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BREAKDOWN OF MODULATION EQUATIONS FOR DISPERSIVE SYSTEMS

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1 Introduction

Consider a linear, one dimensional, constant coefficients, dispersive system in one dependent variable, for example the linear Korteweg de Vries equation $u_t + u_{xxx} = 0$. Monochromatic (single frequency) elementary solutions of the K.d.V. equation are of the form $u(x,t) = ae^{i(kx-\omega t)}$, where the constants a, k and $\omega = -k^3$ are the amplitude, the wave number and the wave frequency respectively. Now consider a modulated version of the monochromatic solution, that is a solution of the K.d.V. equation which is locally monochromatic, but which has the property that over long distances or large periods of time the amplitude, the wave number and the wave frecuency can vary. Namely, try a formal expansion for the solution given by:

$u(x,t) = a_\epsilon(\chi,\tau)e^{i\theta}$, with $\theta = \frac{1}{\epsilon}\Theta(\chi,\tau)$, $\chi = \epsilon x$, $\tau = \epsilon t$, and ϵ the ratio of a typical period to the time scale of the modulations, where we can define k and ω as slowly varying parameters by: $k = \theta_x = \Theta_\chi$ and $\omega = -\theta_t = -\Theta_\tau$.

Then substituting this form for the solution in the K.d.V. equation one obtains at differents order of ϵ the following modulation equations:

$k_\tau + \omega_\chi = 0$, (conservation of waves), $\omega = \Omega(k)$, (the dispersion relation), and $(a^2)_\tau + ((c_g)a^2)_\chi = 0$, (the transport equation), where $c_g = \frac{d\Omega}{dk}$ (the group velocity) is the velocity with which the wave number, k, and the amplitude, a, propagate. In the non-linear case the solutions are no longer sinusoidal as

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in the prior example but the existence of periodic solutions in $\theta = kx - \omega t$ can be shown explicitly in the simpler cases. The main non linear effect is not the difference in the functional form, but rather the appearance of amplitude dependence in the dispersion relation. Superposition of solutions is not available to generate more general solutions, but modulation theory can still be applied. Now the equations for the amplitude and the wave number are coupled.

For a very simple example in a nonlinear case, consider the equation for the modulation of a solution without oscillations (zero phases) (i.e.: just mean level) of the equation $u_t + (f(u))_x + u_{xxx} = 0$

$$u = \beta(\chi, \tau) + \epsilon u_1(\chi, \tau) + \cdots, \tag{1.1}$$

Then the Modulation Equation is

$$\beta_\tau + f(\beta)_\chi = 0. \tag{1.2}$$

This equation can be solved exactly by characteristics (see **[W1]**) and it is well known that — for general initial data — multiple values develop after a finite time if $f'(\beta) \neq$ constant .

The question of what happens with the solution of the underlying dispersive equation after the Modulation Equations break down is generally open.

The goal of this work is to study the corresponding phenomena of oscillatory behavior of solutions of difference approximations, that are dispersive and to understand, for the general non-linear case what happens with the solution of a dispersive system when the modulated equations break down.

In Section 2 we give some details about the resolution of the breakdown for the linear and non-linear cases.

In Section 3 we consider dispersive difference systems.

In Section 4 we show some numerical experiments.

2 Breakdown

It is well know that solutions of non-linear hyperbolic equations, in one space variable, generally break down after a finite elapse of time (even for smooth initial data) and multiple values arise. It is also known that the solutions of these equations can be continued beyond the time of the breakdown as single valued weak solutions in the integral sense of a conservation law. These solutions in the integral sense contain discontinuities (the mathematical representation of shock waves); they are uniquely determined by their initial data provided that the discontinuities are constrained to satisfy an entropy condition .

Since in many of the interesting cases the modulation equations are a set of non-linear hyperbolic evolution equations in the wave parameters, even for smooth initial data the solutions eventually cease to make sense as smooth single value functions. However, the introduction of shocks as described above, is not appropriate in this case.

For some $\tau = \tau_c$ a singularity in the derivatives arises at some $\chi = \chi_c$ and beyond that time attempts at continuing the solution produce multiple values near $\chi = \chi_c$

The linear case can be easily understood.

A multiple valued solution of the modulation equations simply means that where there used to be a single phase, now there are several, in fact the multiple valued solution of the modulation equations is a perfectly acceptable solution (in the linear case) provided we reinterpret it adecuately.

Namely, assume we start at time $t = 0$ with a single phase wave

$$u = \tilde{a}_0 \sin \tilde{\theta}_0 + O(\epsilon), \qquad \tilde{a}_0 = \tilde{a}_0(\chi) \;\; and \;\; \tilde{\theta}_0 = \frac{1}{\epsilon}\tilde{\varphi}(\chi). \tag{2.1}$$

Then, if the Modulation Equations for a and θ (with initial values $\tilde{a}_0$ and $\tilde{\theta}_0$, respectively) develop the multiple values (a_j, θ_j) , $1 \leq j \leq n$, we have

$$u \sim \sum a_j \sin \theta_j. \tag{2.2}$$

The only difficulties arise along the lines (caustics) in space-time (χ, τ) where n, the number of phases, changes. There a local expansion accounting for the transition is needed (and possible). All of this can be justified rigorously using the Fourier Transform representation of the solutions.

Physically the situation described above can be understood quite easily. Although the initial configuration might have a single wave locally in each region of space, places "far apart" will generally have completely different wavelengths and amplitudes. As different wavelengths move at different speeds (due to the *dispersive* character of the equations), eventually waves from wide apart regions in the initial conditions may come together. But because the system is linear, these different waves do not interact, and to get the full solution we need only superpose them.

For nonlinear systems the arguments in the prior paragraph almost also apply, except for the last sentence. Now the waves interact as they approach each other, and it is generally not clear what this interaction may produce — except in the case of completely integrable systems.

From the considerations above it seems safe to conclude that the breakdown in the solutions of the Modulation Equations is related to the appearance of new oscillation frequencies in the solution of the p.d.e.

For completely integrable systems the situation is much better understood and it is somewhat similar to that described above for linear systems. Basically, in this case, through the multiple phase Modulation Equations, we know and can describe accurately and precisely how different phases interact. For example: in the case of the KdV equation for $f = 3u^2$, using the work in **[FFMcL]** one can describe what happens as (1.2) breaks:

(i) First a new phase appears and a region in space-time arises where the solution u must be described in terms of the one phase modulation equations. This is true for *all* reasonable functions f — not just for $f = 3u^2$, as we can always do one phase modulation using the results in **[W1]**, **[K]**, **[GP]**.

(ii) When the one phase modulation equations break down, then an additional phase appears. The solution u must now be described in terms of the two phase modulation equations.

The process continues. Each time a break in the n-phase modulation equations occurs, a new phase is added to solve the problem. We recall that the n-phase modulation equations for (1.2), derived in **[FFMcL]**, are a system of first order hyperbolic equations in $(2n + 1)$ parameters[1] which in fact can always be reduced to a *Riemann Invariant* form. These results are confirmed by the work of Lax, Levermore, and Venakides in small dispersion problems. **[V1]**,**[V2]**,**[V3]**,**[V4]**,**[Le1]**,**[LLe1]**

[1] i.e.: one mean level, n wavenumbers and n amplitudes.

Even in this case there are strong differences in behavior with the linear case. To begin with: in the linear case the sequence in the number of phases at any given point is generally n=1, 3, 5,... and not n=0, 1, 2,... as here. Furthermore, in the linear case no new oscillations are created: they are simply redistributed in space. If at any given time several phases are encountered someplace, it is because in the initial conditions those same frequencies were somewhere — even if not together — and have moved into the same region of space. On the other hand, nonlinear interactions can and do generate new phases. In fact, when we start with (1.2), *no* oscillations are present initially ($n = 0$). Nevertheless, after some time oscillations appear ($n > 0$).

3 Dispersive Difference Systems

A similar problem to the one described in the prior section appears when one looks at the behavior of semi-discrete dispersive systems. For example, consider the finite difference approximation to the Burgers equation

$$u_t + (u^2)_x = 0, \tag{3.1}$$

given by the semi-discrete dispersive scheme

$$\dot{u}_n + \frac{1}{2h}(u_{n+1}^2 - u_{n-1}^2) = 0. \tag{3.2}$$

This leads to the interpretation of the oscillations appearing in numerically dispersive algorithms (approximating non-linear hyperbolic p.d.e.'s) as modulated waves, where the wave frequencies and wave numbers are large and the modulations occur over $O(1)$ distances and times.

The oscillatory nature of the solutions of dispersive difference schemes was discovered , accidentaly, by von Neumann [Nv] in 1944 in the course of a calculation of compressible flows with shocks in one space dimension, employing centered difference schemes. The solutions contained, as expected, a shock but they also contained post-shock oscillations on the mesh scale. Von Neumann conjectured that these mesh-scale oscillations were to be interpreted as the heat energy produced by the irreversible action of the shock wave and that as $\Delta x, \Delta t \to 0$, the solutions of the difference equations would converge weakly to exact discontinuous solutions of the equations governing the flow of compressible fluids. If the equation $u_t + (\frac{1}{2}u^2)_x = 0$ is a guide to the equations of compressible flow, then there is reason to doubt the validity of von Neumann's conjeture. For more on this see **[GL]**, **[La2]** and **[La3]**. [2] To see how (3.2) approximates (3.1) in the limit $h \to 0$ interpret $u_n(t) = \tilde{u}(nh, t)$. Using Taylor expansions then one obtains

$$\tilde{u}_t + (\tilde{u}^2)_x + \frac{1}{6}(h)^2(\tilde{u}^2)_{xxx} \approx 0, \tag{3.3}$$

an eqtation very much like aa non-linear K. d .V. equation. From this the dispersive nature of the approximation becomes clear. Note however, that the dispersive terms are nonlinear in (3.3).

4 Numerical experiments

We describe here some numerical experiments for the difference scheme (3.2). We solve the equation using an Adams Bashforth method, that adjusts the step size and order so as to control the local error per unit step.

It is clear from all the previous sections that one of the basic points of our program relates to the analysis of functions (the solutions of the nonlinear dispersive systems) which are locally oscillatory (say quasiperiodic) in terms of some phases and amplitudes. The question is, given the functions (say, as output of a numerical algorithm for solving the nonlinear dispersive system), how can we recover these phases and amplitudes?

Basically we have a function $u = u(x, t; \epsilon)$ for which we postulate a representation of the form

$$u = \sum_{\nu} a_\nu(x,t)e^{\frac{i}{\epsilon}\varphi_\nu(x,t)}, \quad 0 < \epsilon << 1, \tag{4.1}$$

with[3] $\varphi_\nu = -\varphi_{-\nu}$ and $a_\nu = \bar{a}_{-\nu}$ for real u. The presumption is that $| a_\nu | \to 0$

[2] "Unfortunately" the scheme considered in this reference is very special, i.e. : it is integrable, so that results like those in **[FFMcL]**, **[LL1]**, **[V1]**, etc. presumably apply to it, but this is far from being the general case.

[3] For example, for a single phase wave: $\nu = n$, $\varphi_\nu = \nu\varphi$ and $\theta = \frac{1}{\epsilon}\varphi$; for a two phase wave: $\nu = (n, m)$ with $\varphi_\nu = n\varphi_1 + m\varphi_2$; etc.

sufficiently fast as $|\nu| \mapsto \infty$. Now, given u, how do we recover the amplitudes a_ν and phases φ_ν? One way to do it is the following:

After we compute $u(x, t_o)$ for some t_o we divide the x interval in small windows (we are interested in the local wave number), and use a discrete Fast Fourier transform (FFT) in that portion of u. From that we can see which k is most important in that region and then we can associate that k to the mean point of that interval. In this fashion we obtain information about the wave number k as a function of x and t.

First we applied this algorithm to the linear case, where the results are known.

To this end, we solved the wave equation $u_t + u_x = 0$ with the following scheme $\dot{u}_n + \frac{1}{2h}(u_{n+1} - u_{n-1}) = 0$, and initial data $u(x) = sin(300cos(2\pi x))$, so that $\epsilon = \frac{1}{300}$.

For this problem the equation for the wave number is $k_\tau + \omega_\chi = 0$, where $\omega = sin(k)$, and the shock first forms at time $t = 0.1$, at $x = 0.11$ and $x = 0.61$. In figure # 1, # 2 and # 3 we can see the evolution of the breakdown, coincedent with the one expected.

We are still working with the non-linear case. Lots of subtleties and interesting fenomena seem to take place.

Still we were able to study some phenomena of oscillatory behavior of the solutions of difference approximations like (3.2) and

$$\dot{u}_n + \frac{1}{3h}(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}) = 0, \qquad (4.2),$$

which has the advantage over (3.2) of preserving the total energy Σu_n^2 (so that unbounded solutions do not arise).

These equations are semi-discrete, ie. continuous in time, discrete in x. We observed that the equations (3.2) and (4.2) have solutions that oscillate with a wave length $O(h)$, ie. on the mesh scale.

For example for (3.2) with $u(x, 0) = sin(2\pi x)$ we propose the following form of the solution after the time of breakdown :

$$u_n(t) = \beta_n(t) + (-1)^n \gamma_n(t), \qquad (4.3)$$

where $\beta_n = \beta(nh, t)$ and $\gamma_n = \gamma(nh, t)$. If we use Taylor's series for γ and

β and replace them in (3.2) we obtain the following system for γ and β(at leading order) :

$$\beta_t + (\beta^2 + \gamma^2)_x = 0,$$

$$\gamma_t - (2\beta\gamma)_x = 0.$$

We took $\beta = 1$ for $x \le 0$, $\beta = -1$ for $x \ge 0$ and $\gamma = 0$ everywhere as an approximation for initial data that model the behavior inmmediatelly after a discontinuity starts forming in the solution of (3.1) —shock—, leading to "breakdown" in the solution of (3.2) (as an approximation to (3.1)) and appearance of oscillations. We solved the system using characteristics and Riemman invariants. A rarefaction wave arises and there are no breakdowns.

The solution is (for $x \le 0$ only, for $x \ge 0$ use the symmetry $x \to -x$, $\beta \to -\beta$, $\gamma \to -\gamma$)

$$\beta(x,t) = 1, \qquad x \le -2t$$

$$\beta(x,t) = \frac{1}{2}((1 + \sqrt{1 + (\frac{1}{2t})^3})^{\frac{2}{3}} + (1 - \sqrt{1 + (\frac{1}{2t})^{3}})^{\frac{2}{3}}), \qquad -2t \le x \le 0$$

$$\gamma(x,t) = 0, \qquad x \le -2t$$

$$\gamma(x,t) = \frac{1}{2}((1 - \sqrt{1 + (\frac{1}{2t})^3})^{\frac{2}{3}} - (1 - \sqrt{1 + (\frac{1}{2t})^{3}})^{\frac{2}{3}}), \qquad -2t \le x \le 0$$

See figure # 4 and compare the solution above with the actual numerical data. The description seems qualitatively correct. The differences can be attributed to the fact that away from the breakdown point we took constant data while in figure # 4 this is not true.

In the case described above the point where the shock in (3.1) would appear is produced is a mesh point. The case in which the shock does not lie on a mesh point presents a totally different behavior: there are no oscillations.

This can easily be understood, and in this case (3.2) allows for discontinuous solutions (a jump from $u_n \simeq \alpha$ to $u_n \simeq -\alpha$ causes no large time derivatives in (3.2).)

If we change the initial data to $u(x, 0) = sin(2\pi x) + c$, then the location of the shock is not fixed, the velocity of that point is $2c$. Now when the shock passes through a mesh point it will produce the maximun number of oscillations. See figure # 5.

For the scheme (4.2) we proposed solutions of period 3. We modulated those solutions and we observed two differents kinds of behavior depending on whether the shock laid on the grid or not. In the former case the solution presented a shock and in the latter a rarefaction. See figures # 6 and # 7.

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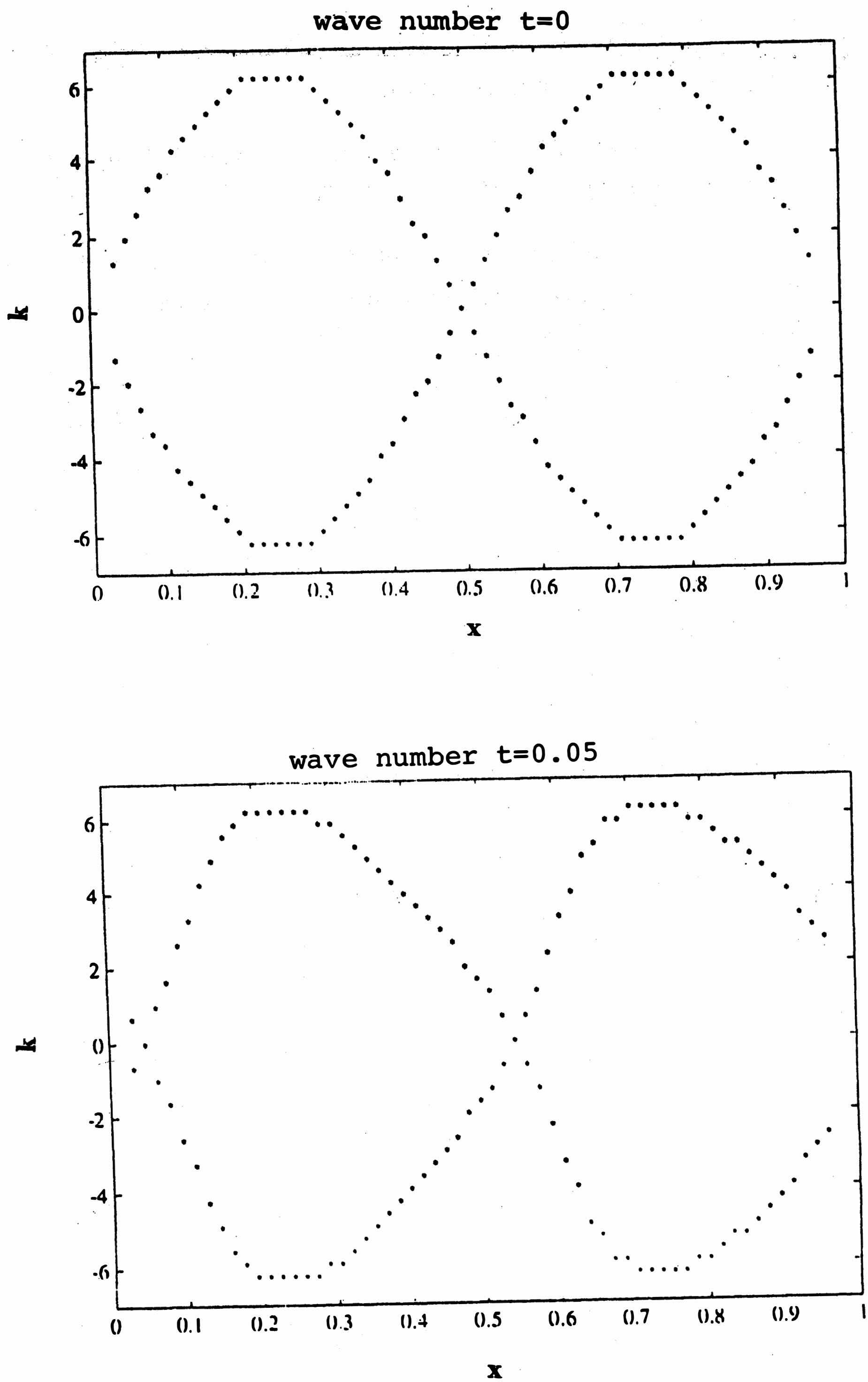
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0.8
0.9
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x
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-4
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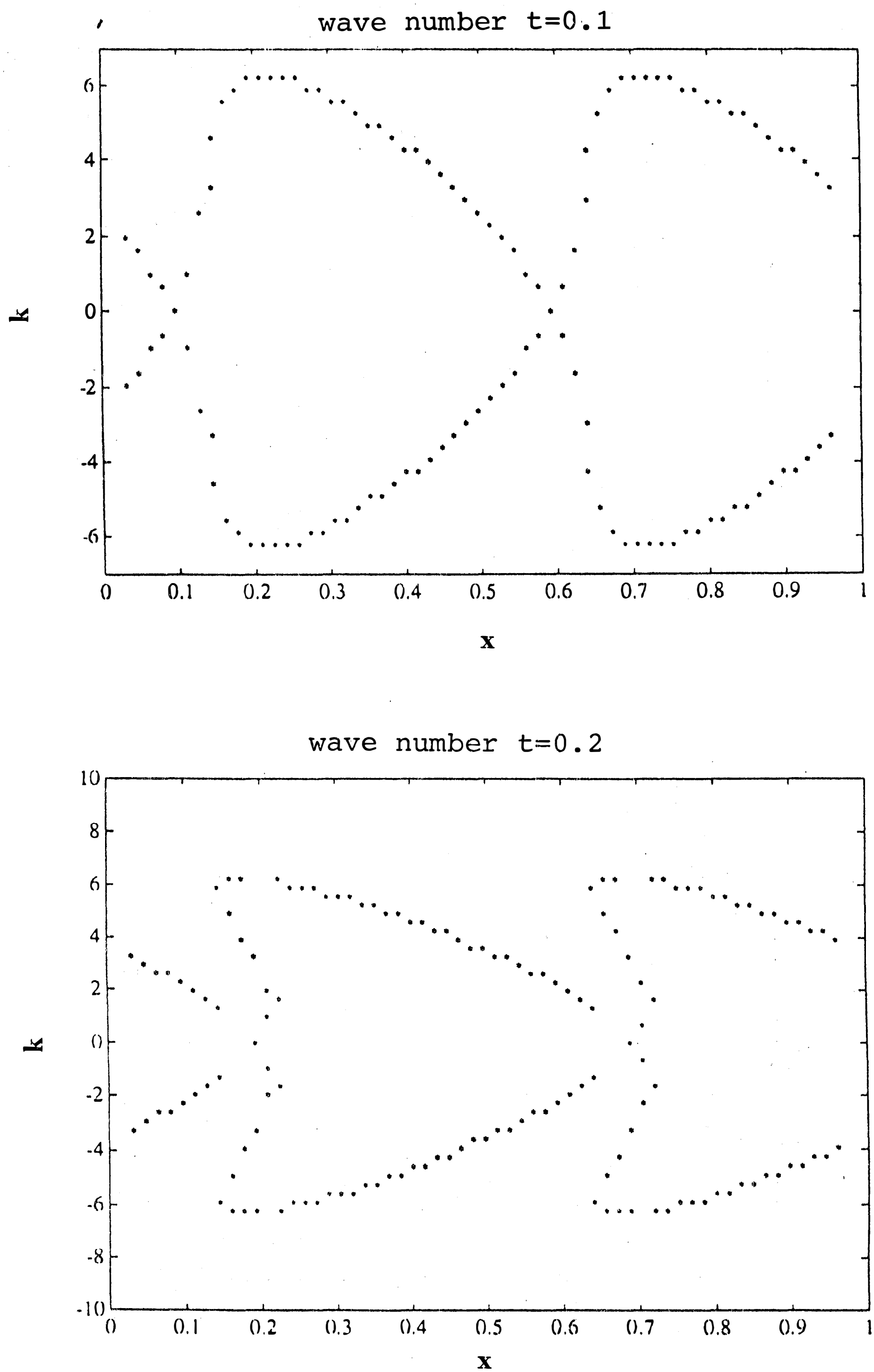
x

figure 1



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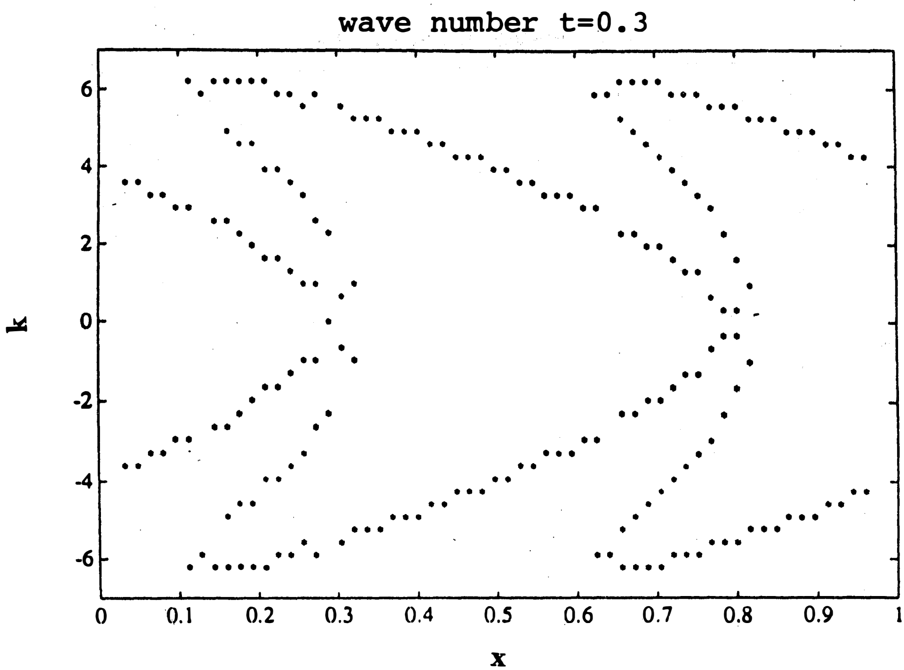
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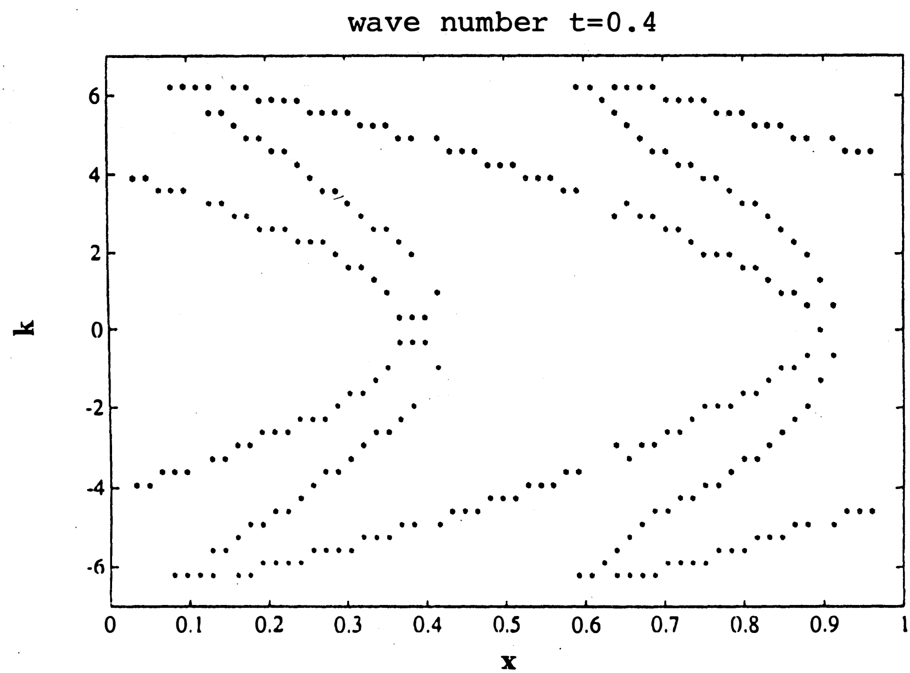
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figure 3

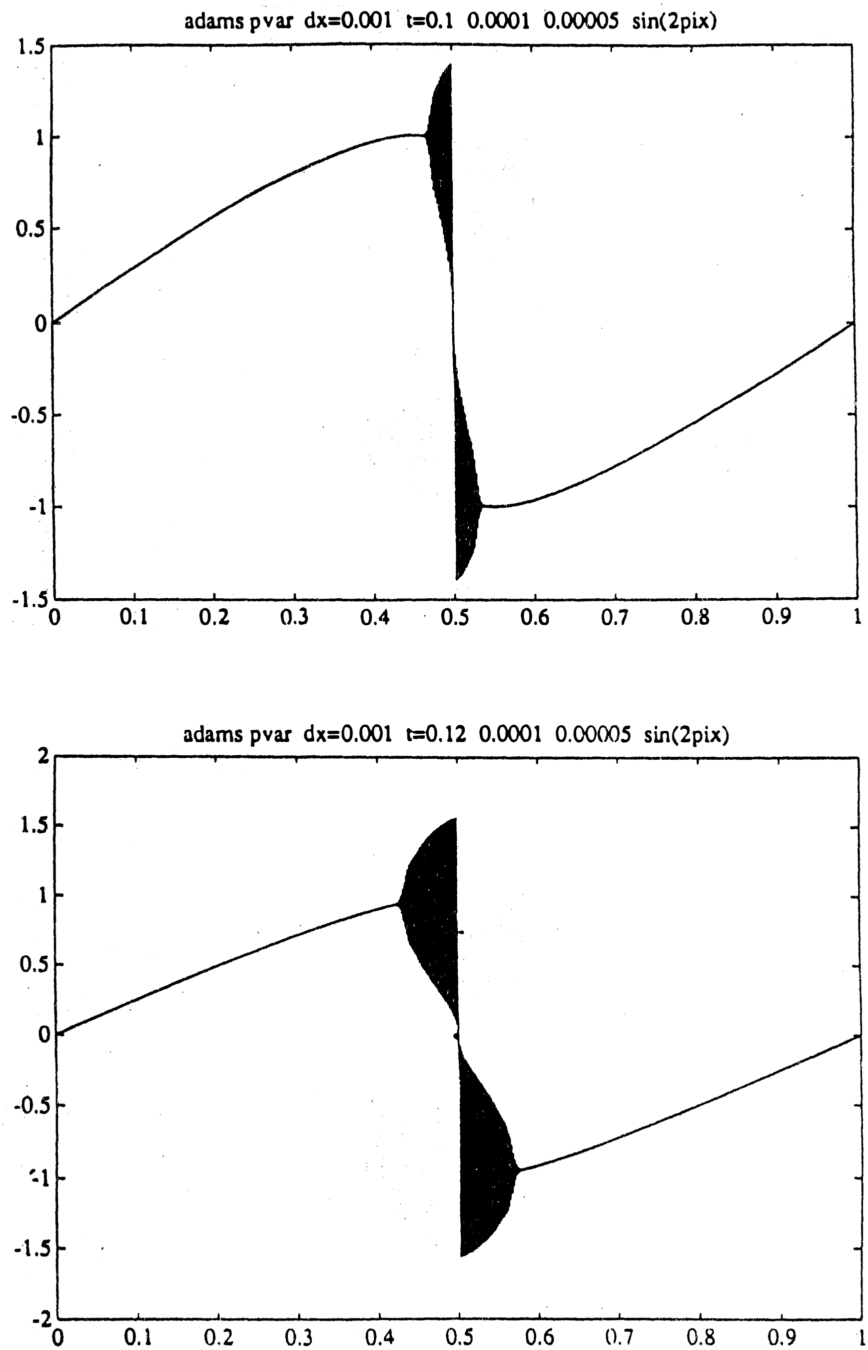
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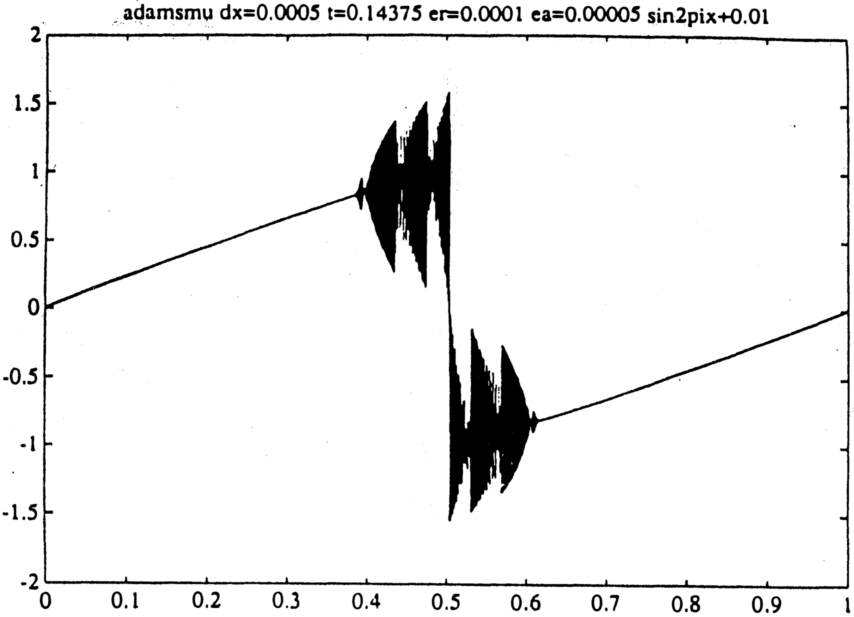
figure 4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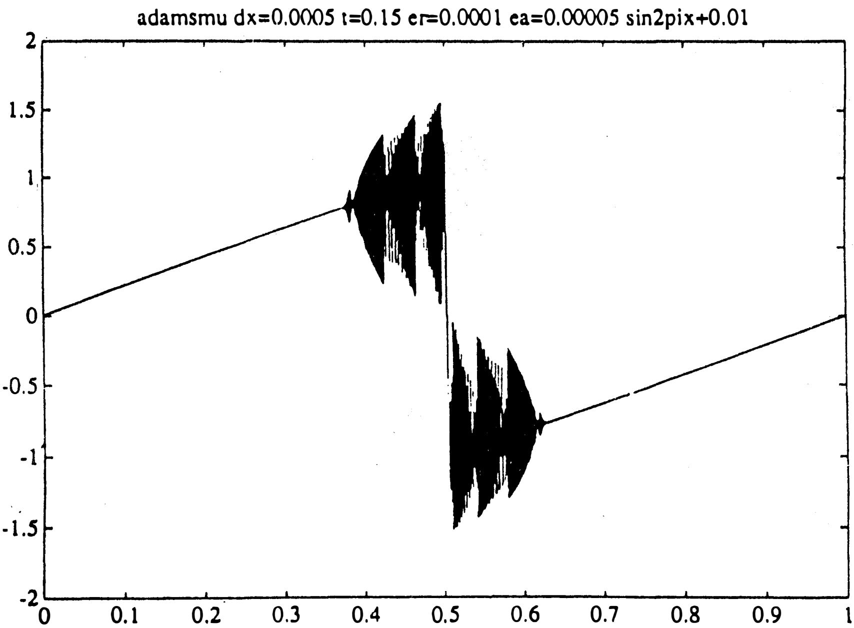
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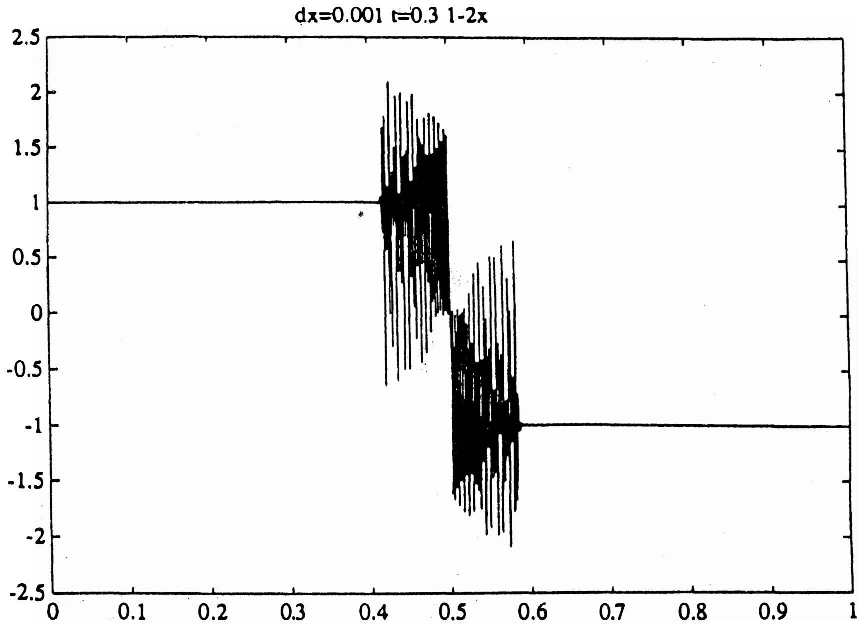
figure 5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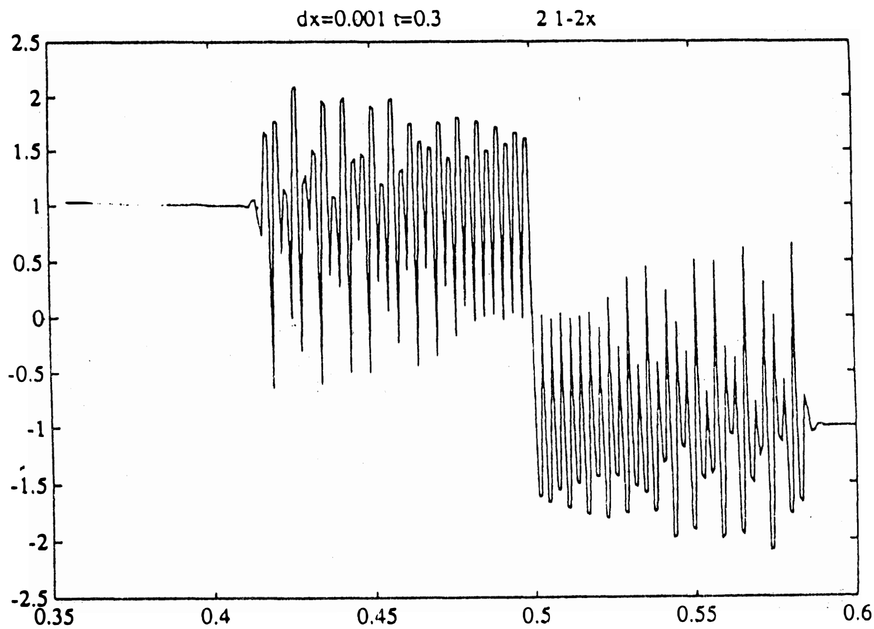
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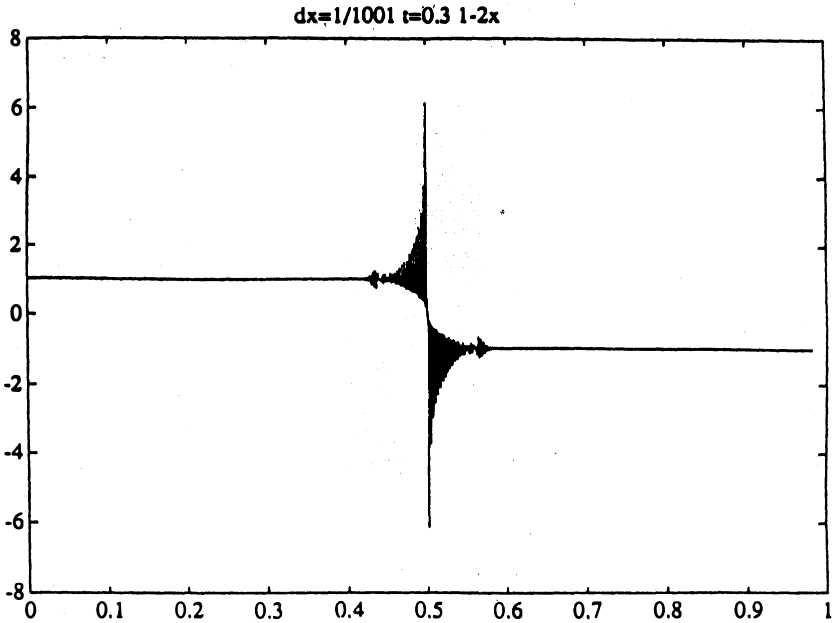
figure 6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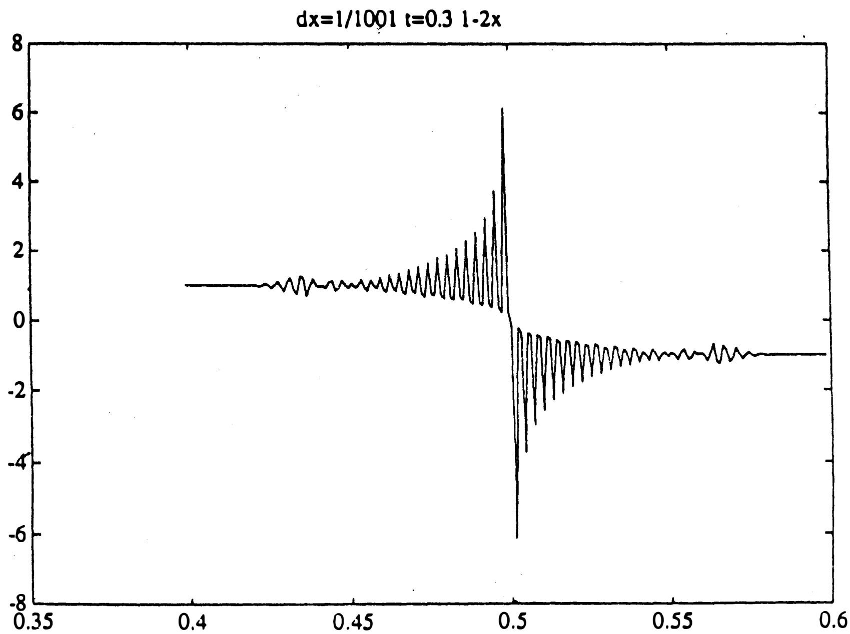
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figure 7