Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

SIMPLIFIED FIXED AND MOBILE BED HYDRODYNAMIC MODELS AS SCALAR CONSERVATION LAWS

PABLO M. JACOVKIS

Abstract

In this lecture the shallow water hydrodynamic models with fixed and mobile bed are analysed. It is shown that, with appropiate simplifying assumptions, they both may be represented by scalar quasilinear hyperbolic equations, written as conservation laws. They may also be easily calibrated, provided that enough data are available. Some generalizations are introduced and further research is suggested.

1 Introduction

The one-dimensional gradually varied unsteady hydrodynamic flow of shallow water in rivers with fixed bed and arbitrary cross sections is governed by the Saint-Venant quasilinear hyperbolic equations

$$\frac{\partial V}{\partial t}+\frac{\partial V^2}{2\partial x^2}+g\frac{\partial Z}{\partial x}+g\frac{Q\mid Q\mid}{D^2}=0 \tag{1}$$

$$\frac{\partial S}{\partial t}+\frac{\partial Q}{\partial x}=0 \tag{2}$$

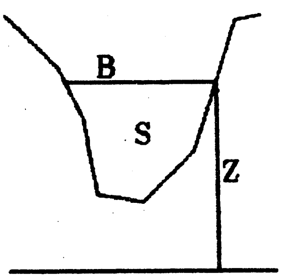

Figure 1: Cross section of a river

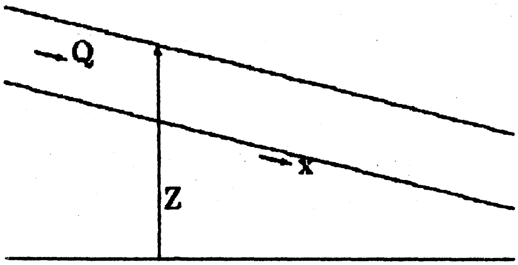

Figure 2: Longitudinal section of a river

where x is the space variable through the longitudinal axis of the channel or river, t is the time. $Q = Q(x,t)$ is the discharge. $S = S(Z(x,t),x)$ is the wetted cross sectional area, $V = V(x,t)$ is the mean velocity in the longitudinal direction. $Z = Z(x,t)$ is the surface elevation measured from a fixed reference level, g is the acceleration of gravity, and $D = D(Z(x,t),x)$ is the conveyance, related to the frictional resistance to the flow (see figures 1 and 2). Equations 1 and 2 represent conservation of momentum and mass, respectively. With suitable initial and boundary conditions they form an initial-boundary value problem for a quasilinear hyperbolic system of partial differential equations, that can be solved numerically. A careful derivation of equations 1 and 2 may be found in [17]; in [13] several numerical methods which have been successfully applied to these equations are introduced and explained.

There is a large number of efficient numerical methods implemented for solving the Saint-Venant equations in whatever available computer, so that, from a practical point of view, the main problem that arises when modelling a reach of a river is the calibration of the conveyances $D(Z(x,t),x)$, which usually can not be measured. For rivers with very irregular cross sections it is difficult to represent conveyances by means of simple functions, so that in general it is necessary to use tables. In this case a large number of parameters must be calibrated, which is a time-consuming and complex task. Besides, many field data are necessary that may not be available.

The process of calibration is in general an iterative process, carried on under the responsibility of an experienced engineer, who begins with some initial "feasible" conveyances and improves the results at each iteration. For getting the initial conveyances, some empirical relationships must be used, for instance

$$D = KSR^{2/3}$$

with $R = S/P$ the hydraulic radius, P the wetted perimeter, K the Strickler coefficient, or

$$C = KR^{1/6} = R^{1/6}/n$$

with $n = 1/K$ being the Manning coefficient, and C the Chézy coefficient, $n = 0.0254d^{1/6}$, and d - measured in meters - being the mean diameter of the bed particles. Some engineering books have tabulated likely values of these coefficients for different kinds of bed types and geometries (see for instance [2] or [7]).

The calibration problem may be treated as an inverse problem, but the number of parameters has been until now too large for this mathematically-oriented approach to be feasible.

If instead of a hydrodynamic fixed bed model we have a mobile bed one, that is, a situation in which the particles may roll and slide down changing the bed level through time, the situation is slightly different but not easier. On one hand, we can not apply equations 1 and 2 to arbitrary cross sections, unless we go from one to two spatial dimensions: in a mobile bed model, we decompose the level $Z(x,t)$ in two variables, the wetted height $h(x,t)$ and the (now varying) bed level $e(x,t)$; in order to have a well determined bed level $e(x,t)$ we must restrict ourselves to channels with rectangular cross sections, as in figure 3, where at each point x the width $B(x)$ is constant for all heigths $h(x,t)$. Then $S = Bh$ and $P = B + 2h$. In this way we have the traditional shallow water model of the form

$$\frac{\partial V}{\partial t} + \frac{\partial V^2}{2\partial x^2} + g\frac{\partial h}{\partial x} + g\frac{\partial e}{\partial x} + g\frac{V \mid V \mid S^2}{D^2} = 0 \tag{3}$$

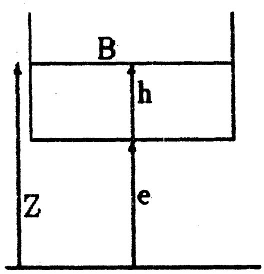

Figure 3: Prismatic rectangular cross section

$$B\frac{\partial h}{\partial t} + \frac{\partial hBV}{\partial x} = 0 \tag{4}$$

to which we must add the equation of conservation of solid mass

$$B\frac{\partial e}{\partial t} + \frac{\partial BG}{\partial x} = 0 \tag{5}$$

with G being the solid discharge per unit width.

Several empirical equations of state have been proposed for G as a (monotonically increasing) function of V and, sometimes, a (monotonically decreasing) function of h. Formulae for G may be found in [5]. The Meyer-Peter and Müller formula, for instance, may be expressed as

$$G = \chi(V^2/h^{1/3} - V_0)^{3/2} \tag{6}$$

where V_0 is a threshold, related to the shear tension on the bed, and χ is an empirical parameter.

Equations 3, 4 and 5 form a system of quasilinear hyperbolic differential equations. In [9] and [10] it may be seen that, to be treated as an initial-boundary value problem, it needs always two boundary conditions upstream and one boundary condition downstream. We note that the fixed bed model needs one boundary condition upstream and one boundary condition downstream for the subcritical regime, and two boundary conditions upstream for the supercritical regime, so that unless a very complex numerical model is used, a transition from supercritical to subcritical regime or from subcritical to supercritical regime requires a change of the numerical model, which is obviously rather inconvenient. On the other hand, a transition is possible in the mobile bed model, as may be seen in [10].

With the mobile bed model given by equations 3, 4, 5, and suitable initial and boundary conditions, this author has modelled in 1981 the mobile bed diversion channels for the Pichi Picún Leufú and Michihuao projected dams, on the Limay river, in southern Argentina.

In general, calibration of a mobile bed model is as hard a work as calibration of a fixed bed model. It is therefore important, for practical applications where not a great deal of accuracy is needed (or where, anyway, it is very difficult to obtain all the necessary field data), to have simplified models, both for the fixed bed and for the mobile bed cases, that guarantee, with a much easier calibration process (that could be treated as an inverse problem) a satisfactory solution. Simplified models will now be deduced for each case, which essentially result in the same mathematical problem: a one-dimensional scalar quasilinear hyperbolic equation, that may be written as a conservation law and that allows the analysis of several interesting phenomena that occur in rivers and channels.

2 The kinematic wave

The kinematic wave model is a simplification of the shallow water or Saint-Venant model obtained when a one-to-one function $Q(Z)$ is assumed. This is a strong assumption, that hydraulic engineers sometimes accept as a convenient approximation or for the sake of expediency. As $S = S(Z(x,t),x)$ is clearly a one-to-one function of Z for each x (for normal cross sectional geometries), Q may be written, following [14], as $Q(S(x,t),x)$. If $c = c(S,x) = \partial Q/\partial S$, we have from equation 2

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} \tag{7}$$

Taking into account that $V = Q/S$, we have

$$c = \frac{\partial(VS)}{\partial S} = V + S\frac{\partial V}{\partial S}$$

so that $c > V$ when fluid velocity increases with the "concentration" (quantity per unit distance) which is the case in fluid dynamics. There are some interesting kinematic wave models of traffic flow (see the pioneer work

Figure 4: A trapecial channel

of [15] or the book [1]), where the opposite assupmtion is made, that is, car velocity decreases with concentration, and $c < V$.

The kinematic wave model is particularly simple if we accept the Chézy law: resistance varies as the square of velocity. A straightforward derivation (see [14]) allows us to write V as $V = (giS/fP)^{1/2}$, where i is the bed slope and f a friction coefficient. The Chézy coefficient is $C = (g/f)^{1/2}$. For channels with simple geometry (for instance, trapecial and prismatic as in figure 4) and composed of a unique kind of bed material, this is a sound assumption, so that we may write equation 7 as a conservation law

$$\frac{\partial S}{\partial t} + \frac{\partial Q(S)}{\partial x} = 0 \tag{8}$$

where now $Q(S) = VS = MS^{3/2}/P^{1/2}$ and $M = (gi/f)^{1/2} = C\sqrt{i}$. We have $S = (B + h\cot(\theta))h$ and $P = B + 2h\csc(\theta)$, and after some computations we check that $\partial Q/\partial S > 0, \partial^2 Q/\partial S^2 > 0$ always, so that equation 8 is genuinely nonlinear in the sense of Lax [11], and it has then a unique entropy solution for initial data $S(x, t_0) = S_0(x) \in L^\infty$.

Equation 7 -or 8 - is treated, in practical applications, as an initial-boundary value problem, for $x_0 \leq x \leq x_F$. A boundary condition $Q(x_0) = g(t)$, or similar, is given at the upstream extreme point x_0. With this analysis, only the value of f must be calibrated.

3 Generalizations of the kinematic wave

It is very easy to model a river basin with an arborescent structure, as shown in figure 5.

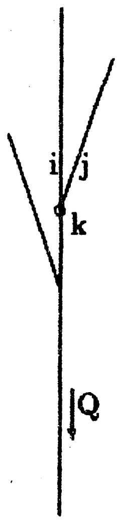

Figure 5: Arborescent structure of a river basin

Using conservation of mass, we have

$$Q_i + Q_j = Q_k$$

at junction points formed by the downstream extreme points of tributaries and the upstream extreme point of the reach to which the tributaries flow. Here Q_i and Q_j are discharges at the downstream extreme points of reaches i and j, and Q_k is the discharge at the upstream point of reach k. For each reach with a "non open" upstream point (that is, a point belonging to a junction of three reaches), we have then a boundary condition; only at open extreme points of the basin it is necessary to give boundary conditions as data. This generalization holds for all conservation laws that maintain a positive flux, not necessarily for models of rivers and channels only. But the approach is not applicable if we have a deltaic structure, instead of an arborescent one: in that case, the complete Saint-Venant system must be introduced. This problem, and an efficient algorithm for treating it, is analysed in [8].

Lateral inflows $q_i(t)$ may also be introduced at points x_i, $i = 1, \ldots, n$, as

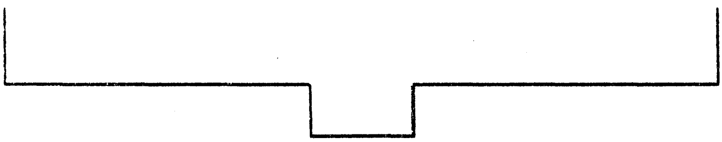

Figure 6: Cross section with flood plain

internal boundary conditions by means of Dirac deltas as, for instance

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = \sum_{i=1}^{n} q_i \delta(x_i).$$

Another interesting problem is to consider a poligonal cross section, as may be seen for instance in figure 1. In this case d^2Q/dS^2 has a finite number of discontinuities, where limits from the left and from the right exist. In a joint research project with E. Tabak, object of a forthcoming publication, we are analysing this case, and the more complex case of a "normal" bed and a flood plain, as seen in figure 6.

Now dQ/dS is discontinous. The theorem of existence and uniqueness of an entropy solution, that may be consulted in [16] may be generalized to the case of the flux function having piecewise continuous second derivative. The case of piecewise continuous first derivative is more involved.

4 The mobile bed scalar equation and some generalizations

The complete mobile bed model 3, 4, 5 may also be conveniently simplified, following an idea of [3] described in English in [12], if we assume discharge Q constant and surface elevation Z horizontal, that is, $\partial Z/\partial x = 0$. This assumption is justified (in this simplification) by taking into account that $\partial Z/\partial x$ is generally of a lesser order of magnitude than $\partial e/\partial x$.

Exner supposes that there exists a relationship

$$\frac{\partial e}{\partial t} = -\epsilon \frac{\partial v}{\partial x}$$

with $\epsilon > 0$. For a prismatic rectangular channel, we have $BhV = Q$, so that

$$\frac{\partial e}{\partial t} + \frac{\epsilon Q}{B(Z-e)^2}\frac{\partial e}{\partial x} = 0$$

and we have a conservation equation if we write this as

$$\frac{\partial e}{\partial t} + \frac{\partial G(e)}{\partial x} = 0 \tag{9}$$

with $G(e) = \epsilon Q/B(Z-e)$.

Clearly $G'(e) > 0$, $G''(e) = \epsilon Q/(2B(Z-e)^3) > 0$, so that equation 9 is genuinely nonlinear. We may note that the velocity of propagation of the sediment wave increases from the base of a bank to its crest, as experiments confirm.

If the dependence of e on V is nonlinear, say

$$\frac{\partial e}{\partial t} = -\epsilon \frac{\partial V^m}{\partial x}$$

with $m \geq 1$ (this case is the case of equation 6, provided that $V_0 = 0$, $m = 3$, and h is not taken into account), we have

$$\frac{\partial e}{\partial t} == -\epsilon \frac{\partial V^m}{\partial x}$$

from which we obtain equation 9 with $G(e) = \epsilon BQ^m/S^m$. In particular

$$\frac{dG}{de} = \frac{m\epsilon Q^m}{B^m(Z-e)^{m+1}} > 0$$

$$\frac{d^2G}{de^2} = m(m+1)\frac{\epsilon Q^m}{B^m(Z-e)^{m+2}} > 0$$

and the conservation law is genuinely nonlinear.

Furthermore, we may generalize to the following form of solid discharge:

$$G = \epsilon(V - V_0)^m \quad \text{if} \quad V > V_0$$
$$= 0 \quad \text{if} \quad V \leq V_0$$

In this case we may have again a flux function that is continuously differentiable but has a second derivative with a discontinuity where both limits, from left and right, exist.

It is also possible to assure existence and uniqueness of an entropy solution for equation 9, and we have again only one parameter to calibrate, namely, ϵ. In fact, experimental work has been performed and published in [4].

As in the case of kinematic wave, a "basin" model is possible, that is, an arborescent structure as indicated in Figure 5, the conserved solid discharge being

$$B_i G_i + B_j G_j = B_k G_k$$

We may also introduce lateral inflow of solid discharge.

5 Further approaches

Several numerical finite-difference methods exist for solving the kinematic wave problem. But it has not been solved with modern shock-capturing Godunov-like methods, as described in [6]. Two lines of research are possible here:

1. The numerical solution of both equations by means of Godunov-like methods, and a further analysis of them with a flux function with some loss of continuity in the first or second derivative. This approach is taken, with respect to the kinematic wave, in the already mentioned joint research (in progress) with E. Tabak.

2. With respect to the mobile bed model, a shock means a "collapse" of a dune. This is a subject worth a detailed study.

Acknowledgement: The author wants to acknowledge Dr. Mario Gradowczyk, who called his attention to the work of F. Exner.

References

[1] W. D. Ashton, *The theory of road traffic flow*, Methuen, London, 1966.

[2] V.-T. Chow, *Open channel hydraulics*, McGraw-Hill, New York, 1959.

[3] F. Exner, Über die Wechselwirkung zwischen Wasser und Geschiebe in Flüssen, *Sitzungsberichte der Österreichischen Akademie der Wissenschaften, Abteilung II*, **134**, 165-203, 1925.

[4] M. H. Gradowczyk, O. J. Maggiolo and R. Raggi, A bed wave moving downstream from obstacles, *Proceedings of the twelfth Congress of the International Association for Hydraulics Research*, Fort Collins, CO, 314-321, 1967.

[5] W. H. Graff, *Hydraulics of sediment transport*, McGraw-Hill, New York, 1971.

[6] A. Harten, Recent developments in shock-capturing schemes, *ICASE Report 91-8*, NASA Langley Research Center, Hampton, VA, 1991.

[7] F. M. Henderson, *Open channel flow*, McMillan, New York, 1966.

[8] P. M. Jacovkis, Modelos numéricos hidrodinámicos en redes fluviales complejas, *Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería*, **6**, 543-572, 1990.

[9] P. M. Jacovkis, One-dimensional hydrodynamic flow in complex networks and some generalizations, *SIAM J. Appl. Math.*, **51**, 948-966, 1991.

[10] P. M. Jacovkis, Análisis de condiciones de contorno y cambio de régimen en modelos hidrodinámicos con fondo móvil, in: S. N. Idelsohn (ed.), *Mecánica computacional*, **Vol. 12**, Asociación Argentina de Mecánica Computacional, AMCA, 325-334, 1991.

[11] P. D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, *Society for Industrial and Applied Mathematics*, Philadelphia, 1973.

[12] S. Leliavsky, *An introduction to fluvial hydraulics*, Dover, New York, 1966.

[13] J. A. Liggett and J.-A. Cunge, Numerical methods of solution of the unsteady flow equation, in: K. Mahmood and V. Yevjevich (eds.), *Unsteady flow in open channels*, Water Research Publications, Fort Collins, CO, 89-182, 1975.

[14] M. J. Lighthill and G. B. Whitham, On kinematic waves I (Flood movements in long rivers), *Proc. Roy. Soc. A* **229**, 281-316, 1955.

[15] M. J. Lighthill and G. B. Whitham, On kinematic waves II (A theory of traffic flow on long crowded roads), *Proc. Roy. Soc. A*, **229**, 317-345, 1955.

[16] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer Verlag, Heidelberg, 1983.

[17] J. J. Stoker, *Water waves*, Interscience, New York, 1957.

Departamento de Computación e
Instituto de Cálculo
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires