Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

DEGREE THEORY AND BIFURCATION OF FREDHOLM MAPS

JACOBO PEJSACHOWICZ

Introduction.

The first studies of bifurcation go back to Euler and Bernoulli. However, a comprehensive theoretical understanding of bifurcation of zeroes of nonlinear Fredholm maps from a trivial branch of zeroes was achieved only recently. In the past 2 years a new approach to the so-called linearized bifurcation was developed by a number of people. It does not use the classical Lyapunov-Schmidt finite dimensional reduction at an isolated potential bifurcation point. Rather, it places emphasis on the computation of those global homotopy invariants of the family of linearizations at points of the trivial branch whose non-vanishing forces the appearance of new zeroes close to the trivial ones. This relates bifurcation to the topological complexity of the parameter space. The invariants in question are derived from the index bundle of the family of linearizations (the total Steifel - Whitney class of the index bundle is an example). They necessarily vanish for maps between finite dimensional spaces and hence they cannot be computed from the Lyapunov-Schmidt reduction. The general discussion of this type of invariants heavily relies on techniques from algebraic topology (cf. [Pe]). I will not do this here. Instead I would like to explain how a suitable degree theory for nonlinear Fredholm maps of index 0 provides a reasonable complete understanding of bifurcation of one parameter families of Fredholm maps and gives some hints about the type of invariants that arise in the general case. This degree theory was recently constructed by P. Fitzpatrick, P. Rabier and myself. In what follows I will sketch the construction of the degree for maps between Banach spaces and relate it to bifurcation from zero and infinity for families of Fredholm maps parametrized by the real line and the circle. Then I will extend the construction to maps between Banach manifolds and discuss the relation with the Elworthy-Tromba degree. Finally I will relate our degree with an interpretation of Casson's invariant as the Euler- Poincaré characteristic of a section of a Hilbert bundle obtained recently by C. Taubes and will consider some related open problems.

Degree and bifurcation of one parameter families of Fredholm maps.

Our interest in degree theory for Fredholm maps arose from an attempt to extend the well known Rabinowitz global bifurcation theorem to one parameter families of nonlinear elliptic operators subject to general boundary conditions of Shapiro-Lopatinskij type. The maps induced by these type of operators in function spaces are generally nonlinear Fredholm. Since the proof of the Rabinowitz theorem uses in an essential way the homotopy invariance of the Leray -Schauder degree for compact vector fields, this theorem cannot be extended to the Fredholm setting in any obvious way. In fact, no integer valued degree theory for Fredholm maps can be homotopy invariant.

To be more specific, let $f : R \times X \to X$ be a one- parameter family of differentiable compact vector fields (i.e., compact perturbations of the identity operator), such that $f(t,0) = 0$. Points of the form $(t,0)$ are the *trivial solutions* of the equation $f(t,x) = 0$. Let $L_t = Df_t(0)$ be the linearization of $f_t(x) \equiv f(t,x)$ at $x = 0$. Assume that for $a < b$ the operators L_a and L_b are isomorphisms. Since they belong to the group $GL_C(X)$ of all linear invertible compact vector fields they have a well defined Leray-Schauder degree. The Leray-Schauder degree of an operator T in $GL_C(X)$ is given by the formula $\deg_{L.S.}(T) = (-1)^m$, where m is the sum of the algebraic multiplicities of the negative eigenvalues of L. The Rabinowitz theorem (cf. [Ra]) states that if the Leray - Schauder degrees of L_a and L_b differ in sign, then the interval $[a,b]$ contains a global bifurcation point t^* for the equation $f(x,t) = 0$. Here global means that the connected component of $(t^*,0)$ in the closure of the set of nontrivial solutions of the above equation is either unbounded or else contains trivial solutions $(0, t)$ with $t \notin [a, b]$. For families of Fredholm maps the above theorem does not hold, as can be easily seen in the following example.

Let us recall that a Kuiper space is a Banach space X such that the space $GL(X)$ of all invertible operators in L(X) is contractible. Hilbert spaces and most of the familiar function spaces, such as the Sobolev and Hölder spaces, are Kuiper spaces. Let L_0 and L_1 be two linear invertible compact vector fields defined on a Kuiper space X having Leray-Schauder degrees of opposite sign. Since $GL(X)$ is connected, these two operators can be joined by a smooth path $L: [0,1] \to GL(X)$. If we now define $f(t, x) = L_t(x)$, then the only solutions of the equation $f(t, x) = 0$ are the trivial ones. Thus no bifurcation arises although there has been a change in sign of the Leray-Schauder degree of the linearizations L_t as t goes from 0 to 1. Incidentally, this also shows that there cannot be a homotopy invariant degree theory that extends the Leray- Schauder degree to any class of maps which includes all linear isomorphisms.

The classical Caccioppoli degree for Fredholm maps [Ca] take values in $\mathbb{Z}_2$, and hence is inadequate for the study of bifurcation. The integer valued degree of Elworthy and Tromba [E.T.] gives no clue as to what should be the right substitute for the homotopy invariance property of the Leray-Schauder degree. For this reason much work has been done in investigating restricted classes of nonlinear Fredholm maps for which a homotopy invariant degree can be constructed. In [F.P.R. I] and [F.P.R. II], we took a different direction by constructing a degree theory for orientable C^2-Fredholm maps in which the possible change in sign of the degree along an admissible homotopy H was perfectly described in terms of a homotopy invariant of the linearizations of H_t at a given point. In what follows I want to briefly discuss this invariant and motivate our notion of orientable map.

Let U be an open subset of a Banach space X and let $f: U \to Y$ be a proper C^2-Fredholm map. In order to define the degree of f by means of the regular value approximation (using generalizations of Sard's Theorem) it is necessary to assign to each regular point of the map f a multiplicity ± 1. This must be done in a coherent manner which will make the sum of the multiplicities of points in the preimage of a regular value independent of the choice of regular value.

If X and Y are of the same finite dimension, this is usually done by assigning multiplicities ± 1 to each of the two connected components of the space $GL(X, Y)$. Fixing bases in X and Y respectively, one can distinguish the two connected components by assigning to each linear isomorphism the sign of the determinant of its associated matrix in the given bases.

If x is a regular point of f, its multiplicity $\epsilon(x)$ is, by definition, sgndet $Df(x)$.

With such a definition, the sum of the multiplicities of points in the inverse image of a regular value of f extends to the well known Brouwer degree. In infinite dimensions, $GL(X, Y)$ does not, in general, split into two components and hence a different approach is needed.

Plainly, the simplest strategy for overcoming this obstacle is to consider only maps whose derivatives take values in a proper subset $\mathcal{F}$ of the set of all Fredholm operators of index 0. A prominent example of one such choice is when X=Y and $\mathcal{F}$ is the set of all linear compact vector fields. The set $GL_C(X)$ of invertible linear compact vector fields has two components which are distinguished by the function ϵ which is defined for $T \in GL_C(X)$ by $\epsilon(T) = \deg_{L.S.}(T)$.This induces an assignment of multiplicities to the regular points of a nonlinear compact vector field which extends to the Leray-Schauder degree.

To understand, in general, what should be required of a choice of $\mathcal{F}$ in order to obtain a coherent assignment of multiplicities at regular points for maps whose derivatives take values in $\mathcal{F}$, we revisit the finite dimensional case and examine a geometric property of the "sgndet" function which is independent of the choice of bases (i.e., of the orientation). The property which I have in mind is that the sign of the determinant of a path of matrices switches by -1 each time the path crosses transversally the one codimensional analytic subset of singular matrices. While the notion of oriented basis cannot be extended to infinite dimensions the above described fact has an appropriate correspondent for paths in the space $\Phi_0(X, Y)$ of all bounded Fredholm operators of index 0.

Indeed, the highest stratum $\mathcal{S}'$ of the set $\mathcal{S}$ of non- invertible linear Fredholm operators is the one codimensional submanifold of $\Phi_0(X, Y)$ consisting of the operators having a one -dimensional kernel. All the other strata are of higher codimension. It follows then, by using standard methods in transversality theory, that any continuous path in $\Phi_0(X, Y)$ with invertible end-points may, by means of a small perturbation, be made to be both smooth and transverse to $\mathcal{S}'$.

By definition, the *parity* of a path $L: I \to \Phi_0(X,Y)$ with invertible end-points is given by $\sigma(L,I) = (-1)^m$, where m is the number of intersection points with $\mathcal{S}'$ of any smooth approximation of L which is transverse to $\mathcal{S}'$. In other words, the parity of a path is the mod-2 intersection index of the path with the stratified set $\mathcal{S}$ (cf. [F. P.I] for a more direct definition). The parity depends only on the homotopy class of the path (relative to ∂I) and is multiplicative under union of intervals, pointwise composition and direct sum of operators. Clearly, the parity of a path of matrices with invertible end-points is nothing but the product of the sign of the determinant at the end-points.

Motivated by this observation, given a subset $\mathcal{F}$ of $\Phi_0(X,Y)$ we shall say that $\mathcal{F}$ is *orientable* provided that the parity of any path in $\mathcal{F}$ with invertible end-points depends only on its end-points. On the set of isomorphisms belonging to an orientable set, a function having the above described property of the sign of the determinant can be defined. Such a function will induce a coherent assignment of multiplicities to regular points of maps with $Df(x) \in \mathcal{F}$ and hence a degree. It can be shown that many of the degree theories for restricted classes of Fredholm maps that appear in the literature correspond to particular choices of orientable subsets $\mathcal{F}$.

In infinite dimensions, the whole set $\Phi_0(X,Y)$ of all linear Fredholm operators cannot be orientable. Hence, in order to construct a degree theory for C^2-Fredholm maps without imposing further restrictions on the values taken by the derivatives one has to consider a more refined notion of oriented map. The idea is the following: we shall think of Df as a family of linear Fredholm operators parametrized by the domain of the map and then we shall assign multiplicities not to the linear operator $Df(x)$ but rather to the parameter x directly.

The map f is defined to be *orientable* provided that given any two regular points of f, the parity of the family Df of derivatives of f along any path in the domain of f joining these two points is independent of the choice of the path. Such a notion of orientation is more sensitive to the topology of the domain of f. For instance, irrespective of the image of the family Df, any map having a simply connected domain is orientable (*). If the map f is orientable, we can assign multiplicities $\epsilon(x)$ to all regular points of f such that the following rule holds: if x and x' are regular points of the map f, then

$$\epsilon(x) \cdot \epsilon(x') = \text{the parity of Df along any path between } x \text{ and } x'.$$

A function ϵ as above will be called an *orientation* of the map f. Once the multiplicity $\epsilon(x_0) = \pm 1$ of some fixed regular point x_0 of f (the base point) is chosen, the orientation ϵ is completely determined by the above rule.

Let U be an open subset of a Banach space X. Let $f: U \to Y$ be an orientable C^2-Fredholm map that is proper on closed, bounded subsets of U and let ϵ be an orientation of f. Suppose that Ω is open and bounded, with $\bar{\Omega} \subset U$. If $y \notin f(\partial\Omega)$ and y is a regular value of $f: \Omega \to Y$, we define the *degree of f in Ω with respect to y and the orientation ϵ by*

$$deg_\epsilon(f,\Omega,y) = \sum_{x \in f^{-1}(y) \cap \Omega} \epsilon(x).$$

If y is a singular value of the map, the degree with respect to y is defined by approximating y with regular values on the grounds of the generalized Sard-Smale theorem for Fredholm maps of index 0 and 1 (this very last part is the only point at which the C^2 assumption is needed).

In this way one obtains a degree theory verifying all of the usual properties of the degree except the homotopy invariance. However, the most interesting aspect of this degree is that its behaviour under admissible homotopies can be perfectly described. This can be done better by introducing base-points.

(*) Actually, the vanishing of $H^1(U; Z_2)$ suffices in order to make orientable all maps with domain U.

Let us assume that f is orientable, and let p be any regular point of f. Let ϵ_p be the unique orientation of f such that $\epsilon_p(p) = 1$. Define $\deg_p(f, \Omega, y) = \deg_{\epsilon_p}(f, \Omega, y)$. With the above definition the homotopy property of the degree can be formulated (in a special case) as follows:

Suppose that $H: [0,1] \times X \to Y$ is a C^2 orientable homotopy of nonlinear Fredholm mappings, which is proper on closed bounded subsets. Assume also that p_0 is a regular point both of H_0 and H_1. If Ω is an open bounded subset of X and H does not vanish on $[0,1] \times \partial\Omega$, and if $L_t = D_xH(t, p_0)$, then

$$deg_{p_0}(H_0, \Omega, 0) = \sigma(L, [0,1]) \cdot deg_{p_0}(H_1, \Omega, 0).$$

It is easy to see from this formula how the bifurcation arises. Suppose that $f: R \times X \to Y$ is a C^2-Fredholm family with a trivial branch of zeroes and assume that L is invertible at 0 and 1. If the interval $[0,1]$ contains no bifurcation points of (1.1), then for small enough r the map f itself will be an orientable homotopy between the restrictions of f_0 and f_1 to $\Omega = B(0,r)$. Taking $0 \in X$ as the base point, by the construction of our degree, we will have that $\deg_0(f_0, \Omega, 0) = 1 = \deg_0(f_1, \Omega, 0)$. But if $\sigma(L, [0,1]) = -1$, this is inconsistent with the homotopy formula. Therefore we get:

Proposition 1 *If $\sigma(L, [0,1]) = -1$, then the interval $[0,1]$ contains a bifurcation point.*

In fact, it can be proved if $\sigma(L, [0,1]) = -1$, then the bifurcating branch is global.

Based on the description of parity as the mod-2 intersection index of a path with the set $\mathcal{S}$ of singular operators we obtain the following simple explanation of bifurcation of one parameter families: *bifurcation arises whenever the path of linearizations crosses non-trivially the set $\mathcal{S}$.*

The above homotopy formula has wider implications in the case in which the parameter space is topologically nontrivial. As an example, let us consider the simplest topologically nontrivial space; the circle S^1, and a family of C^2 -Fredholm maps $f : S^1 \times X \to Y$. Such a family can be also seen as a map $f : [0,1] \times X \to Y$ with $f_0 = f_1$. It follows from the homotopy formula that if $\sigma(L, [0,1]) = -1$ and one has a priori bounds for the zeroes of f, then the degree of f_λ must vanish for all values of $\lambda \in S^1$. From this it is easy to conclude:

Proposition 2 *If f_λ has nontrivial degree for some λ and if $\sigma(L, S^1) = -1$, then there must be a bifurcation point from infinity* (*).

Since the parity of a path of compact vector fields depends only on the values of the path at the end points, it follows that closed curves of compact vector fields always have parity 1. However, closed paths of Fredholm operators may have parity -1 , so that the above described phenomenon appears to be typical of the Fredholm maps. It can be shown that the parity of a closed path is 1 if and only if the path can be deformed out of the set $\mathcal{S}$ of singular Fredholm operators. For families of bounded Fredholm operators parametrized by general compact spaces the above property characterizes the index bundle. This explains why this bundle is relevant to bifurcation problems (cf. [Pe], [F.P.II]).

Further extensions and open problems.

Let M and N be smooth, paracompact Banach manifolds with N connected and let $g: M \to N$ be a C^2-Fredholm map which is proper. The family of derivatives of g is now a Fredholm vector bundle morphism $Dg: T_M \to g^*(T_N)$ between the tangent bundle T_M of M and the pullback by g of the tangent bundle T_N of N. The definition of parity of the morphism Dg along paths in M is clear. Indeed, given a path $\gamma: I \to M$, the pull-back of T_M under γ is a bundle over I which, by the contractibility of I, is trivial. Composing the pullback of the morphism Dg with the trivializations, one gets a path of linear Fredholm operators with fixed domain and range. The parity of this path will not depend on the choice of trivialization. A map $g; M \to N$ will be called orientable if the parity of Dg along paths joining regular points of g is independent of the choice of the path. The orientation is defined as before and the degree of the map g is defined as the sum of the orientations of all points in the inverse image of a regular value. This sum is independent of the choice of regular value and the resulting degree will have the same homotopy property as before.

(*) A point λ in Λ is called a bifurcation point from infinity for f provided that there is no neighbourhood of λ over which there are bounds for the solutions of the equation $f(\lambda, x) = 0$

It is not difficult to prove that for maps between completely orientable (*) Banach manifolds our degree coincides with the Elworthy Tromba degree (cf. [E.T.]). In this setting our homotopy formula fills a mayor gap in that theory. Moreover, in infinite dimensions it seems to be more natural to orient maps rather than manifolds. Orientation of infinite dimensional manifolds has the unpleasant feature that the same manifold can have both orientable and nonorientable Fredholm structure. In our construction, Fredholm structures are not used at all and the resulting degree is defined for all oriented maps between any two Banach manifolds. In finite dimensions our degree coincides with Olum's degree for orientable maps between not necessarily orientable manifolds.

In a recent paper [Ta], C. Taubes gave an interpretation of the Casson invariant as an Euler- Poincaré characteristic of a section of a Hilbert bundle. It appears that an appropriate extension of our degree theory may provide an abstract setting for his arguments and simplify considerable some of the proofs. In [Ta], by identifying representations of the first homotopy group of a three dimensional manifold M (having the homology of a 3-sphere) with flat bundles over the manifold, Taubes gives the following interpretation of the Casson invariant: this invariant is half of the "intersection number with the zero section" of a section ϕ of a Hilbert bundle over the space $\mathcal{B}$ of all gauge equivalence classes of connections on the principal bundle $M \times SU(2)$. This section is induced by the map which associates to each connection its curvature. It is a Fredholm section in the sense that its covariant derivative (with respect to a natural connection) $\nabla\phi$ is Fredholm. Moreover since ϕ, at least locally, is a differential of a functional, the covariant derivatives $\nabla\phi$ are self-adjoint operators. Taubes uses perturbations of ϕ having only regular intersection points with the zero section (by compactness, they are finite in number). Then, using the equivalence class of the trivial connection as a base point, he assigns multiplicities to these points by taking the reduction mod-2 of the spectral flow of $\nabla\phi$ along paths joining them to the base point. He then proves that the sum of the multiplicities does not depend on the perturbation and that indeed it is twice the Casson invariant. Here is a rough idea of how this can be interpreted using our degree. Hilbert bundles are trivial and hence after trivialization the section becomes a map f from $\mathcal{B}$ into a Hilbert space whose zeroes are precisely the intersection points of ϕ with the zero section. The Uhlenbeck compactness theorem will give the properness of the map (at least close to the value 0). There are minor difficulties in comparing $\nabla\phi$ with Df but at the zeroes of f they agree. Since the mod-2 reduction of the spectral flow of a path of self-adjoint Fredholm operators is precisely the parity one should conclude that the algebraic number of intersection points coincides with the degree of f, with respect to the chosen base point. However, things are not quite so simple because $\mathcal{B}$ is not a manifold. It contains a closed singular set given by the equivalence classes of reducible connections. Moreover, the base point is sitting inside of the singular set and because of this Taubes needs an ad hoc argument. Nevertheless, since this singular set is of infinite codimension in $\mathcal{B}$, it seems that our degree theory can be conveniently extended to cover this case.

Amann and Weiss established a simple set of axioms characterizing the Leray-Schauder degree. That such a uniqueness result should also hold for Fredholm maps is strongly suggested by the fact that for maps having Df transversal to $\mathcal{S}$, the distribution of multiplicities is quite rigidly described by the following picture: The set S_f of singular points of f, being the inverse image by Df of $\mathcal{S}$, is a stratified subset of the domain whose highest stratum is a submanifold of codimension one. The multiplicity is constant on each connected component of the complement R_f of S_f. Once the multiplicity is assigned to a fixed connected component, the multiplicity of any other connected component will be the same or the opposite depending on whether a generic curve joining the two components crosses S_f transversally an even or an odd number of times. Hence the uniqueness should follow from a suitable density of transversality theorem. However I don't know any appropriate result of this type. As I already mentioned, everything in our construction works in the case of C^1-Fredholm maps except for the argument involving the density of regular values for the homotopies, i.e., Fredholm maps of index 1. This is an insurmountable obstacle to the extension of the above construction.(•) A method that could possible work in the C^1 case is the original reduction used by Caccioppoli for the mod-2 degree [Ca] together with our definition of orientability. However, such an approach is far from being straightforward .

(*) This means that all Fredholm structures on the manifold are orientable.

(•) Whitney gave an example of a C^1 function $h: R^2 \to R$ such that the set of critical values of h contains an open interval.

References

[Ca] R. Caccioppoli, Sulle corrispondenze funzionali inverse dira mate, Opere scelte , Vol II, Edizioni Cremonese, Roma (1963), pp. 157-177.
[ET] K.D. Elworthy, A.J. Tromba, Degree theory in Banach manifolds, Proc. Symp. Pure Math. Vol 15 (1970), pp.45-94.
[FP.I] P.M. Fitzpatrick, J. Pejsachowicz, Parity and generalized multiplicity. Trans. Amer. Math. Soc., Vol 326 (1991), pp. 281-305.
[FP.II] P.M. Fitzpatrick, J. Pejsachowicz, Nonorientability of the index bundle and several-parameter bifurcation. J. of Functional Anal., Vol 98 (1991), pp. 42-58.
[FPR.I] P.M. Fitzpatrick, J. Pejsachowicz and P. Rabier, Degré topologique pour les opérateurs de Fredholm non linéaires, Compte Rendus Acad. Sc. Paris. t.311, Serie 1, (1990), pp.711-716.
[FPR.II] P.M. Fitzpatrick, J. Pejsachowicz and P. Rabier, Degree theory for proper C2-Fredholm maps I, to appear in J. Reine Angew. Math.
[Pe] J. Pejsachowicz, K-theoretic methods in bifurcation theory. Contemporary Math. Vol 12, (1988), pp. 193-205.
[Ra] P. Rabinowitz, Some global results for nonlinear eigenvalue proble ms, J. Funct. Anal. Vol 7 (1971), pp. 487-513.
[Ta] C.H. Taubes, Casson's invariant and gauge theory, J. Diff. Geom., Vol 31 (1990), pp. 547-599.

DIPARTIMENTO DI MATEMATICA
POLITECNICO DI TORINO
Corso Duca degli Abruzzi 24
10.129 Torino
Italia