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THE DISTRIBUTIONAL CONVOLUTION PRODUCTS OF $(P \pm i0)^\lambda * \delta^{(k)}(P_+)$ AND $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2 + P)$

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Abstract:

In this note we establish the distributional convolution products of the form $(P \pm i0)^\lambda * \delta^{(k)}(P_+)$ *(c.f. (I,2,12), (I,1,40), (I,1,41), (I,1,42) and (I,1,43)) and* $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2 + P)$ *(c.f. (I,3,4), (I,3,5), (I,3,6), (I,3,7) and (I,3,8)).*

We obtain the results by using systematically the Fourier transformation and to obtain $(m^2 + P \pm i0)^{\alpha-1} * \delta^{(k)}(m^2 + P)$ *we have employ the expansion*

$$\delta^{(k)}(m^2 + P) = \sum_{\nu \geq 0} \frac{(m^2)^\nu}{\nu\,!} \delta^{(k+\nu)}(P_+)$$

(c.f. **[2]**, *page 6, formula (I,1,24)).*

The convolution product $(P \pm i0)^\lambda * \delta^{(k)}(P_+)$ *generalizes the result ([1], page 13, formula (1,3,6)).*

I.1 Introduction:

Let P be a non degenerate quadratic form in n variables on the form,

$$P = P(x) = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2, \qquad \text{(I,1,1)}$$

where $n = \mu + \nu$ and $\delta^{(k)}(P_+)$ the derivate of k order of the delta measure of Dirac (cf. **[5]**, page 249).

The distribution $(P \pm i0)^\lambda$ is defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} (P \pm i\varepsilon(x)^2)^\lambda, \qquad \text{(I,1,2)}$$

where $\varepsilon > 0$, $|x|^2 = x_1^2 + \cdots + x_n^2$, $\lambda \in c$.
These distributions are analytic in λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, 2, \ldots$, where they have simple poles (cf. **[5]**, page 275).

In this paper, we give a sense to the products of convolution:

$$(P \pm i0)^\lambda * \delta^{(k)}(P_+) \qquad \text{(I,1,4)}$$

and

$$(m^2 + P \pm i))^\lambda * \delta^{(k)}(m^2 + P), \qquad \text{(I,1,5)}$$

where,

$$(m^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0}(m^2 + P \pm i\varepsilon(x)^2)^\lambda, \qquad (I,1,6)$$

$\varepsilon > 0$, m a positive real number.

Here $*$ designates as usual the convolution.

To obtain (I,1,4) and (I,1,5) we take into account the following results,

$$\operatorname*{Res}_{\alpha=-k} P_+^{\alpha-1} = \frac{(-1)^k}{k\,!}\delta^{(k)}(P_+), \quad ([5], \text{page } 278), \qquad (I,1,7)$$

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm\lambda\pi i}P_-^\lambda, \quad ([5], \text{page } 276), \qquad (I,1,8)$$

$(P \pm i0)^\lambda \cdot (P \pm i0)^\mu = (P \pm i0)^{\lambda+\mu}$, ([7], page 23, formula (I,3,1))
λ and μ are complex numbers such that λ, μ and $\lambda + \mu \neq -\frac{n}{2} - k$,
$k = 0, 1, 2, \ldots,$ (I,1,9)

$(P_+i0)^k = (P - i0)^k = P^k$ ([5], page 276),
k integer non negative; (I,1,10)

$$\{(P \pm i0)^\lambda\}^\wedge = a(\lambda, n)e^{\mp\frac{\nu\pi i}{2}}(Q \mp i0)^{-\lambda-\frac{n}{2}}, \quad [5], \text{page } 284), \qquad (I,1,11)$$

$\{P_+^\lambda\}^\wedge = b(\lambda, n)\left[e^{-\pi i(\lambda+\frac{\nu}{2})}(Q - i0)^{-\lambda-\frac{n}{2}} - e^{\pi i(\lambda+\frac{\nu}{2})}(Q + i0)^{-\lambda-\frac{n}{2}}\right],$
([5], page 284), (I,1,12)

$\{P_-^\lambda\}^\wedge = -b(\lambda, n)\left[e^{-\frac{\pi i\nu}{2}}(Q - i0)^{-\lambda-\frac{n}{2}} - e^{\frac{\pi i\nu}{2}}(Q + i0)^{-\lambda-\frac{n}{2}}\right],$
and ([5], page 284) (I,1,13)

$$(m^2 + P \pm i0)^\lambda = \sum_{k=0}^{\infty} \frac{(m^2)^k}{k\,!}\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - k + 1)}(P \pm i0)^{\lambda-k},$$
(cf. [2],page 4, formula (I,1,13)) (I,1,14)

for $m^2 \leq P(x)$ and $\lambda \neq k - \frac{n}{2} - l$, $l = 0, 1, 2 \ldots.$

Where,

$$a(\lambda, n) = 2^{2\lambda+n}\pi^{n/2}\Gamma(\lambda + \frac{n}{2})[\Gamma(-\lambda)]^{-1}, \qquad (I,1,15)$$

$$b(\lambda, n) = 2^{2\lambda+n}\pi^{\frac{n-2}{2}}\Gamma(\lambda + \frac{n}{2})(2i)^{-1}, \qquad (I,1,16)$$

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } p \geq 0, \\ 0 & \text{if } p \leq 0; \end{cases} \quad ([5], \text{page } 276), \qquad (I,1,18)$$

and $Q = Q(y) = y_1^2 + \cdots + y_\mu^2 - y_{\mu+1}^2 - \cdots - y_{\mu+\nu}^2$, $\mu + \nu = n$.

In (I,1,11), (I,1,12) and (I,1,13). Here $\wedge$ designates the Fourier transform:

$$\widehat{f} = \int_{R^n} .f(x)e^{-i\langle x,y\rangle}\, dx, \tag{I,1,19}$$

where $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.

The distribution $(P \pm i0)^\lambda$ have poles at the point $\lambda = -\frac{n}{2} - k$, $k = 0, 1, 2, \ldots$ and from [5], page 276 we have,

$$\underset{\lambda=-\frac{n}{2}-k}{\text{Residuo}}\ (P \pm i0)^\lambda = e^{\pm\nu\pi i/2}\pi^{\frac{n}{2}}\left[4^k \cdot k\,!\Gamma\left(\frac{n}{2}+k\right)\right]^{-1} L^k\{\delta\}, \tag{I,1,20}$$

where

$$L = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{\mu+1}^2} - \cdots - \frac{\partial^2}{\partial x_{\mu+\nu}^2}, \tag{I,1,21}$$

$\mu + \nu = n$.

On the other hand, from (I,1,8) and taking into account the formula:

$$\Gamma(\lambda)\Gamma(1-\lambda) = \pi\ csc\ \lambda\pi([4], \text{page }) \tag{I,1,22}$$

we have,

$$\frac{P_+^{\lambda-1}}{\Gamma(\lambda)} = -(2\pi i)^{-1}\Gamma(1-\lambda)[e^{(\lambda-1)\pi i}(P - i0)^\lambda - e^{-(\lambda-1)\pi i}(P + i0)^{\lambda-1}]. \tag{I,1,23}$$

From (I,1,23), taking into account (I,1,7), (I,1,11) and the formula

$$\underset{z=-k}{\text{Residuo}}\ \Gamma(z) = \frac{(-1)^k}{k\,!} \quad ([3], \text{vol I, page 2}), \tag{I,1,24}$$

we have,

$$\{\delta^{(k)}(P_+)\}^\wedge = \left\{\lim_{\lambda\to-k}\frac{P_+^{\lambda-1}}{\Gamma(\lambda)}\right\}^\wedge = k\,!(2\pi i)^{-1}(-1)^k.$$

$$\{(P - i0)^{-k-1} - (P + i0)^{-k-1}\}^\wedge = d(n,k,\pi)\left[e^{\nu\pi i/2}(Q + i0)^{-\frac{n}{2}+k+1} - e^{-\nu\pi i/2}(Q - i0)^{-\frac{n}{2}+k+1}\right], \tag{I,1,25}$$

where

$$d(n,k,\pi) = \pi^{\frac{n}{2}}(-1)^k 2^{n-2k-2}(2\pi i)^{-1} \cdot \Gamma\left(\frac{n}{2} - k - 1\right). \tag{I,1,26}$$

On the other hand, the distribution $\delta^{(k)}(P+)$ exists only if $k < \frac{n}{2} - 1$ (c.f. [5], page 250).

We observe that $(m^2 + P \pm i0)^\lambda$ are entire distributional functions of λ. This is the principal difference between the distributions, formally analogue $(P \pm i0)^\lambda$ which have poles at the points $\lambda = -\dfrac{n}{2} - k$, $k = 0, 1, 2 \ldots.$

I.2 The convolution product $(P \pm i0)^\lambda * \delta^{(k)}(P+)$

LEMMA: *Let λ and μ be complex numbers such that λ, μ and $\lambda+\mu \neq -\frac{n}{2}-k$, $k = 0,1,2,\ldots$, then the following formulae are valid,*

$$e^{(\lambda-\mu)\pi i}(P-i0)^\lambda \cdot (P+i0)^\mu + e^{-(\lambda-\mu)\pi i}(P+i0)^\lambda \cdot (P-i0)^\mu = \\ =[1-c(\lambda,\mu)]e^{(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu} \tag{I,2,1}$$

and

$$(P+i0)^\lambda \cdot (P-i0)^\mu + (P-i0)^\lambda \cdot (P+i0)^\mu = \\ =[1+c(\lambda,\mu)](P-i0)^{\lambda+\mu} + [1-c(\lambda,\mu)](P+i0)^{\lambda+\mu} \tag{I,2,2}$$

where

$$c(\lambda,\mu) = 2i\sin\lambda\pi \cdot \sin\mu\pi csc(\lambda+\mu)\pi \tag{I,2,3}$$

and $(P \pm i0)^\lambda$ is defined by the equation (I,1,2).

<u>*Proof*</u> From [5], page 277, formula (3), we have,

$$P_+^\lambda = -(2i\sin\lambda\pi)^{-1}\left[e^{-\lambda\pi i}(P+i0)^\lambda - e^{\lambda\pi i}(P-i0)^\lambda\right] \tag{I,2,4}$$

and

$$P_-^\lambda = (2i\sin\lambda\pi)^{-1}\left[(P+i0)^\lambda - (P-i0)^\lambda\right]. \tag{I,2,5}$$

The distributions P_+^λ and P_-^λ have two sets of singularities, namely,
$\lambda = -1,-2,\ldots-k,\ldots$ and
$\lambda = -\frac{n}{2}, -\frac{n}{2}-1, \ldots, -\frac{n}{2}-k, \ldots$.

Therefore, for $\lambda, \mu, \lambda+\mu \neq -\frac{n}{2}-k$ and $\lambda, \mu, \lambda+\mu \neq -1,-2,-3,\ldots$ and taking into account the formulae (I,1,8) and (I,1,4) we have,

$$-(2i)^{-1}(\sin(\lambda+\mu)\pi)^{-1}\left[e^{-(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu} - e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu}\right] = \\ = P_+^{\lambda+\mu} = P_+^\lambda \cdot P_+^\mu = (2i\sin\lambda\pi)^{-1}\cdot(2i\sin\mu\pi)^{-1}\cdot \\ \left[e^{-\lambda\pi i}(P+i0)^\lambda - e^{\lambda\pi i}(P-i0)^\lambda\right]\cdot\left[e^{-\lambda\pi i}(P+i0)^\lambda - e^{\mu\pi i}(P-i0)^\mu\right] = \\ (2i\sin\lambda\pi^{-1}\cdot(2i\sin\mu\pi)^{-1}\Big\{\left[e^{-(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu}+\right. \\ \left.+e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu}\right] - \left[e^{(\lambda-\mu)\pi i}(P-i0)^\lambda\cdot(P+i0)^\mu+\right. \\ \left.+e^{-(\lambda-\mu)\pi i}(P+i0)^\lambda\cdot(P-i0)^\mu\right]\Big\} \tag{I,2,6}$$

and

$$2i\sin(\lambda+\mu))^{-1}\left[(P+i0)^{\lambda+\mu} - (P-i0)^{\lambda+\mu}\right] = P_-^{\lambda+n} = \\ P_-^\lambda \cdot P_-^n = (2i\sin\lambda\pi)^{-1}\cdot(2i\sin\mu\pi)^{-1}\cdot \\ \left[(P+i0)^\lambda - (P-i0)^\lambda\right]\cdot\left[(P+i0)^\mu - (P-i0)^\mu\right] =$$

$$
\begin{aligned}
=&(2i\sin\lambda\pi)^{-1}\cdot(2i\sin\mu\pi)^{-1}\cdot\\
&\{[(P+i0)^{\lambda+\mu}+(P-i0)^{\lambda+\mu}]-\\
&-[(P+i0)^{\lambda}\cdot(P-i0)^{\mu}+(P-i0)^{\lambda}\cdot(P+i0)^{\mu}]\}
\end{aligned}
$$

From (I,2,6) and (I,2,7) we have,

$$
\begin{aligned}
&e^{(\lambda-\mu)\pi i}(P-i0)^{\lambda}\cdot(P+i0)^{\mu}+e^{-(\lambda-\mu)\pi i}(P+i0)^{\lambda}\cdot(P-i0)^{\mu}=\\
=&e^{-(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu}\left[1+\frac{(2i\sin(\lambda+\mu)\pi)^{-1}}{(2i\sin\lambda\pi)^{-1}(2i\sin\mu\pi)^{-1}}\right]+\\
&+e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu}\left[1-\frac{(2i\sin(\lambda+\mu)\pi)^{-1}}{(2i\sin\lambda\pi)^{-1}(2i\sin\mu\pi)^{-1}}\right]
\end{aligned}
\qquad (I,2,8)
$$

and

$$
\begin{aligned}
&(P+i0)^{\lambda}\cdot(P-i0)^{\mu}+(P-i0)^{\lambda}\cdot(P+i0)^{\mu}=\\
=&(P+i0)^{\lambda+\mu}\left[1-\frac{(2i\sin(\lambda+\mu)\pi)^{-1}}{(2i\sin\lambda\pi)^{-1}(2i\sin\mu\pi)^{-1}}\right]+\\
&+(P-i0)^{\lambda+\mu}\left[1+\frac{(2i\sin(\lambda+\mu)\pi)^{-1}}{(2i\sin\lambda\pi)^{-1}(2i\sin\mu\pi)^{-1}}\right]
\end{aligned}
$$

From (I,2,8) and (I,2,9) we obtain (I,2,9)

$$
\begin{aligned}
&e^{(\lambda-\mu)\pi i}(P-i0)\cdot(P+i0)^{\mu}+e^{-(\lambda-\mu)\pi i}(P+i0)^{\lambda}\cdot(P-i0)^{\mu}=\\
=&[1+c(\lambda,\mu)]e^{-(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu}+[1-c(\lambda,\mu)]\cdot\\
&\cdot e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu}
\end{aligned}
\qquad (I,2,10)
$$

and

$$
\begin{aligned}
&(P+i0)^{\lambda}\cdot(P-i0)^{\mu}+(P-i0)^{\lambda}\cdot(P+i0)^{\mu}=\\
=&[1+c(\lambda,\mu)](P-i0)^{\lambda+\mu}+[1-c(\lambda,\mu)](P+i0)^{\lambda+\mu}.
\end{aligned}
\qquad (I,2,11)
$$

Where $c(\lambda,\mu)$ is defined by the equation (I,2,3).

Formulae (I,2,10) and (I,2,11) are identical with formulae (I,2,1) and (I,2,2).

Theorem *Let λ a complex number such that $\lambda \neq -\frac{n}{2}-k$, $\lambda \neq -k$, k a non negative integer and n dimension of the space such that $\frac{n}{2}-k-1$ be a positive integer, it results the following formula:*

$$(P\pm i0)^{\lambda} * \delta^{(k)}(P+) = K(\lambda,n,k,\pi,i).$$

$$\left[A_{\nu,n}(\lambda)e^{\frac{-\nu\pi i}{2}}(P-i0)^{\lambda+\frac{n}{2}-k-1}+B_{\nu,n}(\lambda)e^{\frac{\nu\pi i}{2}}(P+i0)^{\lambda+\frac{n}{2}-k-1}\right]$$

where, $(P\pm i0)$ is defined by the equation (I,2,12), (I,1,2), $\delta^{(k)}(P+)$ by the equation (I,1,7),

$$K(\lambda,n,k,\pi,i)=b(\lambda,n)d(n,k,\pi,i)\cdot\left[a\left(\lambda+\frac{n}{2}-k-1,n\right)\right]^{-1}, \qquad (I,2,13)$$

$$A_{\nu,n}(\lambda) = e^{\pm\lambda\pi i}\Big[e^{\pi i\nu} - 1 + c\Big(-\lambda - \frac{n}{2}, -\frac{n}{2} + k + 1\Big)\Big] + e^{\lambda\pi i}\Big[-e^{\pi i\nu} + (-1)^{u}\Big(1 + c\Big(-\lambda - \frac{n}{2}, -\frac{n}{2} + k + 1\Big)\Big)\Big], \tag{I,2,14}$$

$$B_{\nu,n}(\lambda) = e^{\pm\lambda\pi i}\Big[e^{-\pi i\nu} - 1 - c\Big(-\lambda - \frac{n}{2}, -\frac{n}{2} + k + 1\Big)\Big] + + e^{-\lambda\pi i}\Big[-e^{-\pi i\nu} + (-1)^{n}\Big(1 - c\Big(-\lambda - \frac{n}{2}, -\frac{n}{2} + k + 1\Big)\Big)\Big], \tag{I,2,15}$$

$a(\lambda,n)$ is defined by (I,1,15), $b(\lambda,n)$ by (I,1,16), $d(n,k,\pi,i)$ by (I,1,26) and $c(\lambda,n)$ by (I,2,3).

Here $*$ designates, as usual, the convolution.

Proof: Let P_+ the generalized function defined by the equation (I,1,17) and $\{\delta^{(k)}(P+)\}^\wedge$ by the equation (I,1,25) where $\wedge$ indicates the Fourier transform.

On the other hand, from [5], page 276 $(P \pm i0)^\lambda$ are entire distributions function in λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0,1,2,\ldots$, $\delta^{(k)}(P+) \in S'$, where S' ([6], page 233) is the dual of S and S is the Schwartz set of functions ([6], page 268), that the following formula is valid

$$\{(P \pm i0)^\lambda * \delta^{(k)}(P+)\}^\wedge = \{(P \pm i0)^\lambda\}^\wedge \cdot \{\delta^{(k)}(P+)\}^\wedge$$

for $\lambda \neq -\frac{n}{2} - k$, $k = 0,1,2,\ldots$. (I,2,16)

From (I,2,16) and taking into account the equations (I,2,8), (I,1,12), (I,1,13), (I,1,25) and (I,1,9) we have,

$$\begin{aligned}
&\{(P \pm i0)^\lambda * \delta^{(k)}(P+)\}^\wedge = [\{P_+^\lambda\}^\wedge + e^{\pm\lambda\pi i}\{P_-^\lambda\}^\wedge] \cdot \{\delta^{(k)}(P+)\}^\wedge = \\
&= \Big\{b(\lambda,n)\big[e^{\pi i\lambda}e^{-\frac{\pi i\nu}{2}}(Q - i0)^{\lambda\frac{u}{2}} - e^{\pi i\lambda}e^{\frac{\pi i\lambda}{2}}e^{\frac{\nu\pi i}{2}}(Q + i0)^{-\lambda-\frac{n}{2}}\big] - \\
&- be^{\pm\lambda\pi i}\big[e^{-\pi i\frac{\nu}{2}}(Q - i0)^{-\lambda-\frac{n}{2}} - e^{\pi i\frac{\nu}{2}}(Q + i0)^{-\lambda-\frac{n}{2}}\big]\Big\} \cdot \\
&\cdot \{d(n,k,\pi) \cdot \big[e^{\frac{\nu\pi i}{2}}(Q + i0)^{-\frac{n}{2}+k+1} - e^{-\frac{\nu\pi i}{2}}(Q - i0)^{-\frac{n}{2}+k+1}\big]\} = \\
&= d(n,k,\pi)b(\lambda,n)\Big\{\big[(e^{-\lambda\pi i}(Q - i0)^{-\lambda-\frac{n}{2}} \cdot (Q + i0)^{-\frac{n}{2}+k+1} + \\
&+ e^{\lambda\pi i}(Q + i0)^{-\lambda-\frac{n}{2}} \cdot (Q - i0)^{\frac{n}{2}+k+1}) - (e^{-\nu\pi i}e^{-\lambda\pi i}(Q - i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1} + \\
&((Q - i0)^{-\lambda-\frac{n}{2}} \cdot (Q + i0)^{-\frac{n}{2}+k+1} + (Q + i0)^{-\lambda-\frac{n}{2}} \cdot (Q - i0)^{-\frac{n}{2}+k+1}) \\
&- e^{\pi i\nu}(Q - i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1} + e^{\pi i\nu}(Q + i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1})\big]\Big\}.
\end{aligned} \tag{I,2,17}$$

From (I,2,1) and (I,2,2), we have

$$\begin{aligned}
&e^{-\lambda\pi i}(Q - i0)^{-\lambda-\frac{n}{2}} \cdot (Q + i0)^{-\frac{n}{2}+k+1} + \\
&e^{\lambda\pi i}(Q + i0)^{-\lambda-\frac{n}{2}} \cdot (Q - i0)^{-\frac{n}{2}+k+1} = \\
&=(-1)^{k+1}\big[e^{(-\lambda-k-1)\pi i}(Q - i0)^{-\lambda-\frac{n}{2}} \cdot (Q + i0)^{\frac{n}{2}+k+1} + \\
&+ e^{-(-\lambda-k-1)\pi i}(Q + i0)^{-\lambda-\frac{n}{2}} \cdot (Q - i0)^{-\frac{n}{2}+k+1}\big] = \\
&=(-1)^{k+1}\big[e^{(-\lambda-\frac{n}{2}-(-\frac{n}{2}+k+1))\pi i}(Q - i0)^{-\lambda-\frac{n}{2}} \cdot (Q + i0)^{-\frac{n}{2}+k+1} + \\
&+ e^{-(-\lambda-\frac{n}{2}-(-\frac{n}{2}+k+1))}(Q + i0)^{-\lambda-\frac{n}{2}} \cdot (Q - i0)^{-\frac{n}{2}+k+1}\big] = \\
&=(-1)^{k+1}\big[(1 + c)e^{-(-\lambda-\frac{n}{2}+(-\frac{n}{2}+k+1))\pi i}(Q + i0)^{\lambda-\frac{n}{2}-\frac{n}{2}+k+1} + \\
&+ (1 - c)e^{(-\lambda-\frac{n}{2}+(-\frac{n}{2}+k+1))\pi i}(Q - i0)^{-\lambda-\frac{u}{2}-\frac{n}{2}+k+1}\big] \\
&=\big[(1 + c)(-1)^{n}e^{\lambda\pi i}(Q + i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1}\big]
\end{aligned} \tag{I,2,18}$$

and

$$(Q-i0)^{-\lambda-\frac{n}{2}}\cdot(Q+i0)^{-\frac{n}{2}+k+1}+(Q+i0)^{-\lambda-\frac{n}{2}}\cdot(Q-i0)^{-\frac{n}{2}+k+1}= \\ =(1+c)(Q-i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1}+(1-c)(Q+i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1}, \qquad \text{(I,2,19)}$$

where $c=c\Big(-\lambda-\dfrac{n}{2}-\dfrac{n}{2}+k+1\Big)$ and $c(\lambda,n)$ is defined by the equation (I,2,3).

Therefore, from (I,2,17) and (I,2,18) we have,

$$\{(P\pm i0)^{\lambda}*\delta^{(k)}(P=)\}^{\wedge}=d(n,k,\pi)\cdot b(\lambda,n).$$

$$\Big[A_{\nu,n}(\lambda)(Q+i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1}+B_{\nu,n}(\lambda)(Q-i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1}\Big], \qquad \text{(I,2,20)}$$

where, $b(\lambda,n)$ is defined by (I,1,16), $d(n,k,\pi)$ by (I,1,26),

$$A_{\nu,n}(\lambda)=e^{\pm\lambda\pi i}\Big[e^{\pi i\nu}-1+c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)\Big]+ \\ +e^{\lambda\pi i}\Big[-e^{\nu\pi i}+(-1)^{n}\Big(1+c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)\Big)\Big]+ \qquad \text{(I,2,21)}$$

$$B_{\nu,n}(\lambda)=e^{\pm\lambda\pi i}\Big[e^{-\pi i\nu}-1-c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)\Big]+ \\ +e^{-\lambda\pi i}\Big[-e^{-\pi i\nu}+(-1)^{n}\Big(1-c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)\Big)\Big], \qquad \text{(I,2,22)}$$

and $c(\lambda,n)$ is defined by the equation (I,2,3).

On the other hand, from (I,1,11) we have,

$$\{(P\pm i0)^{\lambda+\frac{n}{2}-k-1}\}^{\wedge}= \\ =a\Big(\lambda+\frac{n}{2}-k-1,n\Big)e^{\pm\frac{\nu\pi i}{2}}(Q\mp i0)^{-\lambda-\frac{n}{2}-\frac{n}{2}+k+1}, \qquad \text{(I,2,23)}$$

where $a(\lambda,n)$ is defined by the equation (I,1,15) and $\wedge$ indicates the Fourier transform.

From (I,2,20) and taking into account the equation (I,2,23) we have,

$$\{(P\pm i0)^{\lambda}*\delta^{(k)}(P+)\}^{\wedge}=k(\lambda,,k,\pi,i)\cdot \\ \cdot\{A_{\nu,n}(\lambda)e^{-\frac{\nu\pi i}{2}}(P-i0)^{\lambda+\frac{n}{2}-k-1} \\ +B_{\nu,n}(\lambda)(\lambda)e^{\frac{\nu\pi i}{2}}(P+i0)^{\lambda+\frac{n}{2}-k-1}\}^{\wedge} \qquad \text{(I,2,24)}$$

where

$$k(\lambda,n,k,\pi,i)=b(\lambda,n)d(n,k,\pi)\Big[a\Big(\lambda+\frac{n}{2}-k-1,n\Big)\Big]^{-1}, \qquad \text{(I,2,25)}$$

$a(\lambda,n)$ is defined by the equation (I,1,15), $b(\lambda,n)$ by the equation (I,1,16) and $d(n,k,\pi)$ by the equation (I,1,26).

Finally, using the theorem of unicity for the Fourier transform, from (I,1,24) we conclude,

$$(P\pm i0)^{\lambda}*\delta^{(k)}(P+)=k(\lambda,n,k,\pi,i)\cdot \\ \cdot\big[e^{\frac{\nu\pi i}{2}}A_{\nu,n}(\lambda)(P-i0)^{\lambda+\frac{n}{2}-k-1}+B_{\nu,n}(\lambda)e^{\frac{\nu\pi i}{2}}(P+i0)^{\lambda+\frac{n}{2}-k-1}\big],$$

where $k(\lambda, n, k, \pi, i)$, $A_{\nu,n}(\lambda)$ and $B_{\nu,n}(\lambda)$ are defined by (I,1,25), and (I,2,22), respectively.

Formula (I,2,26) proves our assertion (I,2,12).

On the other hand, from (I,2,3), we have,

$$c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)=\frac{-2i\sin\Big(\lambda+\frac{n}{2}\Big)\pi\cdot\sin\Big(\frac{n}{2}-k-1\Big)\pi}{\sin\Big(\lambda+\frac{n}{2}\Big)\pi\cdot\cos\Big(\frac{n}{2}-k-1\Big)\pi+\sin\Big(\frac{n}{2}-k-1\Big)\pi\cdot\cos\Big(\lambda+\frac{n}{2}\Big)\pi}. \qquad (I,2,27)$$

Therefore from (I,2,27) we have,

$$c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)=0, \qquad \text{if } n \text{ is even}, \qquad (I,2,28)$$

and

$$c\Big(-\lambda-\frac{n}{2},-\frac{n}{2}+k+1\Big)=2i\cos\lambda\pi\cdot c\,s\,c\,\lambda\pi, \qquad \text{if } n \text{ is odd}. \qquad (I,2,28)$$

From (I,1,14) and (I,2,15) and taking into account the equations (I,2,28) and (I,2,29) we have the following conclusions:

1. If n and ν are even, then

$$A_{\nu,n}(\lambda)=0 \qquad \text{and} B_{\nu,n}(\lambda)=0, \qquad (I,2,30)$$

2. If n is even and ν is odd, then

$$A_{\nu,n(\lambda)}=-2(e^{\pm\lambda\pi i}-e^{\lambda\pi i}) \quad \text{and}$$
$$B_{\nu,n}(\lambda)=-2(e^{\pm\lambda\pi i}-e^{-\lambda\pi i}). \qquad (I,2,31)$$

3. If n is odd and ν is even, then

$$A_{\nu,n}(\lambda)=\pm 2e^{\pm\lambda\pi i} \text{ and } B_{\nu,n}(\lambda)=\pm 2e^{\pm\lambda\pi i} \qquad (I,2,32)$$

4. If n and ν are odd, then

$$A_{\nu,n}(\lambda)=\pm 2e^{+\lambda\pi i} \text{ and } B_{\nu,n}(\lambda)=\pm 2e^{\lambda\pi i} \qquad (I,2,33)$$

Also, taking into account the equations (I,1,15), (I,1,16) and (I,1,26) we have,

$$K(\lambda,n,k,\pi,i)=-4^{-1}(-1)^k\Pi^{\frac{u-2}{2}}\cdot\frac{\Gamma(\lambda+1)\Gamma\Big(\lambda+\frac{n}{2}\Big)\Gamma\Big(-\lambda-\frac{n}{2}+k+1\Big)}{\Gamma\Big(\lambda+\frac{n}{2}+\frac{n}{2}-k-1\Big)\pi}. \qquad (I,2,34)$$

On the other hand, taking into account the equation (I,1,21) we have,

$$\Gamma\Big(-\lambda-\frac{n}{2}+k+1\Big)=\frac{\Pi}{\sin\lambda\pi}\cdot\frac{1}{(-1)^{\frac{n}{2}-k}\Gamma\Big(\lambda+\frac{n}{2}-k\Big)}, \qquad (I,2,35)$$

if n is even,

and

$$\Gamma\left(-\lambda-\frac{n}{2}+k+1\right)= \frac{\pi}{\cos\lambda\pi}\cdot\frac{1}{(-1)^k(-1)^{\frac{u-1}{2}}\Gamma\left(\lambda+\frac{n}{2}-k\right)}, \tag{I,2,36}$$

if n is odd.

From (I,2,34), (I,2,35) and (I,2,36) we have,

$$K=\overline{K}(\lambda,n,k)\frac{\Pi}{(-1)^{\frac{n}{2}-k}\sin\lambda\pi},\quad \text{if } n \text{ is even}, \tag{I,2,37}$$

and

$$K=\overline{K}(\lambda,n,k)\frac{\Pi}{(-1)^k(-1)^{\frac{u-1}{2}}\cos\lambda\pi},\quad \text{if } n \text{ is odd}, \tag{I,2,38}$$

where

$$\overline{K}(\lambda,n,k)=\frac{-4^{-1}(-1)^k\Pi^{\frac{u-2}{2}}\Gamma(\lambda+1)\Gamma\left(\lambda+\frac{n}{2}\right)}{\Gamma\left(\lambda+\frac{n}{2}+\frac{n}{2}-k-1\right)\Gamma\left(\lambda+\frac{u}{2}-k\right)\Pi}. \tag{I,2,39}$$

Therefore, taking into accout that $n=\mu+\nu$, where μ is the number of the positive squares and ν is the number of the negative squares and taking into account (I,2,30), (I,2,31), (I,2,32), (I,2,33), (I,2,34) and (I,2,38), the equation (I,2,12) can also be explicitely written as the following formulae:

1. $(P\pm i0)^\lambda * \delta^{(k)}(P+)=0$, if n is even and ν even. (I,2,40)
2. $(P\pm i0)^\lambda * \delta^{(k)}(P+)=4^{+1}\overline{K}(\lambda,n,k)\pi(-1)^{\frac{\nu-1}{2}}(-1)^{\frac{n}{2}-k}\cdot(P\pm i0)^{\lambda+\frac{n}{2}-k-1}$, if n is even and ν is odd. (I,2,41)
3. $(P\pm i0)^\lambda * \delta^{(k)}(P+)=\mp 4i(-1)^{\frac{n}{u}2}\overline{K}(\lambda,n,k)\pi e^{\pm\lambda\pi i}\cdot P_-^{\lambda+\frac{n}{2}-k-1}$, if n is odd and ν is even. (I,2,42)
4. $(P\pm i0)^\lambda * \delta^{(k)}(P+)=\pm\frac{4i(-1)^{\frac{\nu-1}{2}}}{(-1)^k(-1)^{\frac{n-1}{2}}}\overline{K}(\lambda,n,k)\pi\cdot P+^{\lambda+\frac{n}{2}-k-1}$ if n is odd and ν odd. (I,2,43)

or equivalently,

5. $(P\pm i0)^\lambda * \delta^{(k)}(P+)=\pm 4(-1)^k(-1)^{\frac{\mu}{2}}\overline{K}(\lambda,n,k)\cdot\pi P+^{\lambda+\frac{n}{2}-k-1}$ if n is odd and μ is even. (I,2,44)

Here $\overline{K}(\lambda,n,k)$ is defined by the equation (I,2,31).

In particular, if $\lambda=l$ is a non-negative integer, from (I,2,40), (I,2,41). (I,2,42), (I,2,43) and (I,2,44) and taking into account (I,1,8) we have,

1. $P^l * \delta^{(k)}(P+)=0$, if n is even and ν even. (I,2,45)
2. $P^l * \delta^{(k)}(P+)=(-1)^{\frac{\mu-1}{2}}M(n,k)P^{\frac{n}{2}+l-k-1}$, if n is even and ν odd. (I,2,46)

3. $P^l * \delta^{(k)}(P+) = \mp(-1)^l(-i)(-1)^k(-1)^{\frac{\nu}{2}}\pi^{\frac{n-2}{2}} \cdot M(n,k) \cdot P_-^{l+\frac{n}{2}-k-1}$ if n is odd and ν is even. (I,2,47)

4. $P^l * \delta^{(k)}(P+) = \mp(-1)^{\frac{\mu}{2}}(-1)i\pi^{\frac{n-2}{2}} M(n,K) \cdot P+^{l+\frac{n}{2}-k-1}$ if n is odd and μ is even. (I,2,48)

where $M(n,k) = \dfrac{\Gamma(l+1)\Gamma(l+\frac{n}{2})\pi^{\frac{n-2}{2}}}{\Gamma(l+\frac{n}{2}+\frac{n}{2}-k-1)\Gamma(l+\frac{n}{2}-k)}$. (I,2,49)

We observe that the formulae (I,2,45) and (I,2,46) appear in [1], page 13, formula (I,3,6).

I.3 The convolution product $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2+P)$

A natural generalization of the Theorem (paragraph I,2, formula (I,2,12) is obtained by taking into the equation (I,1,14) and the formula,

$$\delta^{(k)}(m^2+P) = \sum_{\gamma\geq 0} \frac{(m^2)^\gamma}{\gamma\,!}\delta^{(\gamma+k)}(P+) \quad \text{if } P \geq m^2 \text{ and } \gamma < \frac{n}{2} - k - 1$$

(c.f. [2], page 6, formula (I,1,24)).

In fact, from (I,1,14) and (I,3,1), we have,

$$(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2+P) = \\ = \sum_{\gamma\geq 0} D_\gamma(\lambda, s, m^2)[(P \pm i0)^{\lambda-s} * \delta^{(k+\gamma-s)}(P+)], \quad \text{(I,3,2)}$$

where

$$D_\gamma(\lambda, s, m^2) = \sum_{s=0}^{\gamma} \frac{(m^2)^\gamma}{s\,!(\Gamma - s)\,!} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda - s + 1)} \quad \text{(I,3,3)}$$

Therefore, from (I,3,2) and taking into account the equation (I,2,12) we have proved the following

Theorem: *Let λ a complex number such that $\lambda \neq -\frac{n}{2} - k$, $\lambda \neq -k$, k a non-negative integer and n the dimension of the space such that $\frac{n}{2} - k - 1$ be a positive integer, then the following formula is valid:*

$$(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2+P) = \sum_{\gamma\geq 0} D_\gamma(\lambda, s, m^2) \cdot \\ k(\lambda - s, n, k+\gamma - s, \pi, i)\big[A_{\nu,n}(\lambda - s)e^{\frac{-\nu\pi i}{2}}(P+i0)^{\lambda+\frac{n}{2}-(k+\gamma)-1} + \\ + B_{\nu,n}(\lambda - s)e^{\frac{\nu\pi i}{2}}(P+i0)^{\lambda+\frac{n}{2}-(k+\gamma)-1}\big], \quad \text{(I,3,4)}$$

where, $k(\lambda - s, n, k+\gamma - s, \pi, i)$ is defined by (I,2,13), $A_{\nu,n}(\lambda - s)$ by (I,2,14) and $B_{\nu,n}(\lambda - s)$ by (I,2,15).

In particular, from (I,3,4) when $m^2 = 0$, $\gamma = 0$, and taking into account the formula (I,3,1) we obtain the formula (I,2,12) (paragraph I,2).

On the other hand, taking into account (I,2,40), (I,2,41), (I,2,42), (I,2,43) and (I,2,44) the formula (I,3,4) can also be, explicitely, written in the following manners:

1. $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2 + P) = 0$ *if* n *is even and* ν *even.* (I,3,5)

2. $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2 + P) = \sum_{\gamma\geq 0}\Big\{D_\gamma(\lambda, s, m^2)$ $\cdot$
$\big[4\overline{K}(\lambda - s, n, k + \gamma - s, \pi) \cdot \pi(-1)^{\frac{\nu-1}{2}}(-1)^{\frac{n}{2}-(k+\gamma)}\big]\Big\}(P \pm i0)^{\lambda+\frac{n}{2}-(k+\gamma)-1}$ *if* n *is even and* ν *is odd,* (I,3,6)

3. $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(M^2 + P) = \sum_{\gamma\geq 0}\{D_\gamma(\lambda, s, m^2)\cdot$
$[\mp 4i\overline{K}(\lambda - s, n, k + \gamma - s, \pi) \cdot \pi(-1)^{\frac{\nu}{2}}e^{\pm(\lambda-s)\pi i}](-1)\}P_-^{\lambda+\frac{n}{2}-(k+\gamma)-1}$ *if* n *is odd and* ν *even.* (I,3,7)

4. $(m^2 + P \pm i0)^\lambda * \delta^{(k)}(m^2 + P) = \sum_{\gamma\geq 0}(-1)^{\frac{\mu}{2}}D_\gamma(\lambda, s, m^2)$
$[\pm 4i\overline{K}(\lambda - s, n, k + \gamma - s, \pi) \cdot \pi(-1)^{(k+\gamma)-1}] \cdot P_+^{\lambda+\frac{n}{2}-(k+\gamma)-1}$ *if* n *is odd and* ν *even.* (I,3,8)

Here $\overline{K}(\lambda-s, n, k+\gamma-s, \pi)$ *is defined by the equation (I,2,39) and* $D_\gamma(\lambda, s, m^2)$ *by (I,3,3). On the other hand, from [4], page 566, we have,*

$$(m^2 + P \pm i0)^\lambda = (m^2 + P)_+^\lambda + e^{\pm\lambda\pi i}(m^2 + P)_-^\lambda \tag{I,3,9}$$

where,

$$(m^2 + P)_+^\lambda = \begin{cases}(m^2 + P)^\lambda & \text{if } m^2 + P \geq 0\\ 0 & \text{if } m^2 + P < 0\end{cases} \tag{I,3,10}$$

and

$$(m^2 + P)_-^\lambda = \begin{cases}(-(m^2 + P))^\lambda & \text{if } m^2 + P \leq 0\\ 0 & \text{if } m^2 + P > 0\end{cases} \tag{I,3,11}$$

By making $\lambda = l$ *non-negative integer in (I,3,9) we have,*

$$(m^2 + P + i0)^l = (m^2 + P - i0)^l = (m^2 + P)^l. \tag{I,3,12}$$

Therefore, putting $\lambda = l$ *in (I,3,5), (I,3,6), (I,3,7) and (I,3,8) and taking into account the equations (I,3,39), (I,3,3) and (I,3,12) we have,*

1. $(m^2 + P)^l * \delta^{(k)}(m^2 + P) = 0$ *if* n *is even and* ν *even.*

2. $(m^2 + P)^l * \delta^{(k)}(m^2 + P) = (-1)^{\frac{\mu-1}{2}}\sum_{\gamma\geq 0}(m^2)^\gamma l\,!\overline{M}(n, l, k)\cdot$
$\cdot(P \pm i0)^{l+\frac{n}{2}-(k+\gamma)-1}$, *if* n *is even and* ν *is odd.* (I,3,14)

3. $(m^2 + P)^l * \delta^{(k)}(m^2 + P) = \mp(-1)^l(-i)(-1)^k(-1)^{\frac{\nu}{2}}\cdot$
$\sum_{\gamma\geq 0}(m^2)^\gamma l\,!(-1)^s\overline{M}(n, l, k)(-1)^s P_-^{l+\frac{n}{2}-(k+\gamma)-1}$, *if* n *is odd and* ν *even.* (I,3,15)

4. $(m^2 + P)^l * \delta^{(k)}(m^2 + P) = \pm(-1)^{\frac{\mu}{2}}(-1)i\cdot$
$\sum_{\gamma\geq 0}(m^2)^\gamma l\,!\overline{M}(n, l, k)(-1)^s P_+^{l+\frac{n}{2}-(k+\gamma)-1}$, *if* n *is odd and* ν *even.* (I,3,16)

where

$$\overline{M}(n,l,k) = $$

$$= \sum_{s=0}^{\gamma} \frac{\Gamma\left(l - s + \frac{n}{2}\right)\Pi^{\frac{n-2}{2}}}{s\,!\,(\gamma - s)\,!\,\Gamma\left(l + \frac{n}{2} + \frac{n}{2} - k - \gamma - 1\right)\Gamma\left(l + \frac{n}{2} - k - \gamma\right)}. \qquad (I,3,17)$$

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