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F-QUOTIENTS AND ENVELOPE OF F-HOLOMORPHY

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This note corresponds to results from the article [6]. Let E be a complex Banach space, let F be a closed subspace of E and let $\Pi: E \to E/F$ be the canonical quotient mapping. A Riemann domain over E is a pair (X,ϕ) such that X is a Hausdorff topological space and $\phi: X \to E/F$ is a local homeomorphism.

DEFINITION 1. Let (X,ϕ) be a Riemann domain over E. We say that (X_F,ϕ_F,ψ) is an F-quotient of X if (X_F,ϕ_F) is a Riemann domain over E/F and ψ is a continuous open mapping from X onto X_F such that $\phi_F \circ \psi = \Pi \circ \phi$.

EXAMPLE 2. Let U be an open subset of E. If i and i_Π are respectively, the canonical inclusions $i: U \hookrightarrow E$ and $i_\Pi: \Pi(U) \hookrightarrow E/F$, it is clear that $(\Pi(U),i_\Pi,\Pi)$ is an F-quotient of (U,i).

EXAMPLE 3. Let (X,ϕ) be a Riemann domain over E, let R be the equivalence relation defined on X by $\phi(x)-\phi(y) \in F$, for x, $y \in X$ and denote by X/R the quotient set of X by this equivalence with the quotient topology associated to the mapping ψ from X onto X/R defined by $\psi(x) := \overline{x}$ (where $\overline{x}$ denotes the equivalence class of x). We can define $\phi_F: X/R \to E/F$ by $\phi_F(\overline{x}) := \Pi(\phi(x))$ for every $\overline{x} \in X/R$ and it is easy to see that $(X/R,\phi_F)$ is a Riemann domain over E/F. It is clear that $(X/R,\phi_F,\psi)$ is an F-quotient of (X,ϕ).

For the next examples U will be a connected open subset of E. Let H(U) be the set of all holomorphic functions f: $U \to \mathbf{C}$. As usual, τ_o denotes the compact open topology. Given any subalgebra A of H(U) endowed with a locally convex topology τ, the spectrum of (A,τ) is the set of all nonzero continuous homomorphisms h: $A \to \mathbf{C}$ and is denoted by $S(A,\tau)$. For any $u \in U$, we define $\hat{u}$: $A \to \mathbf{C}$ by $\hat{u}(f) = f(u)$ for every $f \in A$. It is clear that $\hat{U} = \{\hat{u},\ u \in U\} \subset S(A)$. We will consider the spectra of $(H(U),\tau_o)$ and of $(H(\Pi(U)),\tau_o)$. Alexander in [1] endowed $S(H(U),\tau_o)$ with a topology such that $(S(H(U),\tau_o),p)$ is a Riemann domain over E and the same for $(S(H(\Pi(U)),\tau_o),p_\Pi)$.

EXAMPLE 4. There exists a mapping λ: $S(H(U),\tau_o) \to S(H(\Pi(U)),\tau_o)$ which is continuous, open and satisfies $p_\Pi \circ \lambda = \Pi \circ p$, thus $(\lambda(S(H(U),\tau_o)),p_\Pi,\lambda)$ is an F-quotient of $S(H(U),\tau_o)$. (cf. [6]).

Let $\varepsilon_N(U)$ denote the connected component of $S(H(U),\tau_o)$ which contains $\hat{U}$. Alexander studied $\varepsilon_N(U)$ and called it a normal envelope of holomorphy of U (for details we refer to [1], [3] and [7], chap.XIII). Analogously let $\varepsilon_N(\Pi(U))$ be the connected component of $S(H(\Pi(U)),\tau_o)$ which contains $\Pi(U)$. From the example 4, we get that $(\lambda(\varepsilon_N(U)),p_\Pi,\lambda)$ is an F-quotient of $\varepsilon_N(U)$.

To every connected Riemann domain (Y,ρ) over E/F there corresponds a connected Riemann domain $(Y^*.\rho^*)$ over E, called pull back of Y, where $Y^* = \{(y,a) \in Y \times E;\ \rho^*(y) = \Pi(a)\}$ endowed with the topology induced on Y^* by the product topology on $Y \times E$, and $\rho^*(y,a) = a$, for all $(y,a) \in Y^*$. (cf. [4] and [9]).

Let $(\varepsilon_N{}^*(U),\phi^*)$ be the pull-back of $(\varepsilon_N(\Pi(U)),p_\Pi)$.

EXAMPLE 5. There exists a mapping ψ: $\varepsilon_N{}^*(U) \to \varepsilon_N(\Pi(U))$ which is continuous, open, onto and satisfies $p_\Pi \circ \psi = \Pi \circ \phi^*$, thus $(\varepsilon_N(\Pi(U)),p_\Pi,\psi)$ is an F-quotient of $\varepsilon_N{}^*(U)$. (cf. [6]).

For other non trivial examples of F-quotients we refer to [6].

We denote by $H_{F\psi}(X)$ the space of all $g \circ \psi$ as g ranges over $H(X_F)$.

THEOREM 6. *Let* (X,ϕ) *be a Riemann domain over* E *and let* (X_F,ϕ_F,ψ) *be an* F-*quotient of* X. *The mapping* $g \mapsto g \circ \psi$ *is a topological isomorphism between* $(H(X_F),\tau_o)$ *and* $(H_{F\psi}(X),\tau_o)$. (cf. [6]).

We recall that a morphism $j: U \to X$ is an extension of U if for each $f \in H(U)$, there is a unique $\hat{f} \in H(X)$ such that $\hat{f} \circ j = f$. Finally a morphism $j: U \to X$ is said do be an envelope of holomorphy of U if: a) j is an extension of U; b) if $\gamma: U \to Y$ is an extension of U then there is a morphism $\beta: Y \to X$ such that $\beta \circ \gamma = j$, i.e., j is maximal.

In 1972, Hirchowitz published a paper (cf. [5]) where he showed, using germs of holomorphic functions, that every Riemann domain over a Banach space E has an envelope of holomorphy. Independently and at the same time, Schottenloher considered in his thesis a more general situation by defining regular classes and admissible coverings for Riemann domains over a Banach space E. He showed that the envelope of holomorphy of a connected open set U, usually denoted by $\varepsilon(U)$, could be identified with a connected component of the τ_δ spectrum. (cf. [8]).

Hirchowitz remarked in [5] that his construction was also good to obtain the envelope of U relative to special classes of holomorphic functions on U instead of the envelope of U relative to H(U) (the envelope of holomorphy of U). This more general approach, due to Hirchowitz, is presented in a very clear way by Mujica in [7] chapter XIII. He defines an A-envelope of holomorphy, where A is a subclass of the set of all holomorphic functions of a Riemann domain over a Banach space E and proved that it always exists. A natural problem arises when we want to know if each element of A shares with its extension to the A-envelope of holomorphy some special properties. Hirchowitz considered this problem in remark 1.8 of [5].

If F is a closed subspace of E, we denote by $H_F(U)$ the space of all $f \in H(U)$ such that $f = g \circ \Pi$ for some $g \in H(\Pi(U))$. It seems that no relation can be established between the $H_F(U)$-envelope of holomorphy of U and the envelope of holomorphy of $\Pi(U)$ constructed by Hirchowitz. So we have the following definition.

DEFINITION 7. A morphism $j: U \to X$ is said to be an F-extension of U if there exist an F-quotient (X_F, ϕ_F, ψ) of X and a morphism $j_\Pi: \Pi(U) \to X_F$ such that:

a) j_Π is an extension of $\Pi(U)$,

b) $\psi \circ j = j_\Pi \circ \Pi$.

REMARK. In this case, given $g \in H(\Pi(U))$ there exists an extension $\tilde{f} \in H_{F\psi}(X)$ of $f = g \circ \Pi$ which is defined by $\tilde{f} = \tilde{g} \circ \psi$ where $\tilde{g} \in H(X_F)$ is an extension of g.

PROPOSITION 8. *The mapping* $j: U \to \varepsilon_N*(U)$ *defined by* $j(u) = (\widehat{\Pi(u)}, u)$ *for all* $u \in U$ *is an* F-*extension of* U.

We recall that $j: U \to (\varepsilon(U), q)$ defined by $j(u) = \hat{u}$ is the envelope of holomorphy of U. By using the equality $H(\Pi(U))$ $U_W\ A_W$, (cf. [6]) we can show that $j_\Pi: \Pi(U) \to \varepsilon(\Pi(U))$ defined by $j_\Pi(\Pi(U)) := \widehat{\Pi(u)}$ is an extension of $\Pi(U)$. Since the induced topology τ_δ in $H(\Pi(U))$, defined in [8], p.238, is weaker than our induced topology τ_Π, we have $S(H(\Pi(U)), \tau_\delta) \subseteq S(H(\Pi(U)), \tau_\Pi)$ and so the envelope of holomorphy due to Schottenloher is a topological subspace of $\varepsilon(\Pi(U))$. Now since the Schottenloher's envelope of holomorphy of $\Pi(U)$ is a maximal extension it coincides with $(\varepsilon(\Pi(U)), q_\Pi)$.

PROPOSITION 9. *There is a mapping* $\phi: \varepsilon(U) \to \varepsilon(\Pi(U))$ *which is continuous, open and satisfies* $q_\Pi \circ \phi = \Pi \circ q$, *thus* $(\phi(\varepsilon(U)), q_\Pi, \phi)$ *is an* F-*quotient of* $\varepsilon(U)$. (cf. 6).

PROPOSITION 10. *The morphism* $j: U \to \varepsilon(U)$ *defined by* $j(u) = \hat{u}$ *is an* F-*extension of* U. (cf. [6]).

DEFINITION 11. Let (X,ϕ) be a Riemann domain over E. A morphism $j: U \to X$ is said to be an envelope of F-holomorphy of U if:

a) j is an F-extension of U.

b) if $k: U \to Z$ is an F-extension of U, then there is a morphism $\gamma: Z \to X$ such that $\gamma \circ k = j$.

Let $(\varepsilon^*(U), \phi^*)$ be the pull-back of $(\varepsilon(\Pi(U)), q_\Pi)$, where $\phi^*(h,a) = a$, for all $(h,a) \in \varepsilon^*(U)$.

THEOREM 12. *The mapping* $\alpha: U \to \varepsilon^*(U)$ *defined by* $\alpha(u) := (\widehat{\Pi(u)}, u)$ *for all* $u \in U$ *is an envelope of* F-*holomorphy of* U. (cf. [6]).

FINAL REMARKS. As a consequence of the maximality of $\varepsilon^*(U)$ proved in Theorem 12, we know that there are morphisms $\gamma: \varepsilon(U) \to \varepsilon^*(U)$ and $\gamma^*: \varepsilon_N{}^*(U) \to \varepsilon^*(U)$ such that $\gamma \circ j = \alpha$ and $\gamma^* \circ j = \alpha$ (cf.Prop.10 and 9 for the definition of j in each case).

If F is a closed subspace of a Banach space E such that E/F is separable and has the bounded approximation property, (b.a.p.) $S(H(\Pi(U)), \tau_o) = \varepsilon(\Pi(U)) = \varepsilon_N(\Pi(U))$ (cf. [7], cor.58.10).

Consequently we get $\varepsilon^*(U) = \varepsilon_N{}^*(U)$.

We recall that given any separable Banach space G there exists $F \subset l_1$ such that G is isomorphic to l_1/F. If in addition G has the b.a.p. it is clear that l_1/F has the b.a.p. (e.g. $G = l_p$, $1 < p < \infty$). Now, given $U \subset l_1$ and $F \subset l_1$ such that l_1/F has the b.a.p. we have that $\varepsilon^*(U)$ is the pull-back of

$S(H(\Pi(U)),\tau_o)$.

Finally we want to remark that the morphism $j: U \to \varepsilon_N*(U)$ defi_ ned in Proposition 8 is also open and injective. We didn't succeed in our attempt to give reasonable definition of "normal F-envelope of holomorphy of U".

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