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A CHARACTERIZATION OF EXTRINSIC k-SYMMETRIC SUBMANIFOLDS OF R^N

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SECTION 1.

In [2] D.Ferus introduced the notion of extrinsic symmetric submanifold of R^N. This is a submanifold M of R^N that is locally symmetric in the usual sense and such that for each p the local symmetry $\tau_p: M \to M$ extends to an isometry of the ambient which is the identity on the normal space. Ferus proved that such a submanifold has parallel second fundamental form and obtained a complete classification by applying his previous results about submanifolds with this type of second fundamental form.

On the other hand in [9] W.Strübing completed the remarkable results of Ferus by giving a direct proof of the fact that every submanifold of R^N, with parallel second fundamental form is in fact extrinsic symmetric. This was known to be a fact by Ferus' classification. One has then

(1.1) THEOREM. *Let* (M,g) *be a Riemannian manifold and let* $i: M^n \to R^N$ *be an isometric immersion.*

Then M *is an extrinsic symmetric submanifold if and only if the second fundamental form of the immersion is parallel.*

In [8], the notion of extrinsic k-symmetric submanifold of R^N was introduced by extending Ferus' definition to the case of k-symmetric manifolds in the sense of [6],[7],[10]. This definition is global in nature and [8] contains a complete description of these submanifolds if they are compact and k is odd.

The methods in [8] are quite different from those in [2] because extrinsic k-symmetric submanifolds of R^N *do not* have parallel second fundamental form in the sense of [2] and [9].

However one can define a new type of *covariant derivative* for the second fundamental form in terms of the *canonical connection* ∇^c of the k-symmetric manifold M [6, p.23]. For symmetric spaces, i.e. k=2, ∇^c coincides with the Levi Civita connection but this is not the case in general k-symmetric spaces. The proof given by Ferus in [2, p.83] of the above mentioned property of the second fundamental form does not extend to the new definition but, with a different method, one can prove that extrinsic k-symmetric submanifolds in the sense of [8] have *canonically* parallel second fundamental form (see (2.9)). This is the motivation of the following theorem which is the main result of this paper (see Sec.2 for definitions).

(1.2) THEOREM. *Let* $(M,g,\{\theta_x : x \in M\})$ *be a compact connected Riemannian regular* s*-manifold of order* k *and let* S *denote its symmetry tensor. Let* $i: M^n \to R^{n+q}$ *be an isometric imbedding and denote by* α *its second fundamental form. Then* M *is an extrinsic* k*-symmetric submanifold of* R^{n+q} *if and only if*

i) $(\nabla^c_U \alpha) = 0$ *in* M *and*

ii) $\alpha_a(SX,SX) = \alpha_a(X,X)$ *for some point* $a \in M$ *and every* $X \in M_a$.

The paper is organized as follows: In Sec.2 we recall the definition of extrinsic k-symmetric submanifold from [8] and introduce ∇^c proving that, for these submanifolds, one has $\nabla^c\alpha = 0$ (2.9). The results of this section yield a proof of the fact that the conditions are necessary. In Sec.3 we study the nature of the ∇^c-geodesics as curves in R^{n+q} to prove that the conditions are sufficient.

SECTION 2

Let M^n be a compact Riemannian manifold and let $i: M^n \to R^{n+q}$

be an isometric imbedding with the following properties.

i) For each $p \in M$, there is an isometry $\sigma_p : R^{n+q} \to R^{n+q}$ such that $\sigma_p^k = id$, $\sigma_p(p) = p$, $\sigma_p(M_p^\perp)$ = identity on $M_p^\perp$.

ii) $\sigma_p(i(M)) = i(M)$.

iii) Let $\theta_p = (\sigma_p|M)$. The collection $\{\theta_p : p \in M\}$ defines on M a Riemannian regular s-structure of order k [6,p.4-6]. Notice that condition (iii) implies that p is an isolated fixed point of θ_p in M for each $p \in M$.

If these conditions are satisfied we say that M is an *extrinsic* k-symmetric submanifold of R^{n+q}.

We denote by g the Riemannian metric on M and by $<,>$ the inner product on R^{n+q}. ∇ and ∇^E shall denote the corresponding Levi-Civita connections on M and R^{n+q} respectively.

Associated to our isometric imbedding we have the second fundamental form α, the shape operator and the normal connection $\nabla^\perp$.

On our Riemmanian regular s-manifold we may consider the canonical connection ∇^c [6,p.24] and two important tensors namely $D(X,Y) = \nabla_X Y - \nabla_X^c Y$ and S which is defined by $S_X = \theta_{X*|X}$ $\forall$ $X \in M$. These two tensors and the metric are parallel with respect to ∇^c i.e.

$$(2.1) \qquad \nabla^c g = 0 \quad , \quad \nabla^c D = 0 \quad , \quad \nabla^c S = 0 \qquad [6, p.25]$$

For the second fundamental form of an isometric imbedding one defines its "covariant derivative" as

$$(2.2) \qquad (\overline{\nabla}_X \alpha)(Y,Z) = \nabla_X^\perp(\alpha(Y,Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

This derivative is used by Ferus and Strübing in the characterization of extrinsic 2-symmetric submanifolds of R^N. It is obtained from the connections ∇ in TM and $\nabla^{\perp}$ in $TM^{\perp}$.

Here we propose to use a different combination which, as we shall see, will be more convenient for our purposes. Namely we define

(2.3) $$(\nabla^c_X \alpha)(Y,Z) = \nabla^{\perp}_X(\alpha(Y,Z)) - \alpha(\nabla^c_X Y,Z) - \alpha(Y,\nabla^c_X Z)$$

an call it the *canonical covariant* derivative of α.

If our submanifold $M \subset R^{n+q}$ hapens to be a symmetric space one knows that $\nabla^c = \nabla$ and then both derivatives coincide.

As usual we have

$$\nabla^c_X(f\alpha) = (Xf) + f(\nabla^c_X\alpha)$$

$$(\nabla^c_{fX+gY}\alpha) = f(\nabla^c_X\alpha) + g(\nabla^c_Y\alpha)$$

In a coherent way, we can define the *canonical covariant derivative* for the shape operator

(2.4) $$(\nabla^c_X A)_{\xi} Y = \nabla^c_X (A_{\xi} Y) - A_{\nabla^{\perp}_X \xi} Y - A_{\xi} (\nabla^c_X Y)$$

and they are obviously related by

(2.5) LEMMA. $g((\nabla^c_X A)_{\xi} Y,Z) = \langle(\nabla^c_X \alpha)(Y,Z),\xi\rangle$ X,Y,Z *tangent fields on* M, ξ *a normal field.*

Let a be a point in M which we shall keep fixed. Let N_a be a normal neighborhood of a in M and such that $\sigma_a(N_a) = N_a$. Let $X \in M_a$ and $\xi \in M^{\perp}_a$; we shall denote by X* the "adapted" vector field on N_a constructed from X i.e. X* is constructed by ∇^c-parallel traslation along the ∇^c-geodesics through a. It is easy to see that it is a well defined C^{∞} vector field on N_a and that $\nabla^c_U X^*|_a = 0 \ \forall \ U \in M_a$.

Now it is easy to see that we can extend ξ to a normal field ξ^* defined on N_a with the following properties.

(2.6) ξ^* is $\nabla^{\perp}$-parallel along each ∇^c-geodesic through a

(2.7) ξ^* is σ_a-invariant on N_a i.e.

$$\sigma_{a*}|_X(\xi^*_X) = \xi^*_{\sigma_a(X)} \quad \forall \quad X \in N_a .$$

Associated to X^* we can consider other two vector fields on N_a namely SX^* and $\theta_{a*}X^*$. These fields are also parallel along each ∇^c-geodesic starting at a because $\nabla^c S = 0$ and θ_a is ∇^c-affine. Clearly we have $SX^* = \theta_{a*}X^*$ on N_a because they coincide at a and are parallel along each ∇^c-geodesic through a

(2.8) PROPOSITION. *At each point a of the extrinsic* k-*symmetric submanifold we have* $A_\xi(SX) = SA_\xi X \quad \forall \ X \in M_a \quad \xi \in M_a^{\perp}$.

Proof. Let $\gamma(t)$ be a ∇^c-geodesic starting at a and put $\beta(t) = \sigma_a(\gamma(t))$.
We have

$$A_{\xi^*(\beta(t))}[(\sigma_{a*}|_{\gamma(t)})X^*(\gamma(t))] =$$

$$= A_{(\sigma_{a*}\xi^*)(\gamma(t))}[(\sigma_{a*}|_{\gamma(t)})X^*(\gamma(t))] =$$

$$= \sigma_{a*}|_{\gamma(t)}[A_{\xi^*(\gamma(t))}X^*(\gamma(t))] .$$

Making t = 0 now we have $A_\xi SX = S_a(A_\xi X)$. □

(2.9) LEMMA. *For each* $U,X \in M_a$, $\xi \in M_a^{\perp}$ *we have at the point a* $[\nabla^c_{SU}A]_\xi SX = S[(\nabla^c_U A)_\xi X]$.

Proof. Let $\gamma(t)$ be a ∇^c-geodesic starting at a with $\dot\gamma(0) = U$. By definition $(\nabla^c_{SU}A)_\xi SX = \nabla^c_{SU}(A_{\xi^*}SX^*)$ since $\nabla^{\perp}_{SU}\xi^* = 0 = \nabla^c_{SU}X^*$. Now $\nabla^c_{SU}(A_{\xi^*}SX^*) = \nabla^c_{(\sigma_{a*}U)}[A_{\sigma_{a*}(\xi^*(\gamma))}\sigma_{a*}X^*(\gamma)] =$

$= \nabla^c_{(\sigma_{a*}U)}[\sigma_{a*}(A_{\xi^*(\gamma)}X^*(\gamma))] = \sigma_{a*}[\nabla^c_U(A_{\xi^*}X^*)] = S[(\nabla^c_U A)_\xi X]$, since $\nabla^{\perp}_U \xi^* = 0 = \nabla^c_U X^*$. □

With the aid of these lemmas we can prove

(2.10) THEOREM. *If* $i\colon M^n \to R^{n+q}$ *is a extrinsic* k-*symmetric submanifold then* A_ξ *is parallel with respect to the canonical connection. i.e.* $(\nabla^c_U A) = 0 \quad \forall\ U \in M_p \quad \forall\ p \in M$.

Proof. Let us take our point $a \in M$ and its normal neighborhood N_a as above. By (2.8) we have

$$A_{\xi^*}X^* = S^{-1} A_{\xi^*} SX^*$$

and then $\nabla^c_{SU}(A_{\xi^*} X^*) = \nabla^c_{SU}(S^{-1} A_{\xi^*} SX^*)$.

Now $\nabla^c_{SU}(A_{\xi^*} X^*) = (\nabla^c_{SU}A)_\xi X$,

and since $\nabla^c_{SU}S^{-1} = 0$ we have

$$\nabla^c_{SU}(S^{-1}A_{\xi^*}SX^*) = S^{-1}[\nabla^c_{SU}(A_{\xi^*}SX^*)] = S^{-1}[(\nabla^c_{SU}A)_\xi SX] = (\nabla^c_U A)_\xi X$$

by (2.9).

Then we have proved $(\nabla^c_{SU}A)_\xi X = (\nabla^c_U A)_\xi X$

and since ξ and X are arbitrary we get $\nabla^c_{(I-S)U} A = 0$

which, since I-S is non singular, implies $\nabla^c_U A = 0$. □

(2.11) COROLLARY. *If* $i\colon M^n \to R^{n+q}$ *is a extrinsic* k-*symmetric submanifold then its second fundamental form is canonically parallel i.e.* $(\nabla^c_U \alpha) = 0 \quad \forall\ U \in M_p\ , \quad \forall\ p \in M$. □

SECTION 3

In this section we prove that the conditions of theorem (1.2) are sufficient

(3.1) LEMMA. *Let* $(M^n, g, \{\theta_x : x \in M\})$ *be a Riemannian regular s-manifold and let* $i: M^n \to R^{n+q}$ *be an isometric immersion with the following properties*

i) $(\nabla^c_U \alpha) = 0$ *in* M.

ii) *For some point* $a \in M$, $\alpha_a(SX,SX) = \alpha_a(X,X) \quad \forall \; X \in M_a$.

Then at each point $p \in M$ *and for every* $X,Y \in M_p$, $\alpha_p(SX,SY) = \alpha_p(X,Y)$.

Proof. This is straightforward and left to the reader. □

Let $C: I \to R^{n+q}$ be a regular C^∞ curve. We say that C is a *Frenet curve in* R^{n+q} *of osculating rank* $r \geq 1$ if C is parametrized with respect to arc lengh, defined in an open non empty interval I and for each $t \in I$ the derivatives
$\dot{C}(t),\dots,C^{(r)}(t)$ are linearly independent and
$\dot{C}(t),\dots,C^{(r+1)}(t)$ are linearly dependent.

(3.2) PROPOSITION. *Let* $(M, g, \{\theta_x : x \in M\})$ *and* $i: M^n \to R^{n+q}$ *an isometric imbedding with the same hypothesis of* (3.1) *and let* γ *be a* ∇^c*-geodesic in* M.

Then, except by a linear change of parameter, $C(t) = i(\gamma(t))$ *is a Frenet curve in* R^{n+q} *of osculating rank* r *for some* $1 \leq r \leq n+q$ *and its Frenet curvatures are constant.*

Let $\gamma(0) = a$ *and consider, in the interval where it is defined, the Frenet curve* $C_1(t) = i(\theta_a(\gamma(t)))$. *Then* C_1 *has the same osculating rank as* C(t) *and the corresponding Frenet curvatures are equal.*

Proof. Let $\gamma(t)$ be a ∇^c-geodesic in N_a starting at $a \in M$. It is clear that $g(\dot\gamma,\dot\gamma)$ is constant and then, by a linear change of parameter, (which does not change the fact that γ is geodesic) we can assume that $g(\dot\gamma,\dot\gamma) = 1$. This means that C is parametrized by arc lengh. Since i is an imbedding we can identify M and i(M) and then $C(t) = \gamma(t)$.

Consider the first two derivatives of C,

$$\dot C(t) = \dot\gamma(t)$$

$$\ddot C(t) = \nabla^E_{\dot\gamma}\, \dot\gamma = D(\dot\gamma,\dot\gamma) + \alpha(\dot\gamma,\dot\gamma).$$

Then, we have $\ddot C = T_2 + N_2$ (tangent and normal components) and by (2.1) and (i)

$$\nabla^c_{\dot\gamma}\, T_2 = 0 = \nabla^\perp_{\dot\gamma}\, N_2.$$

Assume now that we have proved that, for each $j \leqslant i$,

$$C^{(j)} = T_j + N_j \qquad \text{(tangent and normal) with}$$

$$\nabla^c_{\dot\gamma}\, T_j = 0 = \nabla^\perp_{\dot\gamma}\, N_j.$$

We shall see that this is also the case for i+1.

$$(3.3) \qquad C^{(i+1)}(t) = \nabla^E_{\dot\gamma}\, T_i + \nabla^E_{\dot\gamma}\, N_i =$$

$$= \nabla^c_{\dot\gamma}\, T_i + D(\dot\gamma,T_i) + \alpha(\dot\gamma,T_i) - A_{N_i}\, \dot\gamma + \nabla^\perp_{\dot\gamma}\, N_i =$$

$$= [D(\dot\gamma,T_i) - A_{N_i}\, \dot\gamma] + \alpha(\dot\gamma,T_i) = T_{i+1} + N_{i+1}$$

now by (2.1), (i) and the inductive hypothesis we get $\nabla^c_{\dot\gamma}\, T_{i+1} = 0$. Similarly, by (i) and the inductive hypothesis, $\nabla^\perp_{\dot\gamma}\, N_{i+1} = 0$.

Then, for each $k \geqslant 1$,

$$(3.4) \qquad C^{(k)}(t) = T_k(t) + N_k(t) \quad , \quad \nabla^c_{\dot\gamma}\, T_k = 0 = \nabla^\perp_{\dot\gamma}\, N_k.$$

Let I be the open interval where γ is defined. For each $t \in I$

let $r(t)$ be the natural number $(1 \leqslant r(t) \leqslant n+q)$ such that $\dot{C}(t),\dots,C^{(r)}(t)$ are linearly independent and $\dot{C}(t),\dots,C^{(r+1)}(t)$ are linearly dependent. Let $t_o \in I$ be a point such that $r(t_o) \leqslant r(t) \quad \forall \ t \in I$.

There are some real numbers $a_1,\dots,a_{r(t_o)+1}$, not all zero, such that $\sum a_j \ C^{(j)}(t_o) = 0$ (sum from $j=1$ to $r(t_o)+1$).

With these real numbers we define a couple of real C^∞ functions on I.

$$h(t) = \|\sum a_j \ T_j(t)\|^2 \qquad \text{sums from } j=1 \text{ to } r(t_o)+1$$
$$f(t) = \|\sum a_j \ N_j(t)\|^2$$

They satisfy $h(t_o) = f(t_o) = 0$ and by (2.1) and (3.4)

$$h'(t) = 0 \qquad \forall \ t \in I$$

and therefore $h(t) = 0 \ \forall \ t \in I$.

Similarly by (3.4) $f'(t) = 0$ and again, $f(t) = 0 \ \forall \ t \in I$.

We have then, $r(t) = r(t_o) \ \forall \ t \in I$ and therefore, $C(t)$ is a Frenet curve on I. Let $r = r(t_o)$.

We have to prove now that the Frenet curvatures of $C(t)$ are constant on I.

In fact, we shall prove that for each j, $1 \leqslant j \leqslant r$, we can write

$$(3.5) \qquad \begin{array}{ll} V_j(t) = P_j(t) + Q_j(t) & \text{(tangent and normal)} \\ \nabla^c_{\dot{\gamma}} P_j = 0 = \nabla^\perp_{\dot{\gamma}} Q_j \ , & k_{j-1}(t) = \text{constant} \end{array}$$

Let us proceed by induction on j. For $j = 1$

$$V_1(t) = \dot{C}(t) = P_1(t) + Q_1(t) \ , \quad Q_1 = 0$$
$$\nabla^c_{\dot{\gamma}} P_1 = \nabla^c_{\dot{\gamma}} \dot{\gamma} = 0 \ , \quad k_o(t) = \|\dot{C}(t)\| = 1$$

Assume that (3.5) is true for each $j \leqslant i < r$. We have to show this for $i+1$. Now we have

$$(3.6)\qquad V_i'(t) = \nabla^E_{\dot\gamma} P_i + \nabla^E_{\dot\gamma} Q_i = [D(\dot\gamma, P_i) - A_{Q_i}\dot\gamma] + \alpha(\dot\gamma, P_i).$$

We shall show first that k_i = constant and then complete the other parts of (3.5). We know [9, p.39, (10)] that

$$(3.7)\qquad k_i(t) = \| V_i'(t) + k_{i-1}V_{i-1}(t)\| \qquad (> 0 \text{ for } 1 \leqslant i < r)$$

Then, replacing the values of V_{i-1} and the derivative, one gets

$$(3.8)\qquad [k_i(t)]^2 = \| D(\dot\gamma, P_i) - A_{Q_i}\dot\gamma + k_{i-1}P_{i-1}\|^2 + \\ + \|\alpha(\dot\gamma, P_i) + k_{i-1}Q_{i-1}\|^2 = u(t) + v(t)$$

and, by induction, it is easy to see that u and v are constant. Once that we know this we can compute V_{i+1} (recall $k_i > 0$ for $1 \leqslant i < r$).

$$(3.9)\qquad V_{i+1}(t) = \frac{1}{k_i}[V_i'(t) + k_{i-1}(t)V_{i-1}(t)] = \\ = \frac{1}{k_i}[(D(\dot\gamma, P_i) - A_{Q_i}\dot\gamma + k_{i-1}P_{i-1}) + (\alpha(\dot\gamma, P_i) + k_{i-1}Q_{i-1})] = \\ = P_{i+1}(t) + Q_{i+1}(t)$$

and, since k_i = constant, we have $\nabla^c_{\dot\gamma} P_{i+1} = 0$. Similarly one can easily get $\nabla^\perp_{\dot\gamma} Q_{i+1} = 0$. In this way we have proved (3.5). Let us prove now the second part of (3.2). Let $\gamma_1(t) = \theta_a(\gamma(t))$ and let r_1 be the rank of $\dot C_1(t) = i(\gamma_1(t))$. We have its Frenet frame $V_{11}, V_{12}, \ldots, V_{1r_1}$ and we can write (3.5) for the curve C_1

$$(3.10)\qquad \begin{array}{l} V_{1j}(t) = P_{1j}(t) + Q_{1j}(t) \\ \nabla^c_{\dot\gamma_1} P_{1j} = 0 = \nabla^\perp_{\dot\gamma_1} Q_{1j} \quad , \quad k_{1(j-1)} = \text{constant.} \end{array}$$

Our next objective is to prove the following identities.

For each j $\quad 1 \leqslant j \leqslant r$

$$(3.11) \qquad P_{1j}(0) = SP_j(0) \quad , \quad Q_{1j}(0) = Q_j(0) \quad , \quad k_{1j-1} = k_{j-1}.$$

Clearly, they are true for j=1 so we assume that they hold for each $j \leqslant i < r$ and prove them for i+1.

Let us write (3.8) for γ_1.

$$[k_{1i}]^2 = \| D(\theta_{a*}\dot{\gamma}, P_{1i}) - A_{Q_{1i}}\theta_{a*}\dot{\gamma} + k_{1i-1}P_{1i-1}\|^2 + \\ + \|\alpha(\theta_{a*}\dot{\gamma}, P_{1i}) + k_{1i-1}Q_{1i-1}\|^2.$$

At t=0, we have by induction and the remarked properties of D, A_ξ and α that

$$[k_{1i}]^2 = \| S[D(\dot{\gamma}, P_i) - A_{Q_i}\dot{\gamma} + k_{i-1}P_{i-1}]\|^2 + \|\alpha(\dot{\gamma}, P_i) + k_{i-1}Q_{i-1}\|^2 = \\ = [k_i]^2$$

and therefore, since they are positive,

$$k_{1i} = k_i.$$

In order to complete the proof of (3.11), we write, for γ_1, the formula (3.9).

$$V_{1i+1}(t) = \frac{1}{k_i}[D(\theta_{a*}\dot{\gamma}, P_{1i}) - A_{Q_{1i}}\theta_{a*}\dot{\gamma} + k_{i-1}P_{1i-1} + \\ + \alpha(\theta_{a*}\dot{\gamma}, P_{1i}) + k_{i-1}Q_{1i-1}]$$

and again, by taking t=0, we get

$$V_{1i+1}(0) = \frac{1}{k_i}[S(D(\dot{\gamma}, P_i) - A_{Q_i}\dot{\gamma} + k_{i-1}P_{i-1}) + \alpha(\dot{\gamma}, P_i) + \\ + k_{i-1}Q_{i-1}]$$

from which (3.11) follows.

It is easy to see now, that (3.11) implies $r_1 \geqslant r$, because S is non singular.

Now we can define, for j = 2,...,k, new geodesics in M by

$$\gamma_j(t) = \theta_a(\gamma_j(t))$$

and if we call r_j the rank of C_j then

$$r = r_k \geqslant r_{k-1} \geqslant \dots \geqslant r_1 \geqslant r$$

which shows $r_1 = r$. This finishes the proof of (3.2). □

Let us complete now the proof of (1.2).

The conditions are sufficient:

Given the tensor S on M we can define, for each $p \in M$, an isometry $\sigma_p\colon R^{n+q} \to R^{n+q}$ by

$$\sigma_p(v) = \begin{cases} S_p(v) & \text{if} \quad v \in M_p \\ v & \text{if} \quad v \in M_p^{\perp}. \end{cases}$$

As we mentioned before we identify M and $i(M) \subset R^{n+q}$. We have to prove that $\sigma_p(M) \subset M$ and that $\sigma_p|M = \theta_p$ for each $p \in M$. At this point we need to make the following observation due to O. Kowalski (private communication).

Let $M = G/K$ be a compact k-symmetric space where G is the connected component of the identity of the group of symmetries. Let g and k be the Lie algebras of G and K respectively. Let θ be the automorphism of G induced by the symmetry at the origin $0 = [K]$ of M. Then (G,K,θ) is a "*regular homogeneous* s-*manifold*" ([6] p.53). Let $g = k \oplus m$ be the decomposition of g given by ([6] II. 24) which makes G/K *reductive with respect to that decomposition*. Let $\langle X,Y\rangle = -B(X,Y)$, where B is the Killing form on g. This is a scalar product invariant by every automorphism of g. Let m' be the orthogonal complement of k in g with respect to this scalar product. This gives a new decomposition $g = k \oplus m'$.

(3.12) LEMMA. *The two decompositions* $g = k \oplus m$ *and* $g = k \oplus m'$ *coincide.*

Proof. Let θ_* be the automorphism of g induced by θ. Then, by definition, if $A = Id_g - \theta_*$, one has

$$\mathfrak{k} = \ker(A) \quad \text{and} \quad \mathfrak{m} = \operatorname{Im}(A)$$

(These are the Fitting 0-component and Fitting 1-component of $\mathfrak{g}$, relative to A, respectively).

Now θ_* leaves $\mathfrak{m}'$ invariant and then $A^i(\mathfrak{g}) \supset \mathfrak{m}' \quad \forall\ i \geqslant 1$. But since the dimensions of $\mathfrak{m}$ and $\mathfrak{m}'$ coincide we have $\mathfrak{m} = \mathfrak{m}'$. □

(3.13) COROLLARY. *The canonical connection* ∇^c *of the regular homogeneous* s*-manifold* (G,K,θ) *and the canonical connection* $\tilde{\nabla}$ *of* G/K *with respect to the decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}'$ *coincide.*

Proof. This follows from the fact that the canonical connection of a homogeneous space G/K, reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, is uniquely determined by the choice of $\mathfrak{m}$ ([6] p.29, I. 6). □

(3.14) COROLLARY. *Let* $(M,g,\{\theta_x\colon x \in M\})$ *be a compact connected Riemannian regular* s*-manifold of order* k. *Let* ∇^c *be its canonical connection and* p *be a point in* M. *Then given any point* $x \in M$ *there exists a* ∇^c*-geodesic in* M *joining* p *to* x.

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the orthogonal decomposition with respect to the Killing form B on $\mathfrak{g}$. The restriction of (-B) to $\mathfrak{m}$ induces on M a new Riemannian metric h(X,Y) which makes M a *naturally reductive homogeneous* space [5, II,p.203]. One knows ([1, p.55]) that the canonical connection $\tilde{\nabla}$ on M, with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, has the same geodesics that the Riemannian connection corresponding to the metric h. Then the corollary follows from (3.13) and the theorem of Hopf-Rinow [4, p.56]. □

Let γ be this ∇^c-geodesic joining p to x; we may assume $\gamma(0) = p$. Put $\gamma_1 = \theta_p(\gamma)$. By (3.2) γ and γ_1 are Frenet curves in R^{n+q} of the same osculating rank r $(1 \leqslant r \leqslant n+q)$ and their corresponding curvatures are equal and constant.

By keeping the same notation as in the proof of (3.2) we call V_j and V_{1j} the Frenet frames of γ and γ_1 respectively. By the nature of the curvatures in this case it is enough to show that

$$\sigma_p(V_i(0)) = V_{1i}(0) \qquad i = 1,\ldots,r$$

To that end we have plenty of information in the proof of (3.2). Clearly this identity follows from (3.5), (3.10) and (3.11) and then $\sigma_p(M) \in M$. It is now clear that $\sigma_p|M = \theta_p$ and the proof of (1.2) is complete. □

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