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ON THE N-DIMENSIONAL INVERSION LAPLACE TRANSFORM OF RETARDED, LORENTZ - INVARIANT FUNCTIONS

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Abstract

The purpose of this Note is to obtain an n-dimensional inversion Laplace transform of retarded, Lorentz - invariant functions by means of the passage to the limit of the rth-order derivative of the one dimensional Laplace transform.

This formula (IV,2) can be understood as a generalization of the one dimensional formula due to Widder ([7]).

This topic is intimately related with the generalized differentiation, the symbolic treatment of the differential equations with constants coefficients and its application to important physical problems (cf. Leibnitz, Pincherle, Liouville, Riemann, Boole, Heaviside and others).

Our main theorem (Theorem 15, formula (IV,2)) can be related with a result due to E. Post ([6]) and we also obtain an equivalent Leray's formula (cf. (VI,I) and (VI,2)) which expresses the Laplace transform of retarded, Lorentz - invariant functions by means of the mth-order derivative of a K_0-transform.

Our method consists, essentially, in the following two steps. First: the obtainment of an analog of Bochner's formula for Laplace transform of the form (II,1), where ϕ is a function of the Lorentz distance, whose support is contained in the closure of the domain $t_0 > 0$, $t_0^2 - t_1^2 - \ldots - t_{n-1}^2 > 0$.

The formula (II,2) permits evaluate n-dimensional integrals by means of a one dimensional K-transform.

This last result was already employed to solve partial differential equations of the hyperbolic type (cf. [9]).

Second: The passage to the limit of the rth-order derivative of the one dimensional Laplace transform (via the K-transform).

The previous conclusions are related with the classical Functional Analysis and Probability (i.e. the theory of moments, the classical Weierstrass theorem of uniform approximation, on compact sets, of continuous functions by polynomials and the inversion of Laplace - Stieltjes integrals).

Finally, by appealing to the analytical continuation, we can extend our results to the distributional n-dimensional Laplace integrals.

I. Introduction

In a famous Note ([1], Vol. I, p. 105), Serge Bernstein has shown that the classical Weierstrass theorem (about the uniform approximation, on compact sets, of continuous functions by polynomials) is an immediate consequence of the Tchebicheff inequality.

The fundamental idea of Bernstein was ingeniously employed by W. Feller ([2], Vol. 2, p. 219) to prove a simple Lemma which has several applications.

One of the most important application of the Feller's Lemma consists in the obtainment of an inversion formula for a sequence of moments.

First, we enunciate

THE MAIN FELLER'S LEMMA (CF.[2])
Hypothesis

a) The random variables $X_n(\theta)$, $n = 1,2,\ldots$, have the identical mean value $E(X_n) = \theta$, $n = 1,2,\ldots$;
b) $\{\sigma_n(\theta)\}^2 \xrightarrow[n\to\infty]{} 0$;
c) $u(x)$ is a continuous and bounded function

$$|u(x)| \leq M < \infty, \quad -\infty < x < \infty .$$

Thesis. The following formula is valid

$$\lim_{n\to\infty} E\{u(X_n(\theta))\} = u(\theta) , \tag{I,1}$$

and the convergence is uniform in every subinterval where $u(x)$ is an uniform continuous function and $\{\sigma_n(\theta)\}^2$ tends, uniformly, to zero.

One of the most important applications of the Feller's Lemma consists in the obtainment of an inversion formula for a sequence of moments. This formula, due also to Feller, is intimately related to Bernstein polynomials.

A sequence of moments μ_r, $r = 0,1,\ldots$, is a sequence of number μ_r, $r = 0,\ldots$, which are defined by the formula

$$\mu_r = \int_0^1 x^r \, dF(x) , \tag{I,2}$$

where $F(x)$ is a distribution function whose support is contained in $[0,1]$.

More generally, $F(x)$ can be a bounded variation function defined in $[0,1]$.

The hypothesis that $F(x)$ is a distribution function simplifies the evaluations and the enunciates and does not restrain, essentially, the generality.
An application of the Feller's Lemma is the

THEOREM 1
Hypothesis. The function $u(x)$ is continuous and bounded in $[0,\infty)$.

Thesis. The following formula is valid

$$\lim_{n\to\infty} \frac{1}{\Gamma(n)} \int_0^\infty u(x) e^{-\frac{nx}{\theta}} \left(\frac{nx}{\theta}\right)^{n-1} \left(\frac{n}{\theta}\right) dx = u(\theta) . \tag{I,3}$$

The convergence is uniform for θ varying in a compact interval $[a,b]$, $0 \leq a \leq \vartheta \leq b < \infty$.
The next is a famous inversion formula of the Laplace transform.

Theorem 2
Hypothesis. Let

$$g(\lambda) = \int_0^{\infty} e^{-\lambda x} u(x) dx$$

be the Laplace transform of the continuous and bounded function $u(x)$ in $[0, \infty)$.

Thesis. The following inversion formula is valid

$$u(\theta) = \lim_{n \to \infty} \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{n}{\theta}\right)^n g^{(n-1)}\left(\frac{n}{\theta}\right), \tag{I,4}$$

for every θ, $0 < \theta < \infty$.

The formula (I,4) (called the Widder's formula) is due to Stieltjes (cf. [5], p. 382), and it appears in a letter to Hermite, from August 23rd, 1893). Later, E. L. Post (cf. [6], vol. 32, pp. 773-781) rediscoveres the formula (I,15).

In 1934 ([7], vol. 36, pp. 107-200), Widder also proves the formula (I,4), independently of the other previous demonstrations, and he shows that the hypothesis of boundedness and continuity of $u(x)$ are not essentials. Further, he obtains an inversion formula for Laplace - Stieltjes integrals in the general case.

We remark that the Widder's formula (I,4) was extended by Hille and Phillips to the one variable functions which take values on a Banach space ([8], p. 224).

Our purpose is to extend the previous inversion formulas to the n-dimensional integral Laplace transforms and, more specifically, to the Laplace transforms of retarded, Lorentz - invariant functions.

II. The Laplace transforms of retarded, Lorentz - invariant functions

In this paragraph, we shall recall an analog Bochner's formula for Laplace transforms (cf. [9]).

Let $t = (t_0, t_1, \ldots, t_{n-1})$ be a point of $\mathbf{R}^n$. We shall write $t_0^2 - t_1^2 - \ldots - t_{n-1}^2 = u$. By Γ_+ we designate the interior of the forward cone: $\Gamma_+\{t \in \mathbf{R}^n / t_0 > 0,\ u > 0\}$, and by $\overline{\Gamma}_+$ we designate its closure. Similarly, Γ_- designates the domain $\Gamma_- = \{t \in \mathbf{R}^n / t_0 < 0,\ u > 0\}$, and $\overline{\Gamma}_-$ designates its closure. We put $z = (z_0, z_1, \ldots, z_{n-1}) \in \mathbf{C}^n$, where $z_v = x_v + iy_v$, $v = 0, 1, \ldots, n-1$; $\langle t, z\rangle = t_0 z_0 + t_1 z_1 + \ldots + t_{n-1} z_{n-1}$; and $dt = dt_0 dt_1 \ldots dt_{n-1}$. The tube T_- is defined by $T_- = \{z \in \mathbf{C}^n / y \in V_-\}$, where $V_- = \{y \in \mathbf{R}^n / y_0 < 0,\ y_0^2 - y_1^2 - \ldots - y_{n-1}^2 > 0\}$. Similarly, we put $T_+ = \{z \in \mathbf{C}^n / y \in V_+\}$, where $V_+ = \{y \in \mathbf{R}^n / y_0 > 0,\ y_0^2 - y_1^2 \cdots y_{n-1}^2 > 0\}$.

The Laplace transform of $\phi(t)$ is

$$f(z) = L\{\phi\} = \int_{\mathbf{R}^n} e^{-i\langle t,z\rangle} \phi(t) dt\,. \tag{II,1}$$

Let $F(\lambda)$ be a function of the scalar variable λ, and let $\phi(t)$ be a function endowed with the following properties:

a) $\phi(t) = F(u)$,
b) supp $\phi(t) \subset \overline{\Gamma}_+$,
c) $e^{\langle t,y\rangle}\phi(t) \in L_1$ if $y \in V_-$.

We call $\mathcal{R}$ the family of functions $\phi(t)$ which satisfies conditions a), b) and c). Similarly, we call $\mathcal{A}$ the family of functions which satisfy conditions

a') $\phi(t) = F(u)$,

b') supp $\phi(t) \in \overline{\Gamma}_-$,

c') $e^{\langle t,y \rangle}\phi(t) \in L_1$, if $y \in V_+$.

The next theorem is an analog of Bochner's formula for Laplace transforms of the form (II,1), where ϕ is a function of the Lorentz distance, whose support is contained in the closure of the domain $t_0 > 0$, $t_0^2 - t_1^2 - \ldots - t_{n-1}^2 > 0$ (cf. [9], p. 53, Theorem 1, formula (I,2;1)).

THEOREM 3

Hypothesis

a) $\phi(t) \in \mathcal{R}$,

b) $z \in T_-$.

Thesis

$$f(z) = L\{\phi\} = \frac{(2\pi)^{(n-2)/2}}{(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{(n-2)/4}} \cdot$$
$$\int_0^\infty F(\lambda)\lambda^{(n-2)/4} K_{(n-2)/2}\{[\lambda(z_1^2 + \ldots + z_{n-1}^2 - z_0^2)^{1/2}]\}d\lambda. \tag{II,2}$$

Here $K_v(z)$ designates the modified Bessel function of the third kind ([10], Vol. II, p. 427).

We remark that the formulas we have obtained for the Laplace transforms of functions of the family $\mathcal{R}$ are also valid for functions of the class $\mathcal{A}$; the only difference is that, for functions of the class $\mathcal{A}$, the formulas are valid on the assumption Im $z_0 > 0$.

III. The K transformation

Let $f(t)$ be a function defined in $\mathbf{R}^+ = \{t \in \mathbf{R} \,/\, t > 0\}$. By the K-transform of order μ of the function $f(t)$ we mean the function $F(s)$ of the complex variable $s = \sigma + i\omega$, defined by

$$F(s) \triangleq \int_0^\infty f(t)\sqrt{st}K_\mu(st)dt\ , \tag{III,1}$$

where $K_\mu(s)$ designates the modified Bessel function of the third kind ([10], vol.II, p. 427), defined by the formula

$$K_v(z) = \frac{\pi}{2}\,\frac{I_{-v}(z) - I_v(z)}{\sin v\pi}\ , \qquad v \neq \text{integer}\ . \tag{III,2}$$

here

$$I_v(z) = e^{-i(v/2)\pi} J_v(ze^{i(\pi/2)}) = \sum_{p=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{v+2p}}{p\,!\,\Gamma(p+v+1)}\ . \tag{III,3}$$

Now, we shall enunciate the following inversion formula due to Zemanian (cf. [14], pp. 194-195).

Theorem 4
Hypothesis. Let $F(s)$ be the K-transform of $f(t)$ for $s \in \Omega_f$, where $\Omega_f = \{s / \operatorname{Re} s > \sigma_f,\ s \neq 0,\ -\pi < \arg s < \pi\}$ and $\sigma_f =$ the abscissa of definition.

Thesis. Then, in the sense of convergence in $\mathcal{D}'(I)$, I is a nonvoid open set in $\mathbf{R}^n$,

$$f(t) = \lim_{n\to\infty} \frac{1}{\pi} \int_{\sigma-ir}^{\sigma+ir} F(s)\sqrt{st}\, I_\mu(st)\,ds \,, \tag{III,4}$$

where σ is any fixed real positive number in Ω_f and $I_v(z)$ is given by (III,3).

Moreover, Zemanian textually affirms, "There is another inversion formula for our K-transformation. It is a generalization of a result due to Boas" (cf. [15]) and can be stated as follows:

Theorem 5
Hypothesis. Let $F(s)$ be the K-transform of $f(t)$ for $s \in \Omega_f$, and assume that f is concentrated on an interval of the form $T \le t < \infty$, $T > 0$.

Thesis. Then, in the sense of convergence on $\mathcal{D}'(I)$,

$$f(t) = \lim_{k\to\infty} \sqrt{\frac{2}{\pi}} \frac{1}{(2k)!} \left(\frac{2k}{t}\right)^{2k+1} [S^k_{\mu,s} F(s)|_{s=\frac{2k}{t}}] \,. \tag{III,5}$$

See [16] for a proof.

Here, $S^k_{\mu,s}$ designates the following differentiation operator:

$$S^k_{\mu,t}\phi(t) = a_{2k,0} t^{-2k}\phi + a_{2k,1} t^{-2k=1} D\phi + \ldots + a_{2k,2k} D^{2k}\phi \,, \tag{III,6}$$

where the $a_{2k,q}$ are constants depending on the value of μ, and $a_{2k,2k} = 1$.

IV. The n-dimensional inversion Laplace transform of retarded, Lorentz-invariant functions

In virtue of the formula (II,2), we can recuperate the function $f(t)$ by means of the formula (III,5).

That is, we can obtain the n-dimensional inversion Laplace formula of retarded, Lorentz-invariant functions, as the Zemanian manner:

$$f(t) = \lim_{k\to\infty} \sqrt{\frac{2}{\pi}} \frac{1}{(2k)!} \left(\frac{2k}{t}\right)^{2k+1} S^k_{(n-2)/2,\ s} F(s)|_{s=\frac{2k}{t}} \,. \tag{IV,1}$$

Therefore, we can state our main theorem.

Theorem 6
Hypothesis
a) $f(t) \in \mathcal{R}$,
b) $z \in T_-$.

Thesis. The following n-dimensional inversion Laplace formula of retarded, Lorentz - invariant functions is valid:

$$f(t)=\lim_{k\to\infty}\sqrt{\frac{2}{\pi}}\frac{1}{(2k)!}\left(\frac{2k}{t}\right)^{2k+1}S^k_{(n-2)/2,s}F\left(\frac{2k}{t}\right) , \tag{IV,2}$$

where

$$F(s)=\frac{2^{n/2}\pi^{(n-2)/2}}{s^{(n-1)/2}}\int_0^\infty f(t^2)t^{(n-1)/2}\sqrt{st}K_{(n-2)/2}(st)dt , \tag{IV,3}$$

$$s=(z_1^2+\ldots+z_{n-1}^2-z_0^2)^{1/2} , \tag{IV,4}$$

$K_\mu(z)$ is defined by (III,2) and

$$S^k_{\mu,t}\phi(t)=a_{2k,0}t^{-2k}\phi_t a_{2k,1}t^{-2k+1}D\phi+\ldots+a_{2k,2k}D^{2k}\phi , \tag{IV,5}$$

here $D=\frac{d}{dt}$, and the $a_{2k,q}$ are constants depending on the value of μ, and $a_{2k,2k}=1$.

The formula (IV,2) is the n-dimensional version of the due to Widder (1934).

We note that the Widder's formula (I,4) had been extended by Hille and Phillips to the one dimensional variable functions which take values in a Banach space ([8], p. 224) and so, our formula (IV,2) permits generalize the results of Functional Analysis and Semigroups to n-dimensional spaces.

V The equivalence between the Post's inversion Laplace transform and our formula (IV,2)

Emil L. Post, in his work intittled "Generalized Differentiation" (cf. [17], 1930), defines the following operator:

$$A[f(t)]=\lim_{\substack{\Delta x\to+0\\ r\Delta x\to t}}\frac{(-1)^r f^{(r)}\left(\frac{1}{\Delta x}\right)}{r!\,\Delta_x^{r+1}}=\lim_{\substack{\Delta x\to 0\\ r\Delta x\to t}}A[f](r,\Delta x) ; \tag{V,1}$$

and, he explains that it possesses the remarkable property of invert the Laplace transform.

In fact, he shows ([17], p. 772, Theorem XXI), the following result:

Theorem 7

Hypothesis. Let $\psi(t)$ be an continuous function for $t>0$ and let $\int_0^\infty\psi(t)e^{-\xi t}dt$, consider improper in both limits, and convergent for some values of ξ.

Thesis. Then, if

$$\int_0^\infty\psi(t)e^{-\xi t}dt=f(\xi) \tag{V,2}$$

in the semiplane of convergence, we have, for $t>0$,

$$\psi(t)=A[f(t)]=\lim_{\substack{\Delta x\to+0\\ r\Delta x\to t}}\frac{(-1)^r f^{(r)}\left(\frac{1}{\Delta x}\right)}{r!\,\Delta_x^{r+1}} . \tag{V,3}$$

The last result (V,3), can be related with our formulae. (cf. [21], pp. 17-18).

VI. Remarks

i) From formula (6),p. 729 of [17], we have

$$A[u^{(n)}(t)] = \lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} A[u^{(n)}](r, \Delta x) = \frac{t^{-n-1}}{\Gamma(-n)} \cdot \tag{VI,1}$$

The last formula (VI,1) was written by Post, in 1923, without comments.

Now, we can express this result by a distributional manner. In fact, we have the well-known formula

$$\left. \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right|_{\alpha=-n} = \delta^{(n)} , \tag{VI,2}$$

so,

$$A[u^{(n)}(t)] = \delta^{(n)} ; \tag{VI,3}$$

therefore, every one of the previous conclusions can be extended, by a natural way, to the generalized functions or distributions.

ii) We shall refer briefly to a last application of the formula (IV,2), namely, the evaluation of inverse Fourier transforms as limit of Laplace transforms.

Schwartz ([18], p. 264) has evaluated some Fourier transforms by evaluating their Laplace transform (first step), and then passing to the limit (in s') for $y \to 0$, where $y \in V_-$ (second step). The method was later employed by Lavoine ([19]) and Vladimirov ([20], pp. 299-302). It works generally for any $\phi(t) \in \mathcal{R}$ which is, besides, a continuous function of slow growth.

In fact, let $\mathcal{F}[f(t)]$ be the Fourier transform of $f(x)$:

$$f(y) = \mathcal{F}[f] = \int_{\mathbf{R}^n} e^{-i\langle x,y\rangle} f(x) dx . \tag{VI,4}$$

Therefore, we can obtain $f(x)$ by passing to the limit, for $y \to 0$, where $y \in V_-$ in our inversion Laplace transform (IV,2).

Remark.

The complete version of this work appears in [21].

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