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POINTWISE ESTIMATES FOR SOLUTIONS OF DEGENERATE PARABOLIC EQUATIONS

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The results I shall describe in this note are joint work with Richard L. Wheeden. They concern the regularity properties of weak solutions of a certain class of degenerate parabolic equations, the validity of a Harnack principle for non-negative weak solutions and estimates for the fundamental solution. The proofs of the results can be found in references [**G-W1**], [**G-W2**], [**G-W3**] and [**G-W4**]. In order to place the results in proper perspective we recall some results in partial differential equations.

In the late 50's and early 60's a theory for the following class of equations with non-smooth coefficients was developed. Let Ω be a domain in $R^n, Q = \Omega \times (a,b)$ and consider the operator in divergence form

$$Lu = u_t - \sum_{i,j=1}^{n} (a_{ij}(x,t)\, u_{x_i})_{x_j}$$

where the coefficient matrix $A(x,t) = (a_{ij}(x,t))$ is measurable, real, symmetric and there are two positive constants λ, Λ such that

$$\lambda|\xi|^2 \leq \langle A(x,t)\xi,\xi\rangle \leq \Lambda|\xi|^2\,,$$

for every $\xi \in R^n, \langle,\rangle$ being the Euclidean inner product. A function $u \in L^2(Q)$ is a weak solution of $Lu = 0$ if $\nabla_x u \in L^2(Q)$ and

$$\int\int_Q \{-u\,\varphi_t + \langle A(x,t)\nabla u, \nabla\varphi\rangle\}\,dx\,dt = 0$$

for every $\varphi \in C_0^1(Q)$. The reason to assume only measurability of the coefficients is because of the applications to non-linear equations. Nash [**Na**] proved that weak solutions are Hölder continuous. De Giorgi [**DeG**] proved this result in the elliptic case. Moser [**Mo1**],[**Mo2**] established a parabolic Harnack principle for non-negative solutions and derived from it the Hölder continuity. This Harnack principle has a difference with the elliptic one, this is: values of a non-negative solution u at a given time t_1 are only comparable to values of u at a later time $t_2 > t_1$. From the Harnack principle Aronson [**Ar**] proved that the fundamental solution of L behaves like the heat kernel.

The study of linear degenerate elliptic equations in divergence form began in the late 60's and early 70's with the work of Murthy and Stampacchia [**M-S**],and Trudinger [**T1**],[**T2**].

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These equations are of the form $div(A(x)\nabla u) = 0$ where $A(x)$ is a measurable and symmetric matrix satisfying $\lambda w(x)|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda w(x)|\xi|^2$ and $w(x) \geq 0$. These authors found conditions on w under which weak solutions are Hölder continuous and the Harnack principle holds.

The theory of degenerate equations is related to the class of A_∞ weights discovered by Muckenhoupt [**Mu**] in connection with the boundedness of the Hardy-Littlewood maximal operator in weighted L^p spaces. He defined the class A_p. A non-negative locally integrable function w belongs to A_p, $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leq C,$$

for every ball $B \subset R^n$, $|\cdot|$ is Lebesgue measure. For $p = 1$, $w \in A_1$ if there exists a constant $c > 0$ such that

$$\frac{1}{|B|}\int_B w(x)\,dx \leq c \inf_B w,$$

for all balls B. Then $A_\infty = \bigcup_{p\geq 1}^{\infty} A_p$.

The connection between degenerate equations and A_∞ weights was predicted in [**C-F**]. Ten years later this prediction was confirmed by Fabes, Kenig and Serapioni in [**F-K-S**]. They proved that if $w \in A_2$ then the Harnack principle holds, the solutions are Hölder continuous and among A_∞ weights the class A_2 is the best one for which these results hold.

The equations we had studied are degenerate parabolic equations of the form

$$v(x)\,u_t = \sum_{i,j=1}^{n}(a_{ij}(x,t)\,u_{x_j})_{x_i}, \tag{1-1}$$

$$w_1(x,t)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \leq w_2(x,t)|\xi|^2;$$

where the coefficients are measurable functions, the matrix (a_{ij}) is symmetric and v, w_1 and w_2 are non-negative.

In the case when w_1 and w_2 are time independent these equations appear in the following two instances. First, they arise when one pulls back the heat operator via a quasiconformal mapping φ from R^n into R^n. In this case $v(x) = |det\,\varphi'(x)|$ and $w_1 \approx w_2 \approx v^{\frac{n-2}{n}}$. Second, the equation (1-1) appears as a model of the diffusion of temperature in a non-isotropic and non-homogeneous material. The function v represents the product of the density of the material at x times the specific heat at x. The coefficients a_{ij} represent the thermal conductivity.

The problems we considered for these equations are the following:

(a) What conditions on v, w_1 and w_2 imply regularity of solutions, such us continuity or Hölder continuity ?

(b) When is a Harnack principle valid for non-negative solutions of (1-1)?

(c) What is the behavior of the fundamental solution of (1-1)?

The tools we used to attack these problems are weighted norm inequalities of Poincaré type and weighted interpolation inequalities.We say the Poincaré inequality holds for the weights w_1, w_2, with μ-average and exponent $q \geq 2$, if there exists a constant $c > 0$ such that

$$\left(\frac{1}{w_2(B)}\int_B |F(x) - av_{B,\mu}F|^q w_2(x)\,dx\right)^{1/q}$$

$$\leq c|B|^{1/n}\left(\frac{1}{w_1(B)}\int_B |\nabla F(x)|^2 w_1(x)\,dx\right)^{1/2}, \tag{1-2}$$

for every ball B and every $F \in Lip(\bar{B})$, where

$$av_{B,\mu}F = \frac{1}{\mu(B)}\int_B F(x)\mu(x)\,dx,$$

and $Lip(\bar{B})$ denotes the class of Lipschitz functions on $\bar{B}$. Here $B_R(x)$ denotes the Euclidean ball centered at x with radius R, and $w(E) = \int_E w(x)\,dx$ for a measurable set E.

By the results of [**Cha-W**], a sufficient condition for the validity of (1-2) for $q > 2$, with $\mu = 1$ or $\mu = w_2$, when w_2 is a doubling weight and $w_1 \in A_2$, is the following:

$$\left(\frac{|\tilde{B}|}{|B|}\right)^{1/n}\left(\frac{w_2(\tilde{B})}{w_2(B)}\right)^{1/q} \leq c\left(\frac{w_1(\tilde{B})}{w_1(B)}\right)^{1/2}, \tag{1-3}$$

for all balls $\tilde{B}, B, \tilde{B} \subset 2B$, with c independent of the balls. We also know that if (1-2) holds and w_1, w_2 and μ are doubling then (1-3) holds (see [**G-W1**]).

An example of an interpolation inequality is the following,

$$\frac{1}{w(B)}\int_B |u|^{2h} w(x)\,dx$$

$$\leq C\left(\frac{1}{v(B)}\int_B u^2 v\,dx\right)^{h-1}\left(|B|^{\frac{2}{n}}\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx + \frac{1}{v(B)}\int_B u^2 v\,dx\right). \tag{1-4}$$

The study of weighted interpolation inequalities is in [**G-W1**].

We have proved in [**G-W2**] Harnack's inequality and mean value inequalities for solutions of (1-1).

We now recall the Harnack's inequality from [**G-W2**].

THEOREM A. Harnack's inequality. *Suppose that*

(i) $w_1, w_2 \in A_2$*;*

(ii) the Poincaré inequality holds for w_1, w_2 *with* $\mu = 1$ *and exponent* $q > 2$*;*

(iii) the Poincaré inequality holds for $w_1, 1$ *with any* μ *and exponent* $q > 2$.

If u *is a non-negative solution of* (1-1) *in the cylinder* $Q = B_\alpha(x_0) \times (t_0 - \beta, t_0 + \beta)$, *then*

$$\text{(1-5)} \qquad \operatorname*{ess\,sup}_{Q^-} u \leq c_1 \exp\left(c_2\left(\frac{\beta}{\alpha^2}\,\Lambda(B_\alpha(x_0)) + \frac{\alpha^2}{\beta}\,\frac{1}{\lambda(B_\alpha(x_0))}\right)\right) \operatorname*{ess\,inf}_{Q^+} u,$$

where

$$Q^- = B_{\alpha/2}(x_0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta), \qquad Q^+ = B_{\alpha/2}(x_0) \times (t_0 + \frac{1}{4}\beta, t_0 + \beta),$$

$$\text{(1-6)} \qquad \Lambda(B) = \frac{w_2(B)}{|B|}, \quad \text{and} \quad \lambda(B) = \frac{w_1(B)}{|B|}, \qquad \text{for a ball } B.$$

Here the constants c_1, c_2 depend only on the constants which arise in and (i)-(iii).

The inequality (1-5) is sharp. Given the dimension α of the cylinder R , the choice of β leads to different constants in the exponent in (1-5) as well as to different cylinders. The choice of β that minimizes the exponent in (1-5) is given by

$$\beta = \alpha^2 \frac{v(B_\alpha(x_0))}{[w_1(B_\alpha(x_0))\, w_2(B_\alpha(x_0))]^{1/2}}\,.$$

In this case (1-5) becomes

$$\text{(1-7)} \qquad \sup_{R^-} u \leq C_1 \exp\left(C_2[\frac{w_2(B_\alpha(x_0))}{w_1(B_\alpha(x_0))}]^{1/2}\right) \inf_{R^+} u\,,$$

and this inequality is sharp. This shows how the degeneracies affect the classical homogeneity α, α^2. By assuming w_2 does not grow too much with respect to w_1 the continuity of the solutions follows from (1-7). The inequality (1-7) implies the elliptic estimates in [**C-W2**]. The results of [**C-S1**] and [**C-S2**] are special cases of our results in [**G-W2**].

In [**G-W3**] we proved Harnack's inequality when w_1 and w_2 depend also on t. The main tool we have developed to establish Harnack's inequality is an appropriate Sobolev interpolation inequality on cylinders in R^{n+1}, which by iteration leads to mean value inequalities for solutions and then to (1-5). To establish the interpolation inequalities we first use weighted norm inequalities of Poincaré type to derive interpolation inequalities in x on small balls, which are valid uniformly in t. Next, by using a covering argument, we deduce the interpolation inequality in any ball, valid uniformly in t and with a convenient form allowing integration in t. This gives us results for cylinders by Hölder's inequality. This is an extension of the method we developed in [**G-W1**] to prove interpolation inequalities in the time independent case.

As an example of the results in [**G-W3**], we have that in case

$$w_1(x,t) = c_1|x|^{\alpha_1}|t|^{\beta_1}, \qquad w_2(x,t) = c_2|x|^{\alpha_2}|t|^{\beta_2}, \qquad v(x) = |x|^\gamma,$$

an analogue of the Harnack inequality (1-5) holds if

$$-n < \gamma, \alpha_1, \alpha_2 < n, \qquad -1 < \beta_1, \beta_2 < 1,$$

$$0 \leq \alpha_1 - \alpha_2 < 2, \qquad \alpha_1 - \gamma < 2,$$

and

$$1-\frac{\alpha_2-\alpha_1+2}{\gamma-\alpha_1+2}<\frac{\beta_2+1}{\beta_1+1}\leq 1.$$

We note that in case $\alpha_1 = \alpha_2 = 2$, $\beta_1 = \beta_2 = 0$, and $\gamma = 0$, it is shown in [**Chi-Se1**], p.142, that equation (1-1) has solutions which are unbounded.

The results in [**G-W3**] extend those obtained in publication [**G-W2**].

The Harnack inequality and mean value inequalities proved in [**G-W2**] are used in [**G-W4**] to establish bounds for the fundamental solution of (1-1). By using a different method from the one used in [**G-W4**], the author and G. Nelson in [**G-N**] have proved bounds in the case $w_1 \approx w_2$. However, the bounds in [**G-W4**] are in general better than the ones in [**G-N**].

The lower bound for the fundamental solution follows directly from the Harnack inequality in [**G-W2**]. To establish the upper bound we need a certain differential inequality to hold for the test functions used, and this can be achieved for example by assuming that $w_2^{-n/2}$ is in the class strong A_∞ recently introduced by G. David and S. Semmes in [**D-S**]. However, for some classes of weights, the differential inequality can be proved directly without assuming the strong A_∞ condition, and this leads to an upper bound for the fundamental solution. It is not clear how these classes of weights are related to strong A_∞. See the remark 1 for more details.

We say that a non-negative and locally integrable function w in R^n is a doubling weight of order μ (i.e., $w \in D_\mu$) if there exists a constant $c > 0$ such that

$$w(B_{tR}(x)) \leq c\,t^{n\mu} w(B_R(x)),$$

for every $t \geq 1, R > 0$, and $x \in R^n$.

Let $x, y \in R^n$ and define

$$\rho(x,y)^2 = \frac{|x-y|^{n+2}}{\left[w_1\left(B_{|x-y|}(y)\right) w_2\left(B_{|x-y|}(y)\right)\right]^{1/2}},$$

and

$$\Omega_\rho(y,r) = \{x : \rho(x,y)^2 \leq r\}, \qquad r > 0.$$

Let $\Gamma(x,t;y,s) = \Gamma_A(x,t;y,s)$ denote the fundamental solution of (1-1) with pole at (y,s). We shall assume that Γ is a continuous non-negative solution of (1-1) in (x,t) for $(x,t) \neq (y,s)$ which is zero for $t < s$, and satisfies the usual properties (see [**G-W4**]), as well as the fact that if $f \in L^2(R^n)$, then $\int_{R^n} \Gamma(x,t;y,s) f(y)\,dy, t > s$, is a continuous solution of (1-1) with continuous boundary values f as $t \to s$ at the points where f is continuous.

The lower bound is as follows.

THEOREM B. Lower bound. *Let w_1, w_2 be weights satisfying conditions (i)-(iii) of Theorem A, and let $w_i \in D_{d_i}, i = 1, 2$, with $\frac{d_1+d_2}{2} < 1 + \frac{2}{n}$. Then if $t > s$,*

$$\Gamma(x,t;y,s) \geq$$

$$C \max \left(\frac{\exp\left(-c\left(\frac{w_2(\Omega_\rho(y,t-s))}{w_1(\Omega_\rho(y,t-s))}\right)^{1/2}\right)}{|\Omega_\rho(y,t-s)|}, \frac{\exp\left(-c\left(\frac{w_2(\Omega_\rho(x,t-s))}{w_1(\Omega_\rho(x,t-s))}\right)^{1/2}\right)}{|\Omega_\rho(x,t-s)|} \right) \times$$

$$\exp\left(-c\frac{\rho(x,y)^2}{t-s}\left(\frac{w_2(B_{|x-y|}(y))}{w_1(B_{|x-y|}(y))}\right)^{1/2}\right).$$

In order to state the next theorem we recall several definitions, including the definition of the class of strongly A_∞ weights introduced by G. David and S. Semmes in [**D-S**].

A weight w is in A_∞ if there are positive constants c, ϵ such that

$$\frac{w(E)}{w(B)} \leq c\left(\frac{|E|}{|B|}\right)^\epsilon,$$

for every ball B and every measurable subset E of B.

A weight w satisfies a reverse Hölder condition if there are constants $c > 0, \gamma > 1$ such that

$$\left(\frac{1}{|B|}\int_B w(z)^\gamma\, dz\right)^{1/\gamma} \leq c\frac{1}{|B|}\int_B w(z)\, dz,$$

for all balls B. It is well known that the A_∞ and reverse Hölder conditions are equivalent: see, e.g., [**Co-F**].

Given $x, y \in R^n$ we denote by $B_{x,y}$ the ball with diameter $|x - y|$ that contains x and y, and we let

$$\delta(x,y) = \left(\int_{B_{x,y}} w(z)\, dz\right)^{1/n}.$$

Given an arc $\gamma : [0,1] \to R^n$, the w-length of γ is defined by

$$l_\omega(\gamma) = \liminf \sum \delta\left(\gamma(t_{i+1}), \gamma(t_i)\right),$$

where $\{t_i\}$ is any partition of $[0,1]$ and the lim sup is taken as the norm of the partition tends to 0. The metric or geodesic distance associated with the weight w is defined by

$$d(x,y) = \inf\{l_\omega(\gamma) : \gamma \text{ is an arc joining } x \text{ and } y\}.$$

If w is an A_∞ weight then by using the reverse Hölder condition it can be shown that there exists a constant c such that

$$d(x,y) \leq c\,\delta(x,y),$$

for all $x, y \in R^n$.

We say that w is strongly A_∞ (or that w belongs to strong A_∞) if there exists a constant c such that

$$\delta(x,y) \leq c\, d(x,y),$$

for all $x, y \in R^n$. By [**D-S**], every doubling weight which is strongly A_∞ is also an A_∞ weight. Also, every weight in A_1 is strongly A_∞, and if w is the absolute value of the Jacobian of a quasiconformal mapping on R^n then w is strongly A_∞.

THEOREM C. Upper bound. *Let w_1, w_2 be weights satisfying conditions (i)-(iii) of Theorem A and suppose that $w_2^{-n/2}$ is strongly A_∞. Then for $x, y \in R^n$ and $t > s$,*

$$\Gamma(x,t;y,s) \le C \frac{1}{|\Omega_d(x,t-s)|^{1/2}\,|\Omega_d(y,t-s)|^{1/2}} \times$$

$$\exp C\left(\Phi\left(\Omega_d\left(x,t-s\right)\right) + \Phi\left(\Omega_d\left(y,t-s\right)\right)\right)\exp\left(-c\frac{d(x,y)^2}{t-s}\right),$$

where $d(x,y)$ is the geodesic distance associated with $w_2^{-n/2}$, $t > s$,

$$\Omega_d(z,r) = \{\xi : d(\xi,z)^2 \le r\},$$

and

$$\Phi(E) = \frac{1}{\left(\frac{1}{|E|}\int_E w_2^{-n/2}\,dx\right)^{2/n}\frac{1}{|E|}\int_E w_1\,dx} + \left(\frac{w_2(E)}{w_1(E)}\right)^{1/2}.$$

It is easy to see from Hölder's inequality that

$$\left(\frac{w_2(E)}{w_1(E)}\right)^{1/2} \le \Phi(E) \le 2\frac{w_2(E)}{w_1(E)}.$$

Note that $\rho \le c\delta$ if $w_1 \approx w_2 \approx w$ and δ is defined with $w^{-n/2}$. Also $\rho \approx \delta$ if $w_1 \approx w_2 \approx w \in A_{1+\frac{2}{n}}$. In the case that $w_1 \approx w_2 \approx w$, Theorem B gives a better lower bound than the one in [**G-N**], and the hypotheses needed for Theorem B are weaker than the ones needed to establish the lower bound in [**G-N**]. Under the hypothesis of Theorem C, the upper bound obtained in Theorem C is better than the one in [**G-N**]. For example, if $w(x) = |x|^\alpha$ then $w^{-n/2}$ is strongly A_∞ for $\alpha < 2$. Also the weight $w^{-n/2} = |x|^{-\alpha n/2}$ belongs to $D_{1-\frac{\alpha}{2}}$ for $\alpha < 0$, and belongs to $D_1 \equiv A_1$ if $0 \le \alpha < 2$. Therefore, when $\alpha < 0$, the index μ defined in [**G-N**] satisfies $\frac{1}{2\mu-1} = \frac{1}{1-\alpha} < 1$, and consequently the estimate given there for the upper bound is not as good as the one in Theorem C.

Remarks.

1. There are other conditions on the weights under which is possible to show the upper bound for Γ. These conditions are related to the test functions chosen in the proof of the upper bound. If we assume that

(1-8)
$$\left(\frac{1}{|B_{|x-y|}(y)|}\int_{B_{|x-y|}(y)} v(z)\,dz\right)^{1-n}\left(\frac{1}{|\partial B_{|x-y|}(y)|}\int_{\partial B_{|x-y|}(y)} v(z)\,d\sigma(z)\right)^{n} \le c\,v(x)$$

holds for $v = w_2^{-n/2}$, and every x, y, then the upper bound of Theorem C can be obtained with d replaced by ρ_1, where

$$\rho_1(x,y) = \left(\int_{B_{|x-y|}(y)} w_2(z)^{-n/2}\,dz\right)^{1/n}.$$

Here ∂B denotes the boundary of the ball B.

If v is strongly A_∞ then by the results of **[D-S]** the isoperimetric inequality

$$\int_B v(z)\,dz \le c\left(\int_{\partial B} v(z)^{\frac{n-1}{n}}\,d\sigma(z)\right)^{\frac{n}{n-1}}$$

holds. Therefore if v satisfies (1-8) and is strongly A_∞ then it follows by Hölder's inequality that

$$\left(\frac{1}{|B_{|x-y|}(y)|}\int_{B_{|x-y|}(y)} v(z)\,dz\right)^{1-n}\left(\frac{1}{|B_{|x-y|}(y)|}\int_{B_{|x-y|}(y)} v(z)\,dz\right)^{n} \le c\,v(x),$$

i.e.,

$$c\,v(x) \ge \frac{1}{|B_{|x-y|}(y)|}\int_{B_{|x-y|}(y)} v(z)\,dz \approx \frac{1}{|B_{|x-y|}(x)|}\int_{B_{|x-y|}(x)} v(z)\,dz,$$

for every y; that is, $v \in A_1$. Also note that if v satisfies (1-8) and $v \in A_1$, then v belongs to "surface" A_1. In fact, by (1-8),

$$\left(\frac{1}{|\partial B_{|x-y|}(y)|}\int_{\partial B_{|x-y|}(y)} v(z)\,d\sigma(z)\right)^{n} \le c\left(\frac{1}{|B_{|x-y|}(y)|}\int_{B_{|x-y|}(y)} v(z)\,dz\right)^{n-1} v(x)$$

$$\le c\,v(x)^{n-1}v(x) = c\,v(x)^n \qquad \text{if } v \in A_1,$$

i.e.,

$$\left(\frac{1}{|\partial B_{|x-y|}(y)|}\int_{\partial B_{|x-y|}(y)} v(z)\,d\sigma(z)\right) \le c\,v(x).$$

If we assume that w_2 satisfies

$$\text{(1-9)}\qquad \left((n+2)\int_{B_{|x-y|}(y)} w_2(z)\,dz - |x-y|\int_{\partial B_{|x-y|}(y)} w_2(z)\,d\sigma(z)\right)^2 |x-y|^n w_2(x)$$

$$\le c\left(\int_{B_{|x-y|}(y)} w_2(z)\,dz\right)^3,$$

then it can be shown that an upper bound for Γ is

$$\frac{C}{|\Omega_{\rho_2}(x,t-s)|^{1/2}\,|\Omega_{\rho_2}(y,t-s)|^{1/2}}\times$$

$$\exp\left(C\left(\Phi_2(\Omega_{\rho_2}(x,t-s)) + \Phi_2(\Omega_{\rho_2}(y,t-s))\right)\right)\exp\left(-c\frac{\rho_2(x,y)^2}{t-s}\right),$$

where

$$\Phi_2(E) = \frac{w_2(E)}{w_1(E)},$$

and

$$\rho_2(x,y) = \left(\frac{|x-y|^{n+2}}{w_2(B_{|x-y|}(y))}\right)^{1/2}.$$

Condition (1-9) holds if for example w_2 is a doubling weight which satisfies

$$w_2(x) \le c\,\frac{1}{|B|}\int_B w_2(z)\,dz \qquad \text{if } x \in B, \text{ for every ball } B.$$

In particular, this holds if $w_2(x_1,...,x_n) = |x_1|^\alpha$ for $\alpha > 0$, or if $w_2(x) = |x|^\alpha$ for $\alpha > 0$. By Hölder's inequality we have $\rho_2 \le \rho_1$, and if $w_2^{-n/2} \in A_\infty$ then $d \le c\rho_1$. If $w_2 \in A_{1+\frac{2}{n}}$ then $\rho_1 \approx \rho_2$. In case $w_2^{-n/2}$ is strongly A_∞ we have $d \approx \rho_1$. By doubling it can be shown that ρ_1 and ρ_2 are quasi-metrics, i.e., there exist constants $K_i \ge 1$ such that $\rho_i(x,y) \le K_i(\rho_i(x,z) + \rho_i(z,y))$ for $i = 1,2$.

2. The problems (a), (b) and (c) mentioned at the beginning of this note are intimately connected. In fact in the non-degenerate case (i.e. $v \approx w_1 \approx w_2 \approx 1$) (b) $\Rightarrow$ (a) by [**Mo1**], [**Mo2**]; (c) $\Rightarrow$ (a) by [**Na**]; (b) $\Rightarrow$ (c) by [**Ar**] and (c) $\Rightarrow$ (b) by [**F-S**]. In establishing the last implication Fabes and Stroock used some ideas in [**Na**].

It would be an interesting problem to establish first the bounds for Γ and then deduce from them Harnack's inequality in the degenerate case.

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