q(x_j, r)) (\frac{1}{v(Q(x_j,r)} \int_{Q(x_j,r)} u^2 v dx)^{h-1}$$

$$\cdot \{\frac{r^2}{w_1(Q(x_j,r))} \int_{Q(x_j,a^2 r)} |\nabla_\lambda u|^2 w_1 dx + \frac{1}{v(Q(x_j,r))} \int_{Q(x_j,r)} u^2 v dx\}$$

$$\leq c(\frac{s}{r})^\nu w_2(Q(\xi,s))[(\frac{r}{2a^2 s})^{-\alpha} \frac{1}{v(Q(\xi,s))} \int_{(1+\epsilon)Q} u^2 v dx]^{h-1}$$

$$\cdot \{\frac{s^2}{w_1(Q(\xi,s))} (\frac{r}{2a^2 s})^{-\alpha} \int_{(1+\epsilon)Q} |\nabla_\lambda u|^2 w_1 dx + (\frac{r}{2a^2 s})^{-\alpha} \frac{1}{v(Q(\xi,s))} \int_{(1+\epsilon)Q} u^2 v dx\}.$$

The theorem follows if we choose $b = \nu + 2\alpha$, since $s/r = c\epsilon^{-1}$.

3 Mean value inequalities.

In this section we prove Theorem B and some other mean value inequalities. Since the proofs are similar to the ones given by [GW2], we just point out the differences. Basically, we have to be a little more careful in the iteration argument since there is a factor ϵ in Theorem D.

We asume throughout this section that:

(3.1)
- (a) w_1, w_2, $v \in A_2$
- (b) Poincaré's inequality, (1.17), holds for both of the pairs w_1, w_2 and w_1, v with some $q > 2$ and $\mu = 1$
- (c) $w_2 v^{-1} \in A_\infty(v)$.

Denote $R_{r,s} = Q(x_o, r) \times (t_o - s, t_o + s)$ and let $R = R_{r,s}$, $R' = R_{\rho,\sigma}$ with $r/2 < \rho < r$ and $s/2 < \sigma < s$ and define

(3.2) $$C = c\frac{r^{2+b} s}{(r-\rho)^{2+b}(s-\sigma)}$$

where b is given by Theorem D and c is a constant that may vary, but which only depends on the weights and on h, where $h > 1$ is the index for which Theorem D holds for both w_2 and v on the left hand side.

We also write $\lambda(Q) = w_1(Q)/v(Q)$ and $\Lambda(Q) = w_2(Q)/v(Q)$. We start this section with the proof of (C.1). This estimate will be important in deducing a mean value inequality for subsolutions of (1.1).

Proof of (C.1): If $u \in H$ define

$$\varphi(x,t) = \eta^2(x,t)[\int_0^{u(x,t)} H_M'(s)^2 ds + u(x,t)H_M'(u(x,t))^2]\chi(t,\tau_1,\tau_2),$$

where $\eta \in C_\circ^\infty(R)$ will be specified later, $t_\circ - s < \tau_1 < \tau_2 < t_\circ + s$ and $\chi(t,\tau_1,\tau_2)$ denotes the characteristic function of (τ_1,τ_2). The fact that the function φ is in $H_\circ$ follows as a consequence of the following result: if f is a piecewise smooth function on the real line with $f' \in L^\infty(-\infty,\infty)$ and if $u \in H$, then $f \circ u \in H$. Here we use the convention that $f'(u) = 0$ if $u \in L$ where L denotes the set of corner points of f (the proof follows the steps of theorem 7.8 of [GT] and it also shows that $\nabla_\lambda(f \circ u) = f'(u)\nabla_\lambda u$ and $(f(u))_t = f'(u)u_t$). The proof of the above fact also verifies that in our case $\varphi \geq 0$ in the $H_\circ$-sense since $H_M(s) = 0$ for $s < 0$.

Since u is a subsolution, we have

$$\iint_R (\langle A\nabla u, \nabla\varphi\rangle + u_t\varphi v)dxdt \leq 0. \tag{3.3}$$

Note that by another limiting argument

$$u_t[\eta^2\int_0^u H_M'(s)^2ds] = [u\eta^2\int_0^u H_M'(s)^2ds]_t - u(\eta^2)_t\int_0^u H_M'(s)^2ds - \eta^2 H_M'(u)^2u_tu,$$

and then by definition of φ, for $\tau_1 < t < \tau_2$,

$$u_t\varphi = [u\eta^2\int_0^u H_M'(s)^2ds]_t - (\eta^2)_t u\int_0^u H_M'(s)^2ds$$

and

$$\nabla\varphi = 2\eta\nabla\eta[\int_0^u H_M'(s)^2ds + uH_M'(u)^2] + \eta^2[H_M'(u)^2\nabla u + f_M'(u)\nabla u],$$

where $f_M(s) = sH_M'(s)^2$ (note that $\nabla(f_M(u)) = f_M'(u)\nabla u$, since f_M is piecewise smooth with $f_M' \in L^\infty$). If we substitute the two last equations in (3.3) we get, with $Q = Q(x_\circ, r)$,

$$\int_Q\int_{\tau_1}^{\tau_2}[u\eta^2\int_0^u H_M'(s)^2ds]_t v dxdt + \int_Q\int_{\tau_1}^{\tau_2}\eta^2 H_M'(u)^2\langle A\nabla u,\nabla u\rangle dxdt$$
$$\leq \int_Q\int_{\tau_1}^{\tau_2}[(\eta^2)_t u\int_0^u H_M'(s)^2ds]v dxdt - 2\int_Q\int_{\tau_1}^{\tau_2}\eta\langle A\nabla u,\nabla\eta\rangle[\int_0^u H_M'(s)^2ds + uH_M'(u)^2]dxdt$$
$$- \int_Q\int_{\tau_1}^{\tau_2}\eta^2\langle A\nabla u,\nabla u\rangle f_M'(u)dxdt.$$

We can drop the last term on the right since the integrand is non-negative. The second term on the right is majorized in absolute value by

$$\int_Q \int_{\tau_1}^{\tau_2} |\langle A\nabla u, \nabla\eta\rangle| 4\eta H_M'(u)^2 u\,dxdt = 4\int_Q \int_{\tau_1}^{\tau_2} |\langle AH_M'(u)\eta\nabla u, uH_M'(u)\nabla\eta\rangle|\,dxdt$$
$$\leq 4\frac{\epsilon}{2}\int_Q \int_{\tau_1}^{\tau_2} \langle A\nabla(H_M(u)), \nabla(H_M(u))\rangle\eta^2 dxdt + \frac{4}{2\epsilon}\int_Q \int_{\tau_1}^{\tau 2} \langle A\nabla\eta, \nabla\eta\rangle u^2 H_M'(u)^2 dxdt$$

where we used the fact that $|\langle Ax, y\rangle| \leq \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} \leq \frac{\epsilon}{2}\langle Ax, x\rangle + \frac{1}{2\epsilon}\langle Ay, y\rangle$. If we pick $\epsilon = \frac{1}{4}$ we get

$$(3.4)\qquad \int_Q \int_{\tau_1}^{\tau_2} [u\eta^2 \int_0^u H_M'(s)^2 ds]_t v\,dxdt + \frac{1}{2}\int_Q \int_{\tau_1}^{\tau_2} \eta^2\langle A\nabla(H_M(u)), \nabla(H_M(u)\rangle dxdt$$
$$\leq 8\int_Q \int_{\tau_1}^{\tau_2} \langle A\nabla\eta, \nabla\eta\rangle u^2 H_M'(u)^2 dxdt + \int_Q \int_{\tau_1}^{\tau_2} [(\eta^2)_t u \int_0^u H_M'(s)^2 ds] v\,dxdt.$$

Choose η to be zero in a neighborhood of $\{\partial Q \times (t_\circ - s, t_\circ + s)\} \cup \{Q \times (t = t_\circ - s)\}$, $\eta \equiv 1$ in R_+', $0 \leq \eta \leq 1$, $|\nabla_\lambda \eta| \leq c/(r-\rho)$, $|\eta_t| \leq c/(s-\sigma)$ (see page 537 of [FL1]). If we pick τ_1 so close to $t_\circ - s$ that $\eta(x, \tau_1) = 0$ for all $x \in Q$, drop the second term on the left of (3.4)(which is non-negative) and use lemma 5 of [AS] it follows that

$$(3.5)\qquad ess\ sup_{\tau_2 \in (t_\circ - \sigma, t_\circ + s)} \int_{Q'} u(x, \tau_2) \int_0^{u(x,\tau_2)} H_M'(s)^2 ds\ v dx$$
$$\leq c\iint_R u^2 H_M'(u)^2 [\frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}] dxdt.$$

If we fix $\tau_2 \in (t_\circ - \sigma, t_\circ + s)$ and τ_1 as before and if we drop the first term on the left of (3.4) (which we can see is non-negative after performing the integration) we obtain

$$(3.6)\int_Q \int_{\tau_1}^{\tau_2} \eta^2\langle A\nabla(H_M(u), \nabla(H_M(u))\rangle dxdt \leq c\iint_R u^2 H_M'(u)^2 [\frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}] dxdt.$$

Letting $\tau_2 \longrightarrow t_\circ + s$ and using (1.2) we get

$$(3.7)\qquad \iint_{R_+'} |\nabla_\lambda (H_M(u))|^2 w_1 dxdt \leq c\iint_R u^2 H_M'(u)^2 [\frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}] dxdt.$$

Finally note that

$$H_M(u)^2 = \int_0^u (H_M(s)^2)' ds = \int_0^u 2H_M(s)H_M'(s) ds \leq 2\int_0^u sH_M'(s)^2 ds \leq 2u\int_0^u H_M'(s)^2 ds,$$

since $H_M(s) \leq sH_M'(s)$. Combining this with (3.5) and (3.7), (C.1) follows with α, β, α', β' taken there to be r, s, ρ, σ.

Lemma 3.8 *Let $p \geq 2$, R, R' be as defined above and assume (3.1) holds. If u is a subsolution of (1.1) in R, then u_+ is bounded in $R_+' = Q(x_\circ, \rho) \times (t_\circ - \sigma, t_\circ + s)$ and*

$$ess\ sup_{R'_+} u_+^p$$

$$\leq (p^2C)^{\frac{h}{h-1}}(\frac{r^2}{s}\frac{1}{\lambda(Q)}+1)^{\frac{1}{h-1}}(\frac{s}{r^2}\Lambda(Q)+1)^{\frac{h}{h-1}}\iint_R u_+^p(\frac{s}{r^2}w_2+v)dxdt,$$

with C as in (3.2).

Proof: $H_M(u)$ is a function in H since $u \in H$ and H_M is a C^1 function with bounded derivative. Then by Fubini's theorem we have that $H_M(u(.,\tau)) \in \tilde{H}$ for a.e $\tau \in (t_\circ - \sigma, t_\circ + s)$. If we apply Theorem D to the function $F(x) = H_M(u(x,\tau))$, $Q = Q_\rho$ and $\epsilon > 0$ such that $(1+\epsilon)\rho < r$ and combine this with (C.1) we obtain

$$\begin{aligned}
&\frac{1}{w_2(Q_\rho)}\int_{Q_\rho} H_M(u(x,\tau)^{2h}w_2(x)dx\\
\leq\ & c\epsilon^{-b}\{\frac{1}{v(Q_\rho)}\iint_R u^2H_M'(u)^2(\frac{w_2}{(r-(1+\epsilon)\rho)^2}+\frac{v}{s-\sigma})dxdt\}^{h-1}\\
&\{\frac{\rho^2}{w_1(Q_\rho)}\int_{Q_{(1+\epsilon)\rho}}|\nabla_\lambda(H_M(u(x,\tau))|^2w_1(x)dx\\
+\ &\frac{1}{v(Q_\rho)}\iint_R u^2H_M'(u)^2(\frac{w_2}{(r-(1+\epsilon)\rho)^2}+\frac{v}{s-\sigma})dxdt\}
\end{aligned}$$

for a.e. $\tau \in (t_\circ - \sigma, t_\circ + s)$.

Integrate with respect to τ over $(t_\circ - \sigma, t_\circ + s)$ and apply (C.1) to get

$$\begin{aligned}
&\frac{1}{w_2(Q_\rho)}\iint_{R'_+} H_M(u(x,t))^{2h}w_2(x)dxdt\\
\leq\ & c\frac{\epsilon^{-b}}{v(Q_\rho)^{h-1}}(\frac{\rho^2}{w_1(Q_\rho)}+\frac{s+\sigma}{v(Q_\rho)})(\iint_R u^2H_M'(u)^2\frac{w_2}{(r-(1+\epsilon)\rho)^2}+\frac{v}{s-\sigma})dxdt)^h.
\end{aligned}$$

Since $r/2 < \rho < r$ and $s/2 < \sigma < s$, by the doubling property of the weights and the definitions of λ and Λ, it follows that

$$\begin{aligned}
&\frac{1}{w_2(Q_r)}\iint_{R'_+} H_M(u(x,t))^{2h}w_2(x)dxdt\\
\leq\ & c\frac{\epsilon^{-b}}{v(Q_r)^h}(\frac{r^2}{\lambda(Q_r)}+s)(\iint_R u^2H_M'(u)^2(\frac{w_2}{(r-(1+\epsilon)\rho)^2}+\frac{v}{s-\sigma})dxdt)^h
\end{aligned}$$

A similar inequality holds with w_2 replaced by v on the left, and if we add the two inequalities, we obtain

(3.9)
$$\begin{aligned}
&\iint_{R'_+} H_M(u)^{2h}(\frac{w_2}{w_2(Q_r)}+\frac{v}{v(Q_r)})dxdt\\
\leq\ & c\frac{\epsilon^{-b}}{v(Q_r)^h}(\frac{r^2}{\lambda(Q_r)}+s)(\iint_R u^2H_M'(u)^2(\frac{w_2}{(r-(1+\epsilon)\rho)^2}+\frac{v}{s-\sigma})dxdt)^h
\end{aligned}$$

for any ϵ such that $(1+\epsilon)\rho < r$.

Now, note that

$$\frac{w_2}{(r-(1+\epsilon)\rho)^2}+\frac{v}{s-\sigma}\le\frac{r^2}{(r-(1+\epsilon)\rho)^2(s-\sigma)}\{\frac{s}{r^2}w_2+v\},$$

$$\iint_{R'_+}\{\frac{w_2}{w_2(Q_r)}+\frac{v}{v(Q_r)}\}dxdt\approx s,$$

$$\iint_R\{\frac{s}{r^2}w_2+v\}dxdt\approx s\{\frac{s}{r^2}w_2(Q_r)+v(Q_r)\}\approx sv(Q_r)\{\frac{s}{r^2}\Lambda(Q_r)+1\},$$

$$\frac{sr^{-2}w_2(x)+v(x)}{sr^{-2}w_2(Q_r)+v(Q_r)}\le\frac{w_2(x)}{w_2(Q_r)}+\frac{v(x)}{v(Q_r)}.$$

Thus, by raising both sides of (3.9) to the power $1/h$, normalizing and using the fact that $\epsilon^{-b/h}\le\epsilon^{-b}$, we obtain

$$(3.10)\qquad (\iint_{R'_+}H_M(u)^{2h}(\frac{s}{r^2}w_2+v)dxdt)^{1/h}$$
$$\le\ c\epsilon^{-b}\frac{r^2s}{(r-(1+\epsilon)\rho)^2(s-\sigma)}(\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}$$
$$\iint_R u^2H'_M(u)^2(\frac{s}{r^2}w_2+v)dxdt$$

for any ϵ such that $(1+\epsilon)\rho<r$. Since $u_+^{p/2}\chi_{\{0<u<M\}}\le H_M(u)$ and $uH'_M(u)\le\frac{p}{2}u_+^{p/2}$, if we let $M\longrightarrow\infty$ it follows by Fatou's lemma that

$$(3.11)\qquad (\iint_{R'_+}u_+^{ph}(\frac{s}{r^2}w_2+v)dxdt)^{1/h}$$
$$\le\ cp^2\epsilon^{-b}\frac{r^2s}{(r-(1+\epsilon)\rho)^2(s-\sigma)}(\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}$$
$$\iint_R u_+^p(\frac{s}{r^2}w_2+v)dxdt.$$

Now, we have to iterate (3.11). Fix r, s, ρ, σ with $r/2<\rho<r$ and $s/2<\sigma<s$. For $k=1,2,\ldots$ define sequences $\{s_k\}_{k\in N}$ and $\{r_k\}_{k\in N}$ and $\{\epsilon_k\}_{k\in N}$ by $s_1=s$, $s_k-s_{k+1}=\frac{s-\sigma}{2^k}$ for $k\ge1$, $r_1=r$, $r_k-r_{k+1}=\frac{r-\rho}{2^k}$ for $k\ge1$, and $\epsilon_k=\frac{r-\rho}{2^kr_k}=\frac{r_k-r_{k+1}}{r_k}$ for $k\ge1$. Also, define $R_k=Q_k\times(t_\circ-s_k,t_\circ+s)$ for $k\ge1$, where $Q_k=Q(x,r_k)$. Note that $R_1=R$ and $\bigcap_{k=1}^{\infty}R_k=R'_+$. Since $\frac{1}{2}sr^{-2}\le s_kr_k^{-2}\le4sr^{-2}$, if we apply (3.11) with p replaced by ph^{k-1}, $p\ge2$, and $r=r_k$, $\rho=r_{k+1}$ and $\epsilon=\epsilon_{k+1}$ (note that $(1+\epsilon_{k+1})r_{k+1}<r_k$), we obtain

$$(\iint_{R_{k+1}}u_+^{ph^k}(\frac{s}{r^2}w_2+v)dxdt)^{\frac{1}{h^k}}$$

$$\leq \{c(ph^{k-1})^2\epsilon_{k+1}^{-b}\frac{r_k^2 s_k}{(r_k-(1+\epsilon_{k+1})r_{k+1})^2(s_k-s_{k+1})}(\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}\}^{\frac{1}{h^{k-1}}}$$

$$\{\iint_{R_k} u_+^{ph^{k-1}}(\frac{s}{r^2}w_2+v)dxdt\}^{\frac{1}{h^{k-1}}}.$$

But note that

$$\epsilon_{k+1}^{-b}\frac{r_k^2 s_k}{[r_k-(1+\epsilon_{k+1})r_{k+1}]^2(s_k-s_{k+1})}$$
$$= 2^{(k+1)b}\frac{r_{k+1}^b}{(r-\rho)^b}\frac{r_k^2 s_k}{(\frac{r-\rho}{2^k}-\frac{r-\rho}{2^{k+1}})^2(\frac{s-\sigma}{2^k})} \leq c2^{(3+b)k}\frac{r^{2+b}s}{(r-\rho)^{2+b}(s-\sigma)} \leq C2^{(3+b)k},$$

where C is given by (3.2). Thus,

$$(3.12)\qquad (\iint_{R_{k+1}} u_+^{ph^k}(\frac{s}{r^2}w_2+v)dxdt)^{\frac{1}{h^k}}$$
$$\leq \{C(ph^{k-1})^2 2^{(3+b)k}(\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}\}^{\frac{1}{h^{k-1}}}$$
$$\{\iint_{R_k} u_+^{ph^{k-1}}(\frac{s}{r^2}w_2+v)dxdt\}^{\frac{1}{h^{k-1}}}.$$

If we iterate (3.12), we obtain

$$ess\ sup_{R'_+}\ u_+^p$$
$$\leq \prod_{k=1}^{\infty}\{C(ph^{k-1})^2 2^{(3+b)k}(\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}\}^{\frac{1}{h^{k-1}}}\iint_R u_+^p(\frac{s}{r^2}w_2+v)dxdt.$$

Since $\sum_{k=1}^{\infty}\frac{1}{h^{k-1}}=\frac{h}{h-1}$ and $\sum_{k=1}^{\infty}\frac{k}{h^{k-1}}=(\frac{h}{h-1})^2$, it follows that

$$ess\ sup_{R'_+}\ u_+^p \leq (p^2C)^{\frac{h}{h-1}}(\frac{s}{r^2}\Lambda(Q_r)+1)^{\frac{h}{h-1}}(\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{\frac{1}{h-1}}\iint_R u_+^p(\frac{s}{r^2}w_2+v)dxdt,$$

and this proves the lemma. Note that if we apply the above result for $p=2$, it follows that u_+ is bounded on R'_+.

Proof of Theorem B: By Lemma (3.8) we know that u_+ is bounded in $Q_{(1+\epsilon)\rho}\times(t_\circ-\sigma, t_\circ+s)$ for all ϵ such that $(1+\epsilon)\rho<r$. If we define $F(x)=u_+^{p/2}(x,\tau)$ then $F\in\tilde{H}(Q_{(1+\epsilon)\rho})$ for a.e. $\tau\in(t_\circ-\sigma,t_\circ+s)$ and if we follow the proof of lemma (3.8) using (C.2) instead of (C.1), we get (see the comments in the introduction)

$$ess\ sup_{R'_+} u_+^p \leq C^{\frac{h}{h-1}}(\frac{r^2}{s}\frac{1}{\lambda(Q)}+1)^{\frac{1}{h-1}}(\frac{s}{r^2}\Lambda(Q)+1)^{\frac{h}{h-1}}\iint_R u_+^p(\frac{s}{r^2}w_2+v)dxdt$$

for $p \geq 2$. For $0 < p < 2$, define I_p and I_∞ as in lemma (3.4) of [GW2]. The only difference in our case is that

$$I_\infty(\alpha',\beta')^2 \leq c\{\frac{1}{(\alpha-\alpha')^{2+b}(\beta-\beta')}\}^{\frac{h}{h-1}} I_2(\alpha,\beta)^2$$

if $\frac{1}{2} < \alpha' < \alpha < 1$ and $\frac{1}{2} < \beta' < \beta < 1$. Thus, arguing as in lemma (3.4) of [GW2] we prove that if u is a solution of (1.1) and $p > 0$ then

$$(3.13)\quad ess\ sup_{R'_+} u_+^p \leq D(\frac{r^2}{s}\frac{1}{\lambda(Q)}+1)^{\frac{1}{h-1}}(\frac{s}{r^2}\Lambda(Q)+1)^{\frac{h}{h-1}}\iint_R u_+^p(\frac{s}{r^2}w_2+v)dxdt,$$

where D is as in Theorem B.

If we apply (3.13) to both u and $-u$, we obtain Theorem B of the introduction, with α, β, α', β' taken there to be r, s, ρ, σ.

In order to prove Harnack's inequality we need a mean value inequality for u^p when $-\infty < p < \infty$ and u is a non-negative solution.

We begin by noting that if we use (C.3) instead of (C.1) we can prove the following analogue of (3.11):

Lemma 3.14 *Suppose (3.1) holds, $0 < m < u(x,t) \leq M < \infty$ in $R = R_{r,s}$, $r/2 < \rho < r$, $s/2 < \sigma < s$ and $\epsilon > 0$, $(1+\epsilon)\rho < r$. Then, if $p > 1$ and u is a subsolution in R, or if $p < 0$ and u is a supersolution in R,*

$$\begin{aligned}&(\iint_{R'_+} u^{ph}(\frac{w_2}{w_2(Q_r)}+\frac{v}{v(Q_r)})dxdt)^{1/h}\\ \leq\ & c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\rho)^2(s-\sigma)}(\frac{p}{p-1}\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{p}{p-1}\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}\\ &\iint_R u^p(\frac{p}{p-1}\frac{s}{r^2}w_2+v)dxdt.\end{aligned}$$

Moreover, if $0 < p < 1$ and u is a supersolution in R, then

$$\begin{aligned}&(\iint_{R'_-} u^{ph}(\frac{w_2}{w_2(Q_r)}+\frac{v}{v(Q_r)})dxdt)^{1/h}\\ \leq\ & c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\rho)^2(s-\sigma)}(\frac{p}{|p-1|}\frac{s}{r^2}\Lambda(Q_r)+1)(\frac{p}{|p-1|}\frac{r^2}{s}\frac{1}{\lambda(Q_r)}+1)^{1/h}\\ &\iint_R u^p(\frac{p}{|p-1|}\frac{s}{r^2}w_2+v)dxdt.\end{aligned}$$

Both inequalities are still true if we replace the integral averages on the right by the larger integral average

$$\iint_R u^p(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)})dxdt.$$

Theorem 3.15 *Assume (3.1) holds, $r, s > 0$, $r/2 < \rho < r$, $s/2 < \sigma < s$. If u is a non negative solution of (1.1) in R, then for $p > 0$*

$$ess\ sup_{R'} u^p$$

$$\leq C^c(p\frac{s}{r^2}\Lambda(Q_r) + 1)^{\frac{h}{h-1}}(p\frac{r^2}{s}\frac{1}{\lambda(Q_r)} + 1)^{\frac{1}{h-1}}\iint_R u_+^p(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)})dxdt,$$

and for $p < 0$

$$ess\ sup_{R'_+} u^p$$

$$\leq C^{\frac{h}{h-1}}(|p|\frac{s}{r^2}\Lambda(Q_r) + 1)^{\frac{h}{h-1}}(|p|\frac{r^2}{s}\frac{1}{\lambda(Q_r)} + 1)^{\frac{1}{h-1}}\iint_R u^p(\frac{w_2}{w_2(Q_R)} + \frac{v}{v(Q_R)})dxdt,$$

where C is given by (3.2).

Proof: In Lemma (3.17) of [GW2] we replace (3.20) by the result given here in Lemma (3.14) and then argue as in Lemma (3.17) of [GW2].

4 Proof of Theorem E

We start with the following lemma.

Lemma 4.1 *Suppose $Q = Q(\xi, r)$ and φ is a C^1 function such that $\varphi \equiv 1$ in $kQ = Q(\xi, kr)$, $0 < k < 1$, $0 \leq \varphi \leq 1$, $supp\varphi \subset Q$ and*

$$(4.2) \qquad \varphi(x)\varphi(H(t_\circ, x, y)) \leq \varphi(H(t, x, y))$$

for all x, y, t, $t_\circ$ with $0 \leq t \leq t_\circ$. If u is a Lipschitz function, $E = \{x \in Q(\xi, kr) : u(x) = 0\}$ and $|E| \geq \beta|Q|$ for some $0 < \beta < 1$, then if $x \in Q$,

$$(4.3) \qquad |u(x)|\sqrt{\varphi(x)} \leq c\int_Q |\nabla_\lambda u(z)|\sqrt{\varphi(z)}\frac{\delta(x, z)}{|Q(x, \delta(x, z))|}dz,$$

where c is independent of Q, u, x.

Proof: (The general outline of this proof follows the steps of the proof of lemma 4.3 in [FS].) If $x \in Q = Q(\xi, r)$ then $Q(\xi, r) \subset Q(x, 2a^2 r)$ and $Q(x, r) \subset Q(\xi, 2a^2 r)$. Therefore, by doubling, $|Q(x, r)| \simeq |Q|$. Now, we note that there exists $\sigma \in \{-1, 1\}^n$ such that $|E \cap Q^\sigma(x, 2a^2 r)| \geq c\beta |Q^\sigma(x, 2a^2 r)|$. In fact, $E = \bigcup_\sigma (Q^\sigma(x, 2a^2 r) \cap E)$ and so there exists σ such that

$$|Q^\sigma(x, 2a^2 r) \cap E| \geq \beta 2^{-n} |Q| \geq c\beta |Q^\sigma(x, 2a^2 r)|. \tag{4.4}$$

We also claim that there exists $\alpha, \epsilon \in R^n$, independent of x and r, $0 < \epsilon_j < \alpha_j$, $j = 1, ..., n$, such that

$$|E \cap Q^\sigma(x, 2a^2 r) \cap H(2a^2 r, x, \Delta^\alpha_\epsilon(\sigma))| \geq \frac{c\beta}{2} |Q^\sigma(x, 2a^2 r)|. \tag{4.5}$$

To prove this fact, apply (1.14) to $\gamma = \frac{c\beta}{2}$ and find $\alpha, \epsilon \in R^n$, $0 < \epsilon_j < \alpha_j$, $j = 1, ..., n$, such that

$$|H(2a^2 r, x, \Delta^\alpha_\epsilon(\sigma)) \cap Q^\sigma(x, 2a^2 r)| \geq (1 - \frac{c\beta}{2}) |Q^\sigma(x, 2a^2 r)|.$$

Then,

$$\begin{aligned} &|Q^\sigma(x, 2a^2 r)| \geq |(Q^\sigma(x, 2a^2 r) \cap E) \bigcup (Q^\sigma(x, 2a^2 r) \cap H(...))| \\ = \; & |Q^\sigma(x, 2a^2 r) \cap E| + |Q^\sigma(x, 2a^2 r) \cap H(...)| - |E \cap Q^\sigma(x, 2a^2 r) \cap H(...)| \\ \geq \; & |Q^\sigma(x, 2a^2 r)| (c\beta + 1 - \frac{c\beta}{2}) - |E \cap Q^\sigma(x, 2a^2 r) \cap H(...)| \end{aligned}$$

and therefore the claim follows.

We can assume $x \notin E$ and define $\sum = \{y \in \Delta^\alpha_\epsilon(\sigma) : H(2a^2 r, x, y) \in E\}$. Let K be a smooth function supported in $\Delta^{2\alpha}_{\epsilon/2}(\sigma)$, $0 \leq K \leq 1$, $K \equiv 1$ on $\Delta^\alpha_\epsilon(\sigma)$. Suppose $u \in Lip(Q)$. If $y \in \sum$ then

$$|u(x)| \sqrt{\varphi(x)} = |u(x) - u(H(2a^2 r, x, y))| K(y) \sqrt{\varphi(x)},$$

and if we integrate on $\sum$, we obtain

$$|u(x)| \sqrt{\varphi(x)} |\textstyle\sum| = \int_{\sum} |u(x) - u(H(2a^2 r, x, y))| K(y) \sqrt{\varphi(x)} dy.$$

Now we note that $\varphi(H(2a^2 r, x, y)) = 1$ if $y \in \sum$ and using (4.2) we get $\varphi(x) \leq \varphi(H(t, x, y))$ for any $0 \leq t \leq 2a^2 r$. Therefore,

$$\begin{aligned} |u(x)| \sqrt{\varphi(x)} |\textstyle\sum| &\leq \int_{suppK} | \int_0^{2a^2 r} \frac{d}{dt} (u(H(t, x, y))) dt | \sqrt{\varphi(H(t, x, y))} dy \\ \leq \; & \int_{suppK} | \int_0^{2a^2 r} \langle \nabla u(H(t, x, y)), \dot{H}(t, x, y) \rangle dt | \sqrt{\varphi(H(t, x, y))} dy \end{aligned}$$

$$\leq \int_0^{2a^2r} \int_{suppK} |\nabla_\lambda u(H(t,x,y))||y|\sqrt{\varphi(H(t,x,y))}dydt.$$

If we make change of variables $z = H(t,x,y)$ in $\Delta^{2\alpha}_{\epsilon/2}(\sigma)$, then $|\det \frac{\partial z}{\partial y}(t,x,y)| = \prod_{j=1}^n \int_0^t \lambda_j(H(s,x,y))ds$. For $y \in \Delta^{2\alpha}_{\epsilon/2}(\sigma)$, the last product is equivalent to $|Q(x,t)|$ by (1.15). Hence,

$$(4.6) \qquad |u(x)|\sqrt{\varphi(x)} \leq \frac{c}{|\sum|}\int_0^{2a^2r} \frac{1}{|Q(x,t)|}\int_{H(t,x,\Delta^{2\alpha}_{\epsilon/2}(\sigma))} |\nabla_\lambda u(z)|\sqrt{\varphi(z)}dzdt.$$

Note that there exists $c > 0$ such that $H(t,x,\Delta^{2\alpha}_{\epsilon/2}(\sigma)) \subset Q(x,ct)$. In fact, if we define $\gamma(s) = H(s/|y|,x,y)$ then

$$\langle\dot\gamma(s),\xi\rangle^2 = \{\sum_{j=1}^n \lambda_j(H(\frac{s}{|y|},x,y))y_j\xi_j\}^2\frac{1}{|y|^2} \leq \sum_{j=1}^n \lambda_j^2(H(\frac{s}{|y|},x,y))\xi_j^2 = \sum_{j=1}^n \lambda_j(\gamma(s))\xi_j^2$$

$\forall \xi \in R^n$. So, γ is a λ-subunit curve starting from x and attaining $H(t,x,y)$ at the time $s = t|y|$. Therefore by (1.9),

$$\delta(x,H(t,x,y)) \leq ad(x,H(t,x,y)) \leq at|y| \leq ct$$

where $c = 2\alpha a$

Thus, from (4.6), we obtain

$$|u(x)|\sqrt{\varphi(x)} \leq \frac{c}{|\sum|}\int_0^{2a^2r} \frac{1}{|Q(x,t)|}\int_{Q(x,ct)} |\nabla_\lambda u(z)|\sqrt{\varphi(z)}dzdt$$

and, interchanging the order of integration and using the fact that $supp\varphi \subset Q$ (the argument we are going to present next is due to Chanillo, Sawyer and Wheeden), we get

$$(4.7) \qquad |u(x)|\sqrt{\varphi(x)} \leq \frac{c}{|\sum|}\int_Q |\nabla_\lambda u(z)|\sqrt{\varphi(z)}(\int_{c\delta(x,z)}^\infty \frac{dt}{|Q(x,t)|})dz.$$

We claim that $\int_{ch}^\infty \frac{dt}{|Q(x,t)|} \leq c\frac{ch}{|Q(x,h)|}$. To prove this we note that, by (1.8), $\frac{|Q(x,t)|}{t} = \prod_{j=2}^n F_j(x^\star,t)$, and consequently by (1.10), there exists $\epsilon > 0$ such that if $t > \tau$ then

$$\frac{|Q(x,t)|}{t} \geq c(\frac{t}{\tau})^\epsilon\frac{|Q(x,\tau)|}{\tau}.$$

Hence,

$$\int_{ch}^\infty \frac{dt}{|Q(x,t)|} = \int_{ch}^\infty \frac{t}{|Q(x,t)|}\frac{dt}{t} \leq \int_{ch}^\infty \frac{h}{|Q(x,h)|}(\frac{h}{t})^\epsilon\frac{dt}{t} = c\frac{h}{|Q(x,h)|}.$$

Finally, we note that $|\sum| \geq c > 0$, with c independent of x, since, by the change of variables $z = H(2a^2r,x,y)$,

$$|\sum| = \int_{\sum} dy \simeq \int_{H(2a^2r,x,\sum)} \frac{1}{|Q(x,2a^2r)|} dz$$
$$= \frac{|H(2a^2r,x,\sum)|}{|Q(x,2a^2r)|} = \frac{|E \cap H(2a^2r,x,\Delta_\epsilon^\alpha(\sigma)|)}{|Q(x,2a^2r)|} \geq c\beta \frac{|Q^\sigma(x,2a^2r)|}{|Q(x,2a^2r)|} \geq c > 0.$$

The lemma follows by combining the last two last estimates with (4.7).

Proof of Theorem E.

Define $Tf(x) = \int_{R^n} f(y)K(x,y)dy$, where $K(x,y) = \frac{\delta(x,y)}{|Q(x,\delta(x,y))|}$. Fix S a d-ball. In order to show that for a pair of weights $\tilde{v}$, $\tilde{w}$ we have $||Tf||_{L^2(S,\tilde{v})} \leq ||f||_{L^2(S,\tilde{w})}$ (where $||f||_{L^2(S,\tilde{v})} = (\int_S f^2\tilde{v})^{1/2}$) for all $f \geq 0$, $suppf \subset S$, according to [SW], we need to verify that the following conditions hold:

(a) there exists $s > 1$ such that

$$\varphi(I)|I|(\frac{1}{|I|}\int_I \tilde{v}^s dx)^{\frac{1}{2s}}(\frac{1}{|I|}\int_I \tilde{w}^{-s} dx)^{\frac{1}{2s}} \leq c$$

for all d-balls $I \subset 2S$, where $\varphi(I)$ is defined to be

$$\varphi(I) = sup\{K(x,y) : x,y \in I, d(x,y) \geq \frac{1}{2}r(I)\};$$

(b) there is $\epsilon > 0$ such that

$$\frac{|I'|}{|I|} \leq c_\epsilon \frac{\varphi(I)}{\varphi(I')}(\frac{r(I')}{r(I)})^\epsilon$$

for all pairs of d-balls $I' \subset I$.

Note that it is convenient to work with d since the results of [SW] hold for pseudo-metrics (a pseudo-metric d is a quasi-metric satisfying $d(x,y) = d(y,x)$ for all x, $y \in R^n$).

Define $\tilde{v} = \frac{v}{v(S)}$ and $\tilde{w} = \frac{w_1}{w_1(S)}r(S)^2$. Note that if $x, y \in I$ and $d(x,y) \geq \frac{1}{2}r(I)$, then by (1.9)

$$K(x,y) = \frac{\delta(x,y)}{|Q(x,\delta(x,y))|} \leq \frac{2ar(I)}{|Q(x,\frac{1}{2a}r(I))|} \leq c\frac{r(I)}{|Q(x,r(I))|},$$

and since $x \in I$, $|Q(x,r(I))| \simeq |I|$. Therefore,

$$\varphi(I) \leq c\frac{r(I)}{|I|}.$$

So, the expression in (a) is bounded by

$$c\frac{r(I)}{|I|}|I|(\frac{1}{|I|}\int_I(\frac{v}{v(S)})^s dx)^{\frac{1}{2s}}(\frac{1}{|I|}\int_I(\frac{w_1}{w_1(S)}r(S)^2)^{-s}dx)^{\frac{1}{2s}}$$
$$\leq c\frac{r(I)}{r(S)}(\frac{1}{|I|}\int_I(\frac{v}{v(S)})^s dx)^{\frac{1}{2s}}(\frac{1}{|I|}\int_I(\frac{w_1}{w_1(S)})^{-s}dx)^{\frac{1}{2s}},$$

which is equivalent to the expression in condition (1.18) (if we use doubling and (1.9)). This proves (a).

To show (b) we note that if $x, y \in I$ and $d(x,y) \geq \frac{1}{2}r(I)$ then

$$K(x,y) \geq \frac{(2a)^{-1}r(I)}{|Q(x,2ar(I))|} \geq c\frac{r(I)}{|I|}.$$

Thus $\varphi(I) \simeq \frac{r(I)}{|I|}$. Then, if $I' \subset I$, $\frac{\varphi(I)}{\varphi(I')} \simeq \frac{r(I)}{r(I')}\frac{|I'|}{|I|}$ and we obtain (b) with $\epsilon = 1$.

By doubling and (1.9), it follows that

$$\|Tf\|_{L^2(Q,\tilde{v})} \leq c\|f\|_{L^2(Q,\tilde{w})}$$

for all $f \geq 0$, $suppf \subset Q$, where $\tilde{v} = \frac{v}{v(Q)}$ and $\tilde{w} = \frac{w_1}{w_1(Q)}r(Q)^2$.

Suppose u is a Lipschitz function in Q and $|E| = |\{x \in Q(\xi,kr) : u(x) = 0\}| \geq \beta|Q|$, $1/2 < k < 1$. If we combine lemma (4.1) and the fact that $\|Tf\|_{L^2(Q,\tilde{v})} \leq c\|f\|_{L^2(Q,\tilde{w})}$ we obtain

$$(4.8) \quad (\frac{1}{v(Q)}\int_Q |u(x)|^2\varphi(x)v(x)dx)^{\frac{1}{2}} \leq cr(Q)(\frac{1}{w_1(Q)}\int_Q |\nabla_\lambda u(z)|^2\varphi(z)w_1(z)dz)^{\frac{1}{2}}.$$

Given Q and a general Lipschitz function u, there is a number $\mu = \mu(u,Q)$, the median of u in Q, such that if $Q^+ = \{x \in Q : u(x) \geq \mu\}$ and $Q^- = \{x \in Q : u(x) \leq \mu\}$ then $|Q^+| \geq \frac{|Q|}{2}$ and $|Q^-| \geq \frac{|Q|}{2}$. Hence, $u_1 = max\{u - \mu(u,kQ), 0\}$ and $u_2 = max\{\mu(u,kQ) - u, 0\}$ satisfy the hypothesis of Lemma (4.1) for some β depending on k and so if we apply (4.8) to u_1 and u_2 and add both inequalities, we get

$$(4.9) \qquad \int_Q |u(x) - \mu|^2\varphi(x)v(x)dx \leq cr(Q)^2\frac{v(Q)}{w_1(Q)}\int_Q |\nabla_\lambda u(z)|^2\varphi(z)w_1(z)dz.$$

Finally, it is easy to see that in (4.9) μ can be replaced by the average A_Q of u defined in Theorem E. In fact,

$$(4.10) \qquad \begin{aligned} &\int_Q |u(x) - A_Q|^2\varphi(x)v(x)dx \\ \leq\ & 2\int_Q |u(x) - \mu|^2\varphi(x)v(x)dx + 2\int_Q |\mu - A_Q|^2\varphi(x)v(x)dx, \end{aligned}$$

and

$$\begin{aligned} &\int_Q |\mu - A_Q|^2\varphi(x)v(x)dx = (\varphi v)(Q)|\mu - A_Q|^2 \\ =\ & (\varphi v)(Q)|\mu - \frac{1}{\varphi(Q)}\int_Q u(x)\varphi(x)dx|^2 \leq (\varphi v)(Q)(\frac{1}{\varphi(Q)}\int_Q |u(x) - \mu|\varphi(x)dx)^2 \\ \leq\ & \frac{(\varphi v)(Q)}{(\varphi(Q))^2}\int_Q |u(x) - \mu|^2\varphi^2(x)v(x)dx\int_Q \frac{1}{v(x)}dx, \end{aligned}$$

where in the last inequality we used Schwarz's inequality. Since $v \in A_2$ and $0 \le \varphi \le 1$, it follows from (4.9) and (4.10) that

$$\int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx$$
$$\le \; cr(Q)^2 [1 + (\frac{|Q|}{\varphi(Q)})^2] \frac{v(Q)}{w_1(Q)} \int_Q |\nabla_\lambda u(z)|^2 \varphi(z) w_1(z) dz.$$

This finishes the proof of Theorem E if we note that $\varphi(Q) \simeq |Q|$ since $1/2 \le k \le 1$.

The next corollary is also helpful.

Corollary 4.11 *Theorem E is also true with* $A_Q = \frac{1}{(\varphi v)(Q)} \int_Q u \varphi v dx$.

Just note that

$$\int_Q |\mu - A_Q|^2 \varphi v dx = (\varphi v)(Q) |\mu - A_Q|^2$$
$$\le \; (\varphi v)(Q) |\frac{1}{(\varphi v)(Q)} \int_Q |\mu - u| \varphi v dx|^2 \le \int_Q |\mu - u|^2 \varphi v dx,$$

where the last inequality follows by Schwarz's inequality.

5 Harnack's inequality

The proof of Theorem A follows as an application of Bombieri's lemma which we state next. For its proof see section 5 of [GW2].

Lemma 5.1 *Let $R(\rho)$ be a one parameter family of rectangles in R^{n+1}, $R(\sigma) \subset R(\rho)$, $1/2 \le \sigma \le \rho \le 1$ and let ν be a doubling measure in R^{n+1}. Let A, μ, M, m, θ and κ be positive constants such that $M \ge \frac{1}{\mu}$ and suppose that f is a positive measurable function defined in a neighborhood of $R(1)$ satisfying*

$$ess\ sup_{R(\sigma)} f^p \le \frac{A}{(\rho - \sigma)^m} \iint_{R(\rho)} f^p \nu(x) dx dt \tag{5.2}$$

for all σ, ρ, p, $1/2 \le \theta \le \sigma < \rho < 1$, $0 < p < M$ and

$$\nu(\{(x,t) \in R(1) : \log f > s\}) \le (\frac{\mu}{s})^\kappa \nu(R(1)) \tag{5.3}$$

for all $s > 0$. Then there is a constant $\gamma = \gamma(A, m, \kappa) > 0$ such that

$$log(ess\ sup_{R(\theta)} u) \le \frac{\gamma}{(1-\theta)^{2m}} \mu.$$

Hence, in order to prove Theorem A, we need a mean value inequality (that we proved in section 3) and a logarithm estimate which is given by Theorem F (some steps of its proof we will present in this section). The next lemma shows that the test function described on page 537 of [FL1] satisfies the conditions of Theorem E. Then, as we said before, the proof of Theorem (F) follows as Lemma (4.9) of [GW2].

Lemma 5.4 *Given* $Q = Q(\xi, r)$ *and* $0 < k < 1$, *there exists* $\varphi \in C^1$ *such that* $\varphi \equiv 1$ *in* kQ, $0 \leq \varphi \leq 1$, *supp*$\varphi \subset Q$, $|\nabla_\lambda \varphi| \leq \frac{c}{r(1-k)}$ *and* $\varphi(x)\varphi(H(t_\circ, x, y)) \leq \varphi(H(t, x, y))$ *for all* x, y, t, $t_\circ$ *with* $0 \leq t \leq t_\circ$.

Proof: Consider the function φ given by [FL1], page 537:

$$\varphi(x) = \prod_{j=1}^{n} \psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right),$$

where $\psi \in C^\infty(R)$, $0 \leq \psi \leq 1$, $\psi(t) = \psi(-t)$, $\psi \equiv 1$ on $[-k, k]$, $\psi = 0$ outside $]-1, 1[$, $|\psi'(t)| \leq 2(1-k)^{-1}$, for all $t \in R$. Here, we show that φ satisfies the last condition since all the others are proved in [FL1], page 537.

Fix t, $0 < t < t_\circ$, x and y. Define $z = H(t, x, y)$. Then, $z_j = x_j + y_j \int_0^t \lambda_j(H(s, x, y))ds$. Suppose $z_j - \xi_j \geq 0$. If $y_j \geq 0$ then

$$|z_j - \xi_j| \leq x_j - \xi_j + y_j \int_0^{t_\circ} \lambda_j(H(s, x, y))ds = H_j(t_\circ, x, y) - \xi_j.$$

On the other hand, if $y_j < 0$,

$$|z_j - \xi_j| \leq |x_j - \xi_j|.$$

Thus, if $z_j - \xi_j \geq 0$ then $|z_j - \xi_j| \leq |H_j(t_\circ, x, y) - \xi_j|$ or $|z_j - \xi_j| \leq |x_j - \xi_j|$. The same holds if $z_j - \xi_j < 0$. Since $\psi(t)$ can be chosen to be non-increasing for positive t, then $\varphi(z) \geq a_1 \ldots a_n$, where $a_j = \psi(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)})$ or $a_j = \psi(\frac{|H_j(t_\circ, x, y) - \xi_j|}{F_j(\xi^*, r)})$. Since $0 \leq \psi \leq 1$,

$$a_j \geq \psi\left(\frac{|H_j(t_\circ, x, y) - \xi_j|}{F_j(\xi^*, r)}\right)\psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right)$$

for $1 \leq j \leq n$. Therefore,

$$\varphi(z) \geq \varphi(x)\varphi(H(t_\circ, x, y)).$$

The next 3 lemmas are needed in order to show that the hypothesis in Theorem A imply those in Theorems D and E.

Lemma 5.5 *Assume that Poincaré's inequality holds for w_1, w_2 with $q = 2$ and $\mu = 1$. Then*

$$(\frac{r(I)}{r(B)})^2 \frac{w_2(I)}{w_2(B)} \leq c\frac{w_1(I)}{w_1(B)}$$

for any pair of δ-balls I, B, with $I \subset 2B$.

Proof: Suppose $I = Q(u_\circ, r(I))$ and $B = Q(x, r(B))$ and define

$$F(u) = \sum_{j=1}^{n} \frac{|u_j - (u_\circ)_j|}{F_j(u_\circ^\star, r(I))} r(I)\varphi(u)$$

where φ is the function described in lemma (5.4) associated with I (as opposed to B) and $k = 1/2$. If $u \in I$, by (1.8)

$$|\frac{\partial F}{\partial u_k}(u)| \leq \frac{r(I)}{F_k(u_\circ^\star, r(I))} + \frac{\partial \varphi}{\partial u_k}(u)nr(I),$$

for $k \in \{1, ...n\}$, and using the fact that $\lambda_k(u) = \lambda_k(u^\star) \leq \lambda_k(H(u^\star, r(I)))$ if $u \in I$ we get

$$|\lambda_k(u)\frac{\partial F}{\partial u_k}(u)| \leq \frac{F_k(u^\star, r(I))}{F_k(u_\circ^\star, r(I))} + nr(I)\lambda_k(u)\frac{\partial \varphi}{\partial u_k}(u)$$

and by lemma (2.4) and the fact that $|\nabla_\lambda \varphi| \leq c/r(I)$ we have $|\nabla_\lambda F(u)| \leq c\chi_I$.

We have Poincaré's inequality for F, i.e.,

$$(5.6)\qquad \begin{aligned} &(\frac{1}{w_2(B)}\int_{n4^{\eta+1}B} |F(u) - av_{n4^{\eta+1}B}F|^2 w_2(u)du)^{1/2} \\ \leq\ & cr(B)(\frac{1}{w_1(B)}\int_{na^2 4^{\eta+1}B} |\nabla_\lambda F(u)|^2 w_1(u)du)^{1/2}, \end{aligned}$$

where $\eta = \max_{j=1,..n}\{G_j\}$. The right side of (5.6) is bounded by $cr(B)(\frac{w_1(I)}{w_1(B)})^{1/2}$ by doubling and the fact that $|\nabla_\lambda F| \leq c\chi_I$. Now, if $u \in \frac{1}{4}I$ there exists $k \in \{1, ..., n\}$ such that $|u_k - (u_\circ)_k| \geq F_k(u_\circ^\star, \frac{1}{4}r(I))$ and then if $u \in \frac{1}{2}I/\frac{1}{4}I$ (note that $\varphi(u) = 1$)

$$(5.7)\qquad F(u) \geq \frac{F_k(u_\circ^\star, \frac{1}{4}r(I))}{F_k(u_\circ^\star, r(I))} r(I) \geq (\frac{1}{4})^{G_k} r(I) \geq \frac{1}{4^\eta} r(I).$$

Also, if $u \in I$, $F(u) \leq nr(I)$ and therefore

$$av_{n4^{\eta+1}B}F \leq \frac{|I|}{|n4^{\eta+1}B|} nr(I).$$

But, by (1.10), $F_j(x_B^\star, n4^{\eta+1}r(B)) \geq 2n4^\eta F_j(x_B^\star, 2r(B))$, and by (1.11), $|n4^{\eta+1}B| \geq (2n4^\eta)^n|2B| \geq 2n4^\eta|2B|$. Hence, since $I \subset 2B$, $av_{n4^{\eta+1}B}F \leq \frac{r(I)}{2.4^\eta}$ and then if $u \in \frac{1}{2}I/\frac{1}{4}I$ (using also (5.7)),

$$|F(u) - av_{n4^{\eta+1}B}F| \geq cr(I).$$

Therefore, the left hand side of (5.6) is larger than a constant times

$$[\frac{(r(I))^2}{w_2(B)}w_2(\frac{1}{2}I\backslash\frac{1}{4}I)]^{1/2} \geq cr(I)(\frac{w_2(I)}{w_2(B)})^{1/2},$$

where in the last inequality we used the fact that $w_2(\frac{1}{2}I\backslash\frac{1}{4}I) \simeq w_2(I)$, which is shown in the next lemma.

Lemma 5.8 *If w is a doubling weight then $w(Q(u,2s)\backslash Q(u,s))$ is equivalent to $w(Q(u,s))$.*

Proof: Choose $\eta \in Q(u,2s)$ such that $\delta(u,\eta) = \frac{3s}{2}$. By Lemma 2.5,

$$Q(\eta, \frac{3\epsilon s}{2(2a^2)^\zeta}) \subset Q(u, (1+\epsilon)\frac{3s}{2})$$

for any $0 < \epsilon < 1$.

Choose j such that $\delta(u,\eta) = \varphi_j(u^\star, |\eta_j - u_j|)$. Then, if $y \in Q(\eta, \frac{3\epsilon s}{2(2a^2)^\zeta})$,

$$F_j(u^\star, \frac{3s}{2}) = |\eta_j - u_j| \leq |\eta_j - y_j| + |y_j - u_j| \leq F_j(\eta^\star, \frac{3\epsilon s}{2(2a^2)^\zeta}) + |y_j - u_j|.$$

By (1.10) and Lemma 2.4,

$$F_j(u^\star, \frac{3s}{2}) \leq \epsilon F_j(u^\star, \frac{3s}{2}) + |y_j - u_j|.$$

Thus,

$$|y_j - u_j| \geq (1-\epsilon)F_j(u^\star, \frac{3s}{2}) \geq F_j(u^\star, (1-\epsilon)\frac{3s}{2}).$$

If we choose $\epsilon = 1/3$ we have proved that

$$Q(\eta, \frac{s}{2(2a^2)^\zeta}) \subset Q(u,2s)\backslash Q(u,s).$$

The lemma follows by doubling.

Lemma 5.9 *If $w_1 \in A_2$, $v \in A_\infty$ and Poincaré's inequality holds for w_1, v with $q = 2$ and $\mu = 1$, then condition (1.21) holds.*

Proof: If $v \in A_\infty$ there exists $s > 1$ such that

$$(\frac{1}{|I|}\int_I (\frac{v}{v(B)})^s dx)^{1/s} \leq \frac{1}{|I|}\frac{v(I)}{v(B)}.$$

So, since Poincaré's inequality holds for w_1, v with $q = 2$, by Lemma 5.5

$$(\frac{r(I)}{r(B)})^2 (\frac{1}{|I|}\int_I (\frac{v}{v(B)})^s dx)^{1/s} \leq c\frac{1}{|I|}\frac{w_1(I)}{w_1(B)},$$

and the above condition is equivalent to condition (1.18) since $w_1 \in A_2$.

Now we are ready to prove Theorem A.

Proof of Theorem A

Let u be a non-negative solution of (1.1) in the cylinder $R_{\alpha,\beta} = R_{\alpha,\beta}(x_o, t_o) = Q(x_o, \alpha) \times (t_o - \beta, t_o + \beta)$. If we define $T(x,t) = (x, \beta t + t_o)$ and $\bar{u}(x,t) = u(T(x,t))$ then u is a solution in $R_{\alpha,1}(x_o, 0)$ of the equation

$$v(x)\bar{u}_t = div(\bar{A}(x,t)\nabla\bar{u}),$$

where the coefficients matrix $\bar{A} = (\bar{a}_{ij})$ is defined by $\bar{a}_{ij}(x,t) = \beta a_{ij}(x, \beta t + t_o)$ and satisfies the degeneracy condition

$$\bar{w}_1(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2 \le \sum_{j=1}^{n}\bar{a}_{ij}(x,t)\xi_i\xi_j \le \bar{w}_2(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2,$$

if we put $\bar{w}_i = \beta w_i$, for $i = 1, 2$.

Suppose $|p| < [\alpha^{-2}\bar{\Lambda}(Q(x_o,\alpha)) + \alpha^2/\bar{\lambda}(Q(x_o,\alpha))]^{-1}$, where $\bar{\Lambda}(Q) = \bar{w}_2(Q)/v(Q)$, $\bar{\lambda}(Q) = \bar{w}_1(Q)/v(Q)$. Write

$$R^-(\rho) = Q(x_o, \frac{(\rho+1)\alpha}{3}) \times (-\frac{1}{2} - \frac{\rho}{2}, -\frac{1}{2} + \frac{\rho}{2})$$

$$R^+(\rho) = Q(x_o, \frac{(\rho+1)\alpha}{3}) \times (\frac{1}{2} - \frac{\rho}{2}, 1)$$

If we take $1/2 \le \rho < r < 1$ then the mean value inequalities in Theorem (3.15) applied to u give

$$\text{(5.10)} \qquad ess\ sup_{R^-(\rho)}\bar{u}^p \le c\frac{1}{(r-\rho)^m}\iint_{R^-(r)}\bar{u}^p(\frac{\bar{w}_2}{\bar{w}_2(Q_\alpha)} + \frac{v}{v(Q_\alpha)})dxdt,$$

for some $m > 0$, if $p > 0$, where $Q_\alpha = Q(x_o, \alpha)$, and

$$\text{(5.11)} \qquad ess\ sup_{R^+(\rho)}\bar{u}^p \le c\frac{1}{(r-\rho)^m}\iint_{R^+(r)}\bar{u}^p(\frac{\bar{w}_2}{\bar{w}_2(Q_\alpha)} + \frac{v}{v(Q_\alpha)})dxdt,$$

if $p < 0$. Moreover, by Theorem B, $\bar{u}$ is locally bounded and by adding $\epsilon > 0$, we may assume by letting $\epsilon \longrightarrow 0$ at the end of the proof that $\bar{u}$ is bounded below in $R_{\alpha,1}(x_o, 0)$ by a positive constant.

Now, by Theorem F, we have

$$(5.12)\qquad [(\frac{v}{v(Q_\alpha)}+\frac{\bar{w}_2}{\bar{w}_2(Q_\alpha)})\otimes 1](E^+)$$
$$\leq \{\frac{1}{s}\frac{v(Q_\alpha)}{\bar{w}_1(Q_\alpha)}\alpha^2\}^\kappa \leq c\{\frac{1}{s}[\alpha^{-2}\bar{\Lambda}(Q_\alpha)+\alpha^2\frac{1}{\bar{\lambda}(Q_\alpha)}]\}^\kappa,$$

and the same inequality holds for E^-, where E^+, E^- are defined in Theorem F with $u=\bar{u}$, $R=\frac{2}{3}\alpha$, $a=-1$, $b=1$, $t_\circ=0$, $M_2\simeq\bar{\Lambda}(Q_\alpha)/\alpha^2$.

By (5.10) and (5.12), we can apply Bombieri's lemma to the family of rectangles $R^-(\rho)$ with $\mu=\alpha^{-2}\bar{\Lambda}(Q_\alpha(x_\circ))+\alpha^2/\bar{\lambda}(Q_\alpha(x_\circ))$, $M=1/\mu$ and $f=e^{-M_2+V(0)}\bar{u}$, obtaining

$$ess\ sup_{R^-(1/2)}f\leq Ce^{c[\alpha^{-2}\bar{\Lambda}(Q_\alpha)+\alpha^2/\bar{\lambda}(Q_\alpha)]},$$

and this implies that

$$(5.13)\qquad ess\ sup_{R^-(1/2)}\bar{u}\leq Ce^{c[\alpha^{-2}\bar{\Lambda}(Q(x_\circ,\alpha))+\alpha^2/\bar{\lambda}(Q(x_\circ,\alpha))]}e^{-V(0)}.$$

Also, by (5.11) and (5.12), we can apply Bombieri's lemma to the family of rectangles $R^+(\rho)$, $f=e^{-M_2-V(0)}\bar{u}^{-1}$, with μ, M, M_2 and $V(0)$ as before, and we obtain

$$ess\ sup_{R^+(1/2)}\ f\leq Ce^{c[\alpha^{-2}\bar{\Lambda}(Q_\alpha)+\alpha^2/\bar{\lambda}(Q_\alpha)]},$$

which implies that

$$(5.14)\qquad e^{-V(0)}\leq Ce^{c[\alpha^{-2}\bar{\Lambda}(Q(x_\circ,\alpha))+\alpha^2/\bar{\lambda}(Q(x_\circ,\alpha))]}ess\ inf_{R^+(1/2)}\bar{u}.$$

Combining (5.13) and (5.14) it follows that

$$ess\ sup_{R^-(1/2)}\bar{u}\leq c_1e^{c[\alpha^{-2}\bar{\Lambda}(Q(x_0,\alpha))+\alpha^2/\bar{\lambda}(Q(x_0,\alpha))]}ess\ inf_{R^+(1/2)}\bar{u}.$$

Since, $T(R^-(1/2))=R^-$. $T(R^+(1/2))=R^+$ and $\alpha^{-2}\bar{\Lambda}(Q_\alpha)+\alpha^2/\bar{\lambda}(Q_\alpha)=\alpha^{-2}\beta\Lambda(Q_\alpha)+\alpha^2\beta^{-1}/\lambda(Q_\alpha)$, Theorem A follows.

Remark: Using the equivalence between d and δ we can prove the following analogues of Theorem A and B for the metric d.

THEOREM A': Assume (i), (ii), (iii) of Theorem A. If u is a non-negative solution of (1.1) in the cylinder $R=S(x_0,\alpha a^2)\times(t_0-\beta,t_0+\beta)$, then

$$ess\ sup_{R^-}u\leq c_1exp\{c_2[\alpha^{-2}\beta\Lambda(S(x_0,\alpha))+\alpha^2\beta^{-1}\lambda(S(x_0,\alpha))^{-1}]\}ess\ inf_{R^+}u$$

where $R^- = S(x_0, \alpha/2) \times (t_0 - 3\beta/4, t_0 - \beta/4)$, $R^+ = S(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta)$, $\Lambda(S) = w_2(S)/v(S)$ and $\lambda(S) = w_1(S)/v(S)$ for a d-ball S. Here the constants c_1,c_2 depend only on the constants which arise in (i), (ii), (iii).

THEOREM B': Assume hypothesis (i), (ii), (iii) of Theorem A hold. Let $0 < p < \infty$, $\alpha,\ \beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$ and let $S(x_0, \alpha) = S$, $S(x_0, \alpha') = S'$ and $R(\alpha, \beta) = S \times (t_0 - \beta, t_0 + \beta)$, $R'_+(\alpha, \beta) = S' \times (t_0 - \beta', t_0 + \beta)$. If u is a solution of (1.1) in $R(a^2\alpha, \beta)$, then u is bounded in $R'_+(\alpha, \beta)$ and

$$\begin{aligned} & ess\ sup_{R'_+(\alpha,\beta)} |u|^p \\ \leq\ & D(\alpha^2\beta^{-1}\lambda(S)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(S) + 1)^{h/(h-1)} \,⨌_{R(a^2\alpha,\beta)} |u|^p(\alpha^{-2}\beta w_2 + v)\,dx\,dt \end{aligned}$$

where D is as in Theorem B, and $C = c\frac{\alpha^{2+b}\beta}{(\alpha-\alpha')^{2+b}(\beta-\beta')}$. Here $h > 1$, $c > 0$ and $b > 0$ are constants which are independent of $u, p, \alpha, \alpha', \beta, \beta'$.

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