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ON THE MEASURE OF SELF-SIMILAR SETS

PABLO A. PANZONE

ABSTRACT. We exhibit a method by which we can approximate the Hausdorff measure of self-similar sets of a certain class.

0. INTRODUCTION. In 1. we show a procedure for approximating the measure of certain self-similar sets. In 2. we use these methods to show that if K is the Koch curve then $0.26 \leqslant \mathcal{H}^s(K) \leqslant 0.5989 < 2^{s-2}$, $s = \log 4/\log 3$ (example 2). We also calculate the measure of some "regular" self-similar sets in R^2 (see example 1, Th.5). This application contains as particular cases some well known results.

Despite the fact that we repeat arguments and use ideas borrowed from the works of Hutchinson [H] and Marion [M 1], on the whole the method shown seems to be new.

1. THE FUNCTION μ. The Hausdorff metric is defined on the collection of all non empty compact subsets of R^n by

$$d_H(E,F) = \inf \{t: F \subset [E]_t \text{ and } E \subset [F]_t\}$$

where $[E]_t = \{x \in R^n: \inf_{y \in E} \|x-y\| = d(x,E) \leqslant t\}$ and $\|\cdot\|$ $(d(\cdot,\cdot))$ is the usual norm (distance). We shall write $F_j \xrightarrow[H]{} K$ instead of $d_H(F_j,K) \xrightarrow[j \to \infty]{} 0$.

We state here the well-known selection theorem due to Blaschke: Let F be an infinite collection of non empty compact sets all lying in a bounded portion B of R^n. Then there exists a sequence $\{F_j\}$ of distinct sets of F convergent in the Hausdorff metric to a non-empty compact set K, (cf. [F], pg.37).

$|A|$ denotes the diameter of a set $A \subset R^n$ and $\mathcal{H}^s(\cdot)$ its s-Hausdorff measure (cf. [F]).

A convex body is a compact convex set with non-empty interior.

The following is a corollary of Blaschke's theorem.

LEMMA 1. *Let* F_i *be a sequence of compact convex non-empty sets of* R^n *such that*

a) $\lim_{i\to\infty} |F_i| = \alpha > 0$

b) *There exists a compact convex set* F *such that* $F_i \subset F$ *for all* i

Then there exists exists a subsequence F_{i_j} *such that*

i) $F_{i_j} \xrightarrow[H]{} K$, K *compact and convex*

ii) $|K| = \alpha$

iii) $K \subset F$

Proof. By the mentioned Blaschke selection theorem we know that there is a subsequence F_{i_j} such that $F_{i_j} \xrightarrow[H]{} K$ where K is a non-empty compact set. Obviously $K \subset F$. As $F_{i_j} \xrightarrow[H]{} K$, we have $d_H(F_{i_j}, K) < \varepsilon_j$ with $\varepsilon_j \to 0$. But then $K \subset [F_{i_j}]_{\varepsilon_j}$ for all j (notice that $[F_{i_j}]_{\varepsilon_j}$ are compact convex sets) and

$$|[F_{i_j}]_{\varepsilon_j}| \longrightarrow \alpha$$

Thus $|K| \leqslant \alpha$. Suppose that $|K| < \alpha$. Since $F_{i_j} \subset [K]_{\varepsilon_j}$ we have $|F_{i_j}| \leqslant |[K]_{\varepsilon_j}|$, and letting $j \to \infty$ we arrive at a contradiction. This proves ii) and iii).

We now prove that K is the convex set $\cap [F_{i_j}]_{\varepsilon_j}$. Observe that

$[F_{i_j}]_{\varepsilon_j}$ tends to K in the Hausdorff metric because

$$d_H(K, [F_{i_j}]_{\varepsilon_j}) \leq d_H(K, F_{i_j}) + d_H(F_{i_j}, [F_{i_j}]_{\varepsilon_j})$$

Thus given $\varepsilon > 0$ there exists j_o such that

$$[F_{i_j}]_{\varepsilon_j} \subset [K]_\varepsilon \quad \text{if} \quad j \geq j_o$$

Then $\cap [F_{i_j}]_{\varepsilon_j} \subset K$. The inclusion $K \subset \cap [F_{i_j}]_{\varepsilon_j}$ was already established. This finishes the proof of the lemma. ■

Let K be a compact set in R^n such that $H^s(K) < \infty$ $(s > 0)$. Define for $\delta > 0$:

$$\mu(\delta) := \sup \{H^s(K \cap C); \text{ C convex compact and } |C| = \delta\}$$

This function is a basic tool in our method.

THEOREM 1. $\mu(\delta)$ *is continuous from the right and non-decreasing. For any* $\delta > 0$, $\mu(\delta) = H^s(K \cap C^o_\delta)$ *where* C^o_δ *is a particular compact convex set of diameter* δ.
Moreover if for any compact convex set C *we have*

$$H^s(K \cap \partial C) = 0$$

then $\mu(\delta)$ *is continuous.*

Proof. From the definition of $\mu(\delta)$ we know that there exists a sequence C^i of compact convex sets of diameter δ, all lying in a bounded portion of R^n, such that

$$\mu(\delta) = \lim_{i\to\infty} H^s(K \cap C^i)$$

By lemma 1 there exists a compact convex set C^o_δ of diameter δ and a subsequence C^{i_j} of C^i such that

$$C^{i_j} \xrightarrow[H]{} C^o_\delta$$

But $\mu(\delta) \geq H^s(K \cap C^o_\delta) = \lim_{k\to\infty} H^s(K \cap [C^o_\delta]_{1/2^k})$ and

$C^{i_j} \subset [C^o_\delta]_{1/2^k}$ if i_j is large enough and k fixed.

Then $\mu(\delta) = H^s(K \cap C^o_\delta)$.

From this one easily gets that $\mu(\delta)$ is non-decreasing.

Let $\delta_o > 0$ and $\delta_i > 0$; $i = 1,2,3,\ldots$, $\delta_i \to \delta_o$. Then

$$\mu(\delta_j) = H^s(K \cap C^o_{\delta_j}) \quad \text{if} \quad j = 0,1,2,3,\ldots$$

with $C^o_{\delta_j}$ a compact convex set of diameter δ_j lying in a bounded portion of R^n.

By lemma 1 there exists a subsequence of $C^o_{\delta_j}$, which we denote in the same way, such that $C^o_{\delta_j} \xrightarrow[H]{} C^o$, where C^o is a compact convex set of diameter δ_o.

But $H^s(K \cap C^o) = \lim\limits_{i\to\infty} H^s(K \cap [C^o]_{1/2^i})$ and $H^s(K \cap [C^o]_{1/2^i}) \geq$ $\geq H^s(K \cap C^o_{\delta_j}) = \mu(\delta_j)$ if i is fixed and $j \geq j(i)$. Thus

$\overline{\lim\limits_{j\to\infty}}\, \mu(\delta_j) \leq H^s(K \cap C^o) \leq \mu(\delta_o)$.

This proves that $\mu(\delta)$ is continuous from the right.

We show now that if for any compact convex set C

$$H^s(K \cap \partial C) = 0$$

then $\mu(\delta)$ must be continuous.

Recall $\mu(\delta_o) = H^s(K \cap C^o_{\delta_o})$, $C^o_{\delta_o}$ a compact convex set of diameter δ_o.

If $C^o_{\delta_o}$ is not a convex body then $C^o_{\delta_o} = \partial C^o_{\delta_o}$ and by hypothesis

$$\mu(\delta) = 0 \quad \text{if} \quad \delta \leq \delta_o$$

Therefore $\mu(\delta)$ is continuous from the left at δ_o.

Assume $C^o_{\delta_o}$ is a convex body.

Let $]C^o_{\delta_o}[_\varepsilon = \{x: d(x,R^n \setminus C^o_{\delta_o}) > \varepsilon\}$. Thus from the hypothesis we get

$$\mu(\delta_o) = \mathcal{H}^s(K \cap C^o_{\delta_o}) = \mathcal{H}^s(K \cap \partial C^o_{\delta_o}) + \mathcal{H}^s(K \cap \mathrm{int}(C^o_{\delta_o})) =$$

$$= \mathcal{H}^s(K \cap \mathrm{int}(C^o_{\delta_o})) = \lim_{i\to\infty} \mathcal{H}^s(K \cap]C^o_{\delta_o}[_{1/2}i).$$

This implies the continuity of $\mu(\delta)$ at δ_o. ∎

A mapping $Y: R^n \to R^n$ is called a contraction if $\|Y(x)-Y(y)\| \leqslant \leqslant k.\|x-y\|$ for all $x,y \in R^n$, where $0 < k < 1$. Clearly a contraction is a continuous function. A contraction that transforms every subset of R^n to a geometrically similar set is called a similitude. Thus a similitude is a composition of a dilatation, a rotation and a translation.

Let Y_i $i = 1,\dots,m$ be a set of similitudes with contraction ratios k_i. We know that there exists a unique non-void compact set K such that $K = \bigcup_{i=1}^{m} Y_i(K)$ (see [F]). We assume also the following (s is the Hausdorff dimension of K):

I) $0 < \mathcal{H}^s(K) < \infty$ $(s > 0)$

II) $\mathcal{H}^s(Y_i(K) \cap Y_j(K)) = 0$ if $i \neq j$

Such a K will be called a self-similar set.

Notice that if K is a self-similar set then the following equality holds:

$$\sum_{i=1}^{m} k_i^s = 1.$$

By C(A) we denote the convex hull of a set A.

Let K be a self-similar set. It is clear that $Y_i(C(K)) \subset C(K)$ for all i. We rename the sets $Y_{i_1} \circ \dots \circ Y_{i_q}(C(K))$ in the following way: C(K) is called T, $Y_i(C(K))$ is called T_i, $Y_i \circ Y_j\ (C(K)) = Y_i(Y_j(C(K))) = T_{ij}$, etc.

Fix $r \geqslant 1$. Set $G_r := \{T_{i_1\dots i_r};\ i_j = 1,\dots,m\}$. G_r has m^r elements. Notice $Y_{i_1}\circ\dots\circ Y_{i_r}\circ Y_{i_{r+1}}(K) \subset T_{i_1\dots i_r i_{r+1}} \subset T_{i_1\dots i_r}$.

PROPERTY Z. *Let* K *be self-similar. We say that* K *has property* Z *if there exists an index* $i_1 \dots i_{r_o}$ *such that*

$$T_{i_1,\dots i_{r_o}} \subset \text{int } C(K).$$

THEOREM 1'. *Let* K *be a self-similar set having the property* Z. *Then for any compact convex set* C *we have*

$$H^s(K \cap \partial C) = 0$$

and $\mu(\delta)$ *is continuous.*

For the proof we need two auxiliary propositions:

PROPOSITION 1. *Let* C_1, C_2 *be two compact convex sets such that* $C_2 \subset [C_1]_\varepsilon$ *for some* $\varepsilon > 0$. *If* $p \in C_2$, $p \notin \text{int } C_1$ *then* $d(p, \partial C_2) \leqslant \varepsilon$.

Proof. Left to the reader.

PROPOSITION 2. *If the hypotheses of the theorem 1' hold for* K *then* C(K) *is a convex body and the following statement is true: there exist* $\varepsilon_o > 0$ *and an integer number* r_1 ($\geqslant r_o$, r_o *of property* Z) *such that for all convex compact sets* C *and all* $t \leqslant \varepsilon_o$ *the set*

$$[\partial C]_t = \{p : d(p, \partial C) \leqslant t\}$$

does not intersect all elements of G_{r_1}.

Proof. Let r_1 be such that $r_1 \geqslant r_o$ and

(1) Max diameter of elements of $G_{r_1} = (\max k_i)^{r_1} . |K| <$

$$< d(\partial C(K), T_{i_1 \dots i_{r_o}})/2.$$

Let $\varepsilon_c = (\max k_i)^{r_1} . |K|/2$. Take all elements Γ of G_{r_1} such that $\Gamma \cap \partial C(K) \neq \{\emptyset\}$. Call this set G'_{r_1}. Observe that

(2) $$C(\bigcup_{\Gamma \varepsilon G'_{r_1}} \Gamma) = C(K).$$

Let C be a compact convex set and assume that $[\partial C]_{\varepsilon_o}$ intersects all elements of G_{r_1}. For each set $\Gamma \in G'_{r_1}$ take a point $q_j \in \Gamma \cap [\partial C]_{\varepsilon_o}$. Thus $C(\bigcup_j q_j) \subset [C]_{\varepsilon_o}$ and $C(\bigcup_j q_j) \subset C(K)$. But by (1) and (2) $C(K) \subset [C(\bigcup_j q_j)]_{2\varepsilon_o}$. Using prop.1 we have that if $p \in C(K)$, $p \notin \text{int } C(\bigcup_j q_j)$ then

(3) $$d(p, \partial C(K)) \leqslant 2\varepsilon_o.$$

Therefore $T_{i_1 \ldots i_{r_o}} \subset \text{int } C(\bigcup_j q_j)$. By (1) and (3)

(4') $$d(T_{i_1 \ldots i_{r_o}}, \partial C(\bigcup_j q_j)) > 2\varepsilon_o.$$

For $p \in \partial C \cap C(\bigcup_j q_j)$ we have by proposition 1:

$$d(p, \partial C(\bigcup_j q_j)) \leqslant \varepsilon_o.$$

Since $d(p, T_{i_1 \ldots i_{r_o}}) \geqslant d(q, T_{i_1 \ldots i_{r_o}}) - d(p,q)$ holds for any q, taking $q \in \partial C(\bigcup_j q_j)$ we get $d(p, T_{i_1 \ldots i_{r_o}}) > 2\varepsilon_o - \varepsilon_o = \varepsilon_o$. This, together with (4') yields

$$d(\partial C, T_{i_1 \ldots i_{r_o}}) \geqslant d(\partial C(\bigcup_j q_j), T_{i_1 \ldots i_{r_o}}) - \varepsilon_o > \varepsilon_o.$$

Thus one obtains $d(T_{i_1 \ldots i_{r_o}}, \partial C) > \varepsilon_o$ and therefore $[\partial C]_{\varepsilon_o}$ cannot intersect $T_{i_1 \ldots i_{r_o}}$.

The proof is completed if we notice that there are elements of G_{r_1} contained in $T_{i_1 \ldots i_{r_o}}$. ■

Proof of Theorem 1'. Let C be a convex compact set and $t > 0$. We define

$$W(t,C) := \mathcal{H}^s(K \cap [\partial C]_t).$$

Suppose $t < \varepsilon_o$. Then

(5) $$W(t,C) \leqslant \Sigma' \; H^s(Y_{i_1}(\ldots(Y_{i_{r_1}}(K))\ldots) \cap [\partial C]_t)$$

where Σ' means the sum over all indexes $i_1 \ldots i_{r_1}$ such that $T_{i_1 \ldots i_{r_1}} \cap [\partial C]_t \neq \{\emptyset\}$.

But

(6) $$H^s(Y_{i_1}(\ldots(Y_{i_{r_1}}(K))\ldots) \cap [\partial C]_t) =$$

$$= k^s_{i_1} \cdot \ldots \cdot k^s_{i_{r_1}} . H^s(K \cap [\partial C^{i_1 \ldots i_{r_1}}]_{t/k_{i_1} \ldots k_{i_{r_1}}})$$

where $C^{i_1 \ldots i_{r_1}}$ is a convex compact set. More precisely $C^{i_1 \ldots i_{r_1}} = Y^{-1}_{i_{r_1}}(\ldots(Y^{-1}_{i_1}(C))\ldots)$.

Using (5), (6), the identity $\Sigma \; k^s_{i_1} \ldots k^s_{i_{r_1}} = 1$ and prop.2 we have

$$W(t,C) \leqslant (\Sigma' \; k^s_{i_1} \ldots k^s_{i_{r_1}}) . H^s(K \cap [\partial C']_{t/(\min k_i)^{r_1}}) \leqslant$$

$$\leqslant (1 - (\min k_i)^{r_1 s}) . W(t/(\min k_i)^{r_1}, C')$$

where C' is one of the convex sets $C^{i_1 \ldots i_{r_1}}$.

Thus we have proved that there exists $\varepsilon_o > 0$, an integer r_1 and a fixed α, $0 < \alpha < 1$, such that for any compact convex set C and any $t < \varepsilon_o$ there is a compact convex set C' such that

(7) $$W(t,C) \leqslant \alpha . W(t/(\min k_i)^{r_1}, C').$$

Using (7) and the fact that $W(t,C) \leqslant H^s(K) < \infty$ for any C and $t > 0$, we get

$$\lim_{t \to o} W(t,C) = 0. \qquad \blacksquare$$

COROLLARY 1. *The Lebesgue measure of the boundary of a compact convex set in* R^n *is equal to zero.*

To prove this well know fact take K as an hypercube and apply Theorem 1'.

REMARK 1. Let K be a self-similar set. Suppose that property Z does not hold, then it is easy to see that

$$(\text{int } C(K)) \cap K = \{\emptyset\} \quad \text{ie.} \quad K \subset \partial C(K).$$

1.1. THE FUNCTIONS u, U, $\tilde{U}$.

Now we define functions u, U and $\tilde{U}$ which approximate in some sense the function μ. For defining these functions we need other auxiliary functions.

Recall that G_r is the set of all possible $T_{i_1 \ldots i_r}$ with r ($\geq$ 1) fixed.

Let $P(G_r)$ be the family of nonvoid subsets of G_r. Define J_r: $P(G_r) \longrightarrow R$ in the following way:

if $\{T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}\}$ is an element of $P(G_r)$ then

$J_r(\{T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}\}) :=$

$$:= (k^s_{i_1} \cdot \ \ldots \ \cdot k^s_{i_r}) + \ldots + (k^s_{j_1} \cdot \ \ldots \ \cdot k^s_{j_r}).$$

It is not difficult to check that $J_r\,(P(G_r))$ is a finite set of points of R such that if $\alpha \in J_r\,(P(G_r))$ then $0 < \alpha \leq 1$, and $1 \in J_r(P(G_r))$. Also $J_r(P(G_r)) \subset J_{r+1}(P(G_{r+1}))$ for all $r \geq 1$. Besides, for each $\varepsilon > 0$ there exists $r_o \geq 1$ such that for all $r \geq r_o$, if $x \in [0,1]$ then there exists $\alpha \in J_r(P(G_r))$ such that $|x-\alpha| < \varepsilon$.

We shall define functions H_r, h_r on the set $J_r(P(G_r))$, ie.

$H_r, h_r : J_r(P(G_r)) \longrightarrow R$.

Let $\alpha \in J_r(P(G_r))$, we define

$$G_r^{\alpha} := J_r^{-1}(\alpha)$$

and

$$H_r(\alpha) := \min_{\beta \varepsilon G_r^{\alpha}} \left(\max_{\Gamma,\Gamma' \varepsilon \beta} |\Gamma \cup \Gamma'| \right) = $$

$$= \min_{\beta \varepsilon G_r^{\alpha}} (\text{diameter of } \beta) ;$$

$$h_r(\alpha) := \min_{\beta \varepsilon G_r^{\alpha}} \left(\max_{\Gamma,\Gamma' \varepsilon \beta} d(\Gamma,\Gamma') \right)$$

where $d(\cdot,\cdot)$ is the distance between sets. Remember that Γ,Γ' are elements of the form $T_{i_1 \ldots i_r}$.

From the definitions of H_r and h_r it is clear that $h_r(\alpha) \leqslant \leqslant H_r(\alpha) \leqslant |K|$ and $H_r(1) = |K|$. It is not difficult to see that $H_r(\alpha) - h_r(\alpha) < \varepsilon$ for all $\alpha \in J_r(P(G_r))$ if r is big enough. Also $H_{r+1}(\alpha) \leqslant H_r(\alpha)$.

Let $0 < \varepsilon_1 < \varepsilon_2$. We define functions U_r, $\tilde{U}_r$ and u_r which approximate $\mu(\delta)$ on $[\varepsilon_1, \varepsilon_2]$.

Let

$$U_r(\delta) := \max \{\alpha : h_r(\alpha) \leqslant \delta\} ,$$

$$u_r(\delta) := \max \{\alpha : H_r(\alpha) \leqslant \delta\}.$$

Thus $U_r(\delta)$ is defined for $\delta \geqslant \min_{\alpha \varepsilon J_r(P(G_r))} h_r(\alpha)$ and $u_r(\delta)$ is defined for $\delta \geqslant \min_{\alpha \varepsilon J_r(P(G_r))} H_r(\alpha)$. It is easy to see that there exist r_o and $\alpha \in J_{r_o}(P(G_{r_o}))$ such that $H_{r_o}(\alpha) < \varepsilon_1$. Thus U_r and u_r are defined on $[\varepsilon_1, \infty)$ if $r \geqslant r_o$.

Let $\tilde{h}_r(\alpha) := H_r(\alpha) - ((\max k_i)^r . |K| . 2)$ and

$\tilde{U}_r(\delta) := \max \{\alpha : \tilde{h}_r(\alpha) \leq \delta\}$. Thus $\tilde{U}_r(\delta)$ is defined for

$\delta \geq \min\limits_{\alpha \varepsilon J_r(P(G_r))} H_r(\alpha) - ((\max k_i)^r.|K|.2)$.

Moreover $u_r(\delta + ((\max k_i)^r.|K|.2)) = \tilde{U}_r(\delta)$ and therefore $\tilde{U}_r$ is defined on $[\varepsilon_1,\infty)$ if $r \geq r_o$.

All functions $u_r(\delta)$, $U_r(\delta)$ and $\tilde{U}_r(\delta)$ are jump functions with a finite number of jumps, continuous from the right non-decreasing and positive.

The following theorem shows how the above functions are related among them and with $\mu(\delta)$.

THEOREM 2. *Let* K *be a self-similar set and* $u_r(\delta)$, $U_r(\delta)$, $\tilde{U}_r(\delta)$ *as above. Then*

a) $u_r(\delta)/\delta^s \leq \mu(\delta)/(\delta^s.H^s(K)) \leq U_r(\delta)/\delta^s \leq \tilde{U}_r(\delta)/\delta^s$ *for* $\delta \geq \min \{H_r(\alpha);\ \alpha \in J_r(P(G_r))\}$.

b) $|\tilde{U}_r(\delta) - u_r(\delta)| \longrightarrow 0$ *uniformly on* $[\varepsilon_1,\varepsilon_2]$ *as* $r \to \infty$ *if* $\mu(\delta)$ *is continuous on* $(0,\infty)$.

c) $\lim\limits_{r\to\infty} (\sup\limits_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} u_r(\delta)/\delta^s) = \lim\limits_{r\to\infty} (\sup\limits_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} \tilde{U}_r(\delta)/\delta^s) =$

$= (\sup\limits_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} \mu(\delta)/\delta^s)/H^s(K)$ *if* $\mu(\delta)$ *is continuous at* ε_2.

d) b) *and* c) *hold if we replace* $\tilde{U}_r$ *by* U_r.

Proof. We show first that

$$u_r(\delta) \leq \mu(\delta)/H^s(K) \leq U_r(\delta) \quad \text{if} \quad \delta \geq \min_{\alpha\varepsilon J_r(P(G_r))} H_r(\alpha).$$

From theorem 1 we know that $\mu(\delta) = H^s(C^o_\delta \cap K)$ where C^o_δ is a compact convex set of diameter δ. But C^o_δ interesects l elements of G_r: $T_{i_1\ldots i_r}, \ldots, T_{j_1\ldots j_r}$.

Then, because of the self-similarity of K:

$$\mu(\delta) = \mathcal{H}^s(C^o_\delta \cap K) \leqslant [(k_{i_1}\ldots k_{i_r})^s + \ldots + (k_{j_1}\ldots k_{j_r})^s].\mathcal{H}^s(K) =$$

$$= \alpha.\mathcal{H}^s(K).$$

Also $h_r(\alpha) \leqslant |C^o_\delta| = \delta$. Then $\mu(\delta) \leqslant U_r(\delta).\mathcal{H}^s(K)$.

To prove the remaining inequality let $u_r(\delta) = \alpha$. Then $H_r(\alpha) \leqslant \delta$ and there exist 1 elements of G_r, say $T_{i_1\ldots i_r},\ldots,T_{j_1\ldots j_r}$, such that

i) $J_r(\{T_{i_1\ldots i_r},\ldots,T_{j_1\ldots j_r}\}) =$
$= (k_{i_1}\ldots k_{i_r})^s + \ldots + (k_{j_1}\ldots k_{j_r})^s = \alpha.$

ii) $H_r(\alpha) = |T_{i_1\ldots i_r} \cup \ldots \cup T_{j_1\ldots j_r}|.$

Using $\mathcal{H}^s(Y_i(K) \cap Y_j(K)) = 0$ if $i \neq j$ it follows that $u_r(\delta) \leqslant \mu(\delta)/\mathcal{H}^s(K)$.

Now we prove that $U_r(\delta) \leqslant \tilde{U}_r(\delta)$ if $\delta \geqslant \min_{\alpha\varepsilon J_r(P(G_r))} H_r(\alpha)$.

For this we only have to prove that $\tilde{h}_r(\alpha) \leqslant h_r(\alpha)$ if $\alpha \in J_r(P(G_r))$. Fix α. From the definition of $h_r(\alpha)$ we then have 1 elements of G_r, say $T_{i_1\ldots i_r},\ldots,T_{j_1\ldots j_r}$, such that

i) $J_r(\{T_{i_1\ldots i_r},\ldots,T_{j_1\ldots j_r}\}) = \alpha$

ii) $h_r(\alpha) = \max_{\Gamma,\Gamma'\varepsilon\{T_{i_1\ldots i_r},\ldots,T_{j_1\ldots j_r}\}} (d(\Gamma,\Gamma'))$

where $d(\cdot.\cdot)$ is the distance between sets.

But any element of $\{T_{i_1\ldots i_r},\ldots,T_{j_1\ldots j_r}\}$ has diameter less than or equal to $(\max k_i)^r.|K|$. Thus

$$|T_{i_1\ldots i_r} \cup \ldots \cup T_{j_1\ldots j_r}| \leqslant h_r(\alpha) + (\max k_i)^r.|K|.2$$

and therefore $H_r(\alpha) \leqslant h_r(\alpha) + (\max k_i)^r.|K|.2$ ie. $\tilde{h}_r(\alpha) \leqslant h_r(\alpha)$.
This proves a).

To prove c) we need the following: if $\mu(\delta)$ is continuous at ε_2, then $\lim_{r\to\infty} u_r(\varepsilon_2) = \mu(\varepsilon_2)/H^s(K)$. Suppose this is not true, then for some $\varepsilon > 0$ and a subsequence r_j

$$u_{r_j}(\varepsilon_2) < (\mu(\varepsilon_2)/H^s(K)) - \varepsilon.$$

But then

$$\mu(\varepsilon_2-((\max k_i)^{r_j}.|K|.2))/H^s(K) \leqslant \tilde{U}_{r_j}(\varepsilon_2-((\max k_i)^{r_j}.|K|.2)) =$$

$$= u_{r_j}(\varepsilon_2) < \mu(\varepsilon_2)/H^s(K) - \varepsilon$$

which is, for $j \longrightarrow \infty$ an absurd.

Let $\varepsilon > 0$. Let r_1 be such that $\mu(\varepsilon_2+((\max k_i)^{r_1}.|K|.2)) - \mu(\varepsilon_2) < \varepsilon.H^s(K)$, $\mu(\varepsilon_2)/H^s(K)-u_r(\varepsilon_2) < \varepsilon$ if $r \geqslant r_1$ and $|1/x^s-1/y^s| < \varepsilon$ if $|x-y| \leqslant ((\max k_i)^{r_1}.|K|.2)$ and $x,y \in [\varepsilon_1,\infty)$.

Let $\tau = \sup_{\delta\varepsilon(0,\varepsilon_2]} \mu(\delta)/H^s(K)$.

Now we prove c). Due to the fact that $\tilde{U}_r$ is non-decreasing and continuous from the right we have that $\sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} \tilde{U}_r(\delta)/\delta^s$ is taken on a particular point δ_o of $[\varepsilon_1,\varepsilon_2]$.

Thus if $r \geqslant r_1$ we have

$$\sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} \tilde{U}_r(\delta)/\delta^s = \tilde{U}_r(\delta_o)/\delta_o^s = u_r(\delta_o+((\max k_i)^r.|K|.2))/\delta_o^s.$$

There are two possibilities: $(\delta_o + ((\max k_i)^r.|K|.2)) = \delta_o'$ belongs to $[\varepsilon_1,\varepsilon_2]$ or not.

Suppose that it belongs. Then

$$u_r(\delta_o')/\delta_o^s = u_r(\delta_o').(1/\delta_o^s-1/\delta_o'^s)+u_r(\delta_o')/\delta_o'^s \leqslant \tau.\varepsilon + \sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} u_r(\delta)/\delta^s.$$

If δ'_o does not belong to $[\varepsilon_1,\varepsilon_2]$ then

$$u_r(\delta'_o)/\delta_o^s = ((u_r(\delta'_o)-u_r(\varepsilon_2))/\delta_0^s) + u_r(\varepsilon_2).(1/\delta_o^s-1/(\varepsilon_2)^s) +$$

$$+ u_r(\varepsilon_2)/(\varepsilon_2)^s \leqslant 2.\varepsilon/(\varepsilon_1)^s + \tau.\varepsilon + \sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} u_r(\delta)/\delta^s.$$

Thus c) is proved.

We end the proof of theorem 2 proving that b) holds.

Suppose that $\tilde{U}_r(\delta)-u_r(\delta)$ does not tend to zero uniformly on $[\varepsilon_1,\varepsilon_2]$. Then we would have a sequence of points $\delta_j \in [\varepsilon_1,\varepsilon_2]$ and a sequence $r_j \longrightarrow \infty$, such that

$$0 < \theta \leqslant \tilde{U}_{r_j}(\delta_j) - u_{r_j}(\delta_j) = u_{r_j}(\delta_j + q_j) - u_{r_j}(\delta_j)$$

where $q_j := (\max k_i)^{r_j}.|K|.2.$ Then

$$\mu(\delta_j+q_j)/H^s(K) - \mu(\delta_j-q_j)/H^s(K) = \mu(\delta_j+q_j)/H^s(K) \pm u_{r_j}(\delta_j+q_j)\pm$$

$$\pm \tilde{U}_{r_j}(\delta_j-q_j) - \mu(\delta_j-q_j)/H^s(K) \geqslant \theta$$ for all j and this contradicts the uniform continuity of $\mu(\delta)$ on $[\varepsilon_1-\varepsilon,\varepsilon_2+\varepsilon]$. ■

1.2. THE FUNCTION f.

Set $f(\delta) := \mu(\delta)/\delta^s$.

THEOREM 3. *Let* K *be a self-similar set. Then*

$$f(\delta) \leqslant 1 \text{ for all } \delta \in (0,\infty).$$

Proof. Suppose the statement is false. Then there exists a compact convex set C_δ of diameter δ such that

$$H^s(K \cap C_\delta)/|K \cap C_\delta|^s \geqslant H^s(K \cap C_\delta)/|C_\delta|^s \geqslant \beta > 1.$$

From the self-similarity of K (property II above) we obtain

$$H^s(Y_i(K \cap C_\delta) \cap Y_j(K \cap C_\delta)) = 0 \quad \text{if} \quad i \neq j.$$

Also $H^s(Y_i(K \cap C_\delta)) = k_i^s . H^s(K \cap C_\delta)$.

Thus for all i we have

$$H^s(Y_i(K \cap C_\delta))/|Y_i(K \cap C_\delta)|^s = H^s(K \cap C_\delta)/|K \cap C_\delta|^s \geq \beta > 1.$$

By induction, for any $l = 1,2,\ldots,$ we get:

a) $$H^s(\underbrace{Y_i \circ \ldots \circ Y_j}_{=l}(K \cap C_\delta) \cap \underbrace{Y_{i'} \circ \ldots \circ Y_{j'}}_{=l}(K \cap C_\delta)) = 0$$

if the l-tuples $i \ldots j$ and $i' \ldots j'$ are different.

b) $$H^s(Y_i \circ \ldots \circ Y_j(K \cap C_\delta))/|Y_i \circ \ldots \circ Y_j(K \cap C_\delta)|^s \geq \beta > 1$$

for all l-tuples.

Set $$A_n := \bigcup_{\substack{\text{all the} \\ \text{l-tuples with } l \geq n}} \underbrace{Y_i \circ \ldots \circ Y_j}_{=l}(K \cap C_\delta)$$

Then $A_{n+1} \subset A_n$ and $A_n \searrow A := \bigcap_n A_n$.

Set $$B_n := \bigcup_{\substack{\text{all the} \\ \text{n-tuples}}} \underbrace{Y_i \circ \ldots \circ Y_j}_{=n}(K \cap C_\delta).$$

Clearly $B_n \subset A_n$. Also from a) and b) we have

$$H^s(B_n) = \sum_{\substack{\text{all the} \\ \text{n-tuples}}} H^s(Y_i \circ \ldots \circ Y_j(K \cap C_\delta)) \geq \beta . \sum_{\substack{\text{all the} \\ \text{n-tuples}}} |Y_i \circ \ldots \circ Y_j(K \cap C_\delta)|^s =$$

$$= \beta . \sum_{\substack{\text{all the} \\ \text{n-tuples}}} k_i^s . \ldots . k_j^s |K \cap C_\delta|^s = \beta . |K \cap C_\delta|^s$$

(the last inequality because $(\sum_{i=1}^m k_i^s)^n = \sum_{\substack{\text{all the} \\ \text{n-tuples}}} k_i^s . \ldots . k_j^s = 1$).

But $H^s(A_n) \leq H^s(K)$. Thus $\lim_{n\to\infty} H^s(A_n) = H^s(A) \geq \beta . |K \cap C_\delta|^s > 0$.

Clearly the sets $\underbrace{Y_i \circ \ldots \circ Y_j}_{=l}(K \cap C_\delta)$ for all the l-tuples $l \geq n$, form a Vitali family V_n for A, ie. they are compact sets and for any $\varepsilon > 0$ and any $x \in A$ there exists $Y_i \circ \ldots \circ Y_j (K \cap C_\delta)$ of positive diameter $< \varepsilon$ such that $x \in Y_i \circ \ldots \circ Y_j (K \cap C_\delta)$.

Let n_o and $\varepsilon > 0$ be such that $H^s(A_{n_o}) + \varepsilon < \beta . H^s(A)$. Then there exists a disjoint subfamily V'_{n_o} of V_{n_o} such that ([F], pg.11)

$$(8)\quad H^s(A) \leqslant \Big(\sum_{Y_i \circ \ldots \circ Y_j (K \cap C_\delta) \in V'_{n_o}} |Y_i \circ \ldots \circ Y_j (K \cap C_\delta)|^s \Big) + \varepsilon/\beta = W + \varepsilon/\beta$$

and either $W = \infty$ or $W < \infty$ and

$$H^s\Big(A - \bigcup_{Y_i \circ \ldots \circ Y_j (K \cap C_\delta) \in V'_{n_o}} Y_i \circ \ldots \circ Y_j (K \cap C_\delta)\Big) = 0.$$

But if $W = \infty$ by (8) and b) it follows

$$\beta . H^s(A) \leqslant \sum_{Y_i \circ \ldots \circ Y_j (K \cap C_\delta) \in V'_{n_o}} \beta . |Y_i \circ \ldots \circ Y_j (K \cap C_\delta)|^s + \varepsilon \leqslant$$

$$\leqslant \sum_{Y_i \circ \ldots \circ Y_j (K \cap C_\delta) \in V'_{n_o}} H^s(Y_i \circ \ldots \circ Y_j (K \cap C_\delta)) + \varepsilon \leqslant H^s(K) + \varepsilon$$

and then $H^s(K) = \infty$.

Therefore $W < \infty$. Then, by (8) and b),

$$\beta . H^s(A) \leqslant \sum_{Y_i \circ \ldots \circ Y_j (K \cap C_\delta) \in V'_{n_o}} H^s(Y_i \circ \ldots \circ Y_j (K \cap C_\delta)) + \varepsilon \leqslant$$

$$\leqslant H^s(A_{n_o}) + \varepsilon < \beta . H^s(A).$$ ■

PROPERTY A: Let K be a self-similar set. We say that property A holds for K if there exists $\Delta > 0$ such that for any $x \in K$ and any $B_{x,r}$ (ball centered at x and radius r) with $r \leqslant \Delta$ there exist $y \in K$ and a similitude Y with contraction ratio $k = 1$, $Y: R^n \rightarrow R^n$, such that

a) $Y(B_{y,r} \cap K) = B_{x,r} \cap K$,

b) $(B_{y,r} \cap K) \subset Y_{i_o}(K)$ for some i_o $1 \leqslant i_o \leqslant m$.

LEMMA 2. *Let* K *be a self-similar set having property A. Then for any* δ*,* $0 < \delta \leq \Delta$*, there exists* j*,* $j \in \{1,\dots,m\}$ *, such that* $f(\delta) = f(\delta/k_j)$.

Proof. Suppose $0 < \delta \leq \Delta$. By theorem 1 we know that $\mu(\delta) = $
$= \mathcal{H}^s(K \cap C_\delta)$, where C_δ is a convex compact set of diameter δ.
By property A there exists C'_δ a convex compact set of diameter δ such that $\mathcal{H}^s(K \cap C'_\delta) = \mathcal{H}^s(K \cap C_\delta)$ and $(K \cap C'_\delta) - Y_{i_o}(K) = \{\emptyset\}$ with $1 \leq i_o \leq m$.
Then $Y_{i_o}^{-1}(C'_\delta) = C_{\delta/k_{i_o}}$ is a compact convex set of diameter δ/k_{i_o}. It is easy to check that $\mathcal{H}^s(K \cap C_{\delta/k_{i_o}}) = 1/k_{i_o}^s \cdot \mathcal{H}^s(K \cap C'_\delta)$
Clearly $\mu(\delta/k_{i_o}) \geq \mathcal{H}^s(K \cap C_{\delta/k_{i_o}})$.

Also by theorem 1 $\mu(\delta/k_{i_o}) = \mathcal{H}^s(K \cap C'_{\delta/k_{i_o}})$ where $C'_{\delta/k_{i_o}}$ is a convex compact set of diameter δ/k_{i_o}.

But $\mu(\delta/k_{i_o}) = \mathcal{H}^s(K \cap C'_{\delta/k_{i_o}}) \leq 1/k_{i_o}^s . \mathcal{H}^s(K \cap Y_{i_o}(C'_{\delta/k_{i_o}})) \leq$

$$\leq 1/k_{i_o}^s . \mathcal{H}^s(K \cap C'_\delta) = \mathcal{H}^s(K \cap C_{\delta/k_{i_o}}).$$

Then $\mu(\delta/k_{i_o}) = \mathcal{H}^s(K \cap C_{\delta/k_{i_o}}) = 1/k_{i_o}^s . \mathcal{H}^s(K \cap C'_\delta) =$

$$= \mu(\delta)/k_{i_o}^s .$$ ■

THEOREM 4. *Let* K *be a self-similar set. Then*

i) $\overline{\lim_{\delta \to 0}} f(\delta) = 1$

ii) *Let also* K *have property A. Let* $0 < \varepsilon_1 < \varepsilon_2$ *be such that*

a) $\varepsilon_1 \leq \Delta$ *with* Δ *of property A.*

b) $\varepsilon_1 . (\max 1/k_i) \leq \varepsilon_2$

Then $f(\delta) = 1$ *for some* $\delta \in [\varepsilon_1, \varepsilon_2]$

Proof. We prove first that $\overline{\lim}_{\delta\to 0} f(\delta) = 1$ (cf. [F],T.2.3). Suppose it is false, then there exists $a > 0$ such that $f(\delta) \leqslant 1-a$ if $\delta \in (0,a)$. From the definition of Hausdorff measure of K we have that for any $\varepsilon > 0$ there exists a countable family E_i of compact convex sets of diameter less than ε such that $H^s(K \cap E_i) \neq 0$ for all i, $\sum_i H^s(K \cap E_i) \geqslant H^s(K)$ and

$$H^s(K) + \varepsilon \geqslant \sum_i |E_i|^s \tag{9}$$

But if $\varepsilon < a$, then

$$\sum_i |E_i|^s = \sum_i H^s(K \cap E_i) . |E_i|^s / H^s(K \cap E_i) \geqslant$$

$$\geqslant \sum_i H^s(K \cap E_i)/f(|E_i|) \geqslant \sum_i H^s(K \cap E_i)/(1-a) \geqslant$$

$$\geqslant H^s(K)/(1-a)$$

which is in contradiction with (9) for ε small enough.

We prove now ii). Suppose K has property A. To prove ii) it is only necessary to show that $\sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} f(\delta) = 1$ since $\mu(\delta)$ is continuous from the right and non-decreasing. Now, from Lemma 2 it follows that if $0 < \delta < \varepsilon_1$ then there exists $\delta' \in [\varepsilon_1,\varepsilon_2]$ such that $f(\delta') = f(\delta)$. So $\overline{\lim}_{\delta\to 0} f(\delta) \leqslant$ $\leqslant \sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} f(\delta) \leqslant 1$ and because of i) the proof is complete. ■

1.3.

A combination of theorems 2,3 and 4 gives us a procedure by which we can compute the measure of a self similar set K if property A holds and $T = C(K)$ is known.

The method is as follows: we observe first that the function $J_r : P(G_r) \longrightarrow R$ defined above is a function whose values we can calculate. Thus H_r and h_r are functions which we can also calculate because this involves taking the distance (or the diameter) between sets of the form $Y_{i_1}(\ldots Y_{i_r}(C(K))\ldots) = T_{i_1\ldots i_r}$ (recall $T = C(K)$ is known!).

Thus, the functions $\tilde{U}_r$, U_r and u_r are known.

But these functions are of the form

$$\sum_{i=1}^{1} q_i \cdot S(x-\tau_i)$$

where $\tau_i \in (0,\infty)$, $q_i > 0$ (τ_i and q_i are known!) and

$$S(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Let ε_1, ε_2 be as in theorem 4. Then $\sup\{U_r(\delta)/\delta^s : \varepsilon_1 \leqslant \delta \leqslant \varepsilon_2\} =$

$= \max\{U_r(\delta)/\delta^s : \delta = \varepsilon_1$ or $\delta \in [\varepsilon_1,\varepsilon_2]$ and U_r has a jump at $\delta\}$ and similar expressions hold for $\tilde{U}_r$ and u_r.

Thus $B_r = \sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} U_r(\delta)/\delta^s$, $\tilde{B}_r = \sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} \tilde{U}_r(\delta)/\delta^s$ and

$\beta_r = \sup_{\delta\varepsilon[\varepsilon_1,\varepsilon_2]} u_r(\delta)/\delta^s$ are all numbers which we can calculate.

By theorems 2,3,4 we have $\beta_r \leqslant 1/H^s(K) \leqslant B_r \leqslant \tilde{B}_r$ and $\beta_r \leqslant \beta_{r+1}$ (this because $H_r \geqslant H_{r+1}$). From theorem 2 we know that $\tilde{B}_r - \beta_r \longrightarrow 0$ if $r \longrightarrow \infty$ ie. $1/\tilde{B}_r \leqslant 1/B_r \leqslant H^s(K) \leqslant 1/\beta_r$ and $1/\beta_r - 1/\tilde{B}_r \longrightarrow 0$ if $r \longrightarrow \infty$.

In the next section we compute measures and "approximate measures" of some self-similar sets.

2. EXAMPLE 1

The sets K_n will be self-similar sets in R^2 for each $n \geqslant 3$ and they are defined as follows. Let P_n be a regular poligon of n sides, $|P_n| = 1$. Thus, for example, P_3 is an equilateral triangle whose base has lenght 1, P_4 is a square of side equal to $1/\sqrt{2}$, P_5 is a pentagon, etc.

We define Y_i^n, $i = 1,\ldots,n$, a similitude in the following way: for each vertex V_i^n, $1 \leqslant i \leqslant n$, of the regular poligon P_n, Y_i^n is a contraction of ratio $1/n$ and a translation (ie. there is no rotation) and $Y_i^n(V_i^n) = V_i^n$. K_n is defined to be the (unique) compact set such that $\bigcup_{i=1}^{n} Y_i^n(K_n) = K_n$.

From the definitions of Y_i^n one easily gets the open set condition: the sets Y_i^n (int $C(P_n)$) are disjoint and

$$\bigcup_{i=1}^{n} Y_i^n(\text{int } C(P_n)) \subset \text{int } C(P_n)$$

(see beginning of proof of lemma 4).

Thus, by Hutchinson's theorem (see [F], pg.119) we get that

a) $0 < \mathcal{H}^{s_n}(K\ ) < \infty$

b) $\mathcal{H}^{s_n}(Y_i^n(K_n) \cap Y_j^n(K_n)) = 0 \quad \text{if} \quad i \neq j$

where s_n is the Hausdorff dimension of K_n. Here $s_n = 1$ for all n.

Observing that V_i^n must belong to K_n it follows that $C(K_n) = C(P_n) = P_n$. Recall that $C(K_n) = T^n$, $Y_i^n(C(K_n)) = T_i^n$, etc.

Notice that property Z holds for K_n.

We will compute the measures of the sets K_n:

THEOREM 5. $\mathcal{H}^1(K_n) = 1$ *for all* $n \geqslant 3$.

2.1.

Our proof of this theorem will need some lemmas.

To motivate the reading of these auxiliary propositions the reader may go directly to the proof of theorem 5 in next section. Figures 7 and 8 show how K_3 and K_5 look like. We denote with $\mu(\delta,n)$ the function $\mu(\delta)$ of K_n.

LEMMA 3. *Let* n,j *be positive integers. Then*

a) $$\frac{1/n}{(1-1/n).\sin(\pi/n)-1/n} \leqslant 1 \quad \textit{if} \quad n \geqslant 5$$

b) $$\frac{2/n}{(1-1/n).\sin(\pi/n)} \leqslant 1 \quad \textit{if} \quad n \geqslant 5$$

c) $$\frac{(j+1)/n}{\sin(j\pi/n)-2/n} \leqslant 1 \quad \textit{if} \quad n \geqslant 7 \textit{ and } 2 \leqslant j \leqslant [n/2]$$

d) $$(1-1/n).\sin(\pi/n) < \sin(2\pi/n)-2/n \quad \textit{if} \quad n \geqslant 6$$

e) $$(1-1/n).\sin^2(\pi/n) \leqslant 2/n \quad \textit{if} \quad n \geqslant 6$$

f) $$\sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n).\sin(\pi/2n) \leqslant$$
$$\leqslant 2/n \quad \textit{if} \quad n \geqslant 7$$

Proof. From Taylor's series of $\sin x$ we obtain

(1) $$\sin x - x \geqslant -x^3/3! \quad \text{if} \quad x \in [0,\pi/2].$$

In the following x denotes real values and n (or j) denote integer values.

a) Let $f(x) := (\pi-2) - \pi/x - \pi^3.(x-1)/(x^3.3!)$. Then $f(x) \geqslant 0$ if $x \in [5,\infty)$ because $f(x)$ is non-decreasing if $x \in [5,\infty)$ and $f(5) > 0$. But using (1) we get for $n \geqslant 5$ that

$$1 \leqslant 1+f(n) \leqslant n.[(1-1/n).\sin(\pi/n)-1/n]$$

and a) follows.

b) Follows from a) immediately.

c) Let $g(n,j) = (n/j)^2.((\pi-1)-3/j)$. Then $g(n/j) \geqslant \pi^3/3!$ if $n \geqslant 8$ and [(n is even and $4 \leqslant j \leqslant n/2$) or (n is odd and

$4 \leqslant j \leqslant (n-1)/2]$ because $g(n,j) \geqslant 4.(\pi-7/4) \geqslant \pi^3/3!$ for the above values of n and j.

If $j = 2$ and $n \geqslant 7$ we get $g(n,2) \geqslant (7/2)^2.(\pi-5/2) \geqslant \pi^3/3!$. If $j = 3$ and $n \geqslant 7$ we get $g(n,3) \geqslant (7/3)^2.(\pi-2) \geqslant \pi^3/3!$.

Thus

(2) $$g(n,j) \geqslant \pi^3/3! \quad \text{if} \quad n \geqslant 7 \quad \text{and} \quad 2 \leqslant j \leqslant [n/2]$$

Thus using (1) and (2) we get

$$0 \leqslant (g(n,j)-\pi^3/3!).(j/n)^3 \leqslant \sin(j\pi/n)-j/n-3/n$$

and c) follows.

d) Let $h(x) := (\pi.(\sqrt{3}-1)-2).x^2-\pi^3.(\sqrt{3}-1)/3!$. Then $h(6) > 0$ and therefore $h(x) > 0$ if $x \geqslant 6$. But using (1) we get if $n \geqslant 6$ that $0 < h(n)/n^3 \leqslant (\sqrt{3}-1).\sin(\pi/n) -2/n \leqslant$

$$\leqslant (2.\cos(\pi/n)-1 + 1/n).\sin(\pi/n) - 2/n$$

and d) follows.

e) and f) Let $f(x) := \sin^2(\pi x) - \sqrt{(1+\cos(\pi/7))/2}.2x$. It is not difficult to prove that $f(x) \leqslant 0$ if $x \in (0,\infty)$. Using this inequality e) and f) follow. ∎

LEMMA 4. *Let* n *be a positive integer. Then*

a) $\mu((1-1/n).\sin(\pi/n),n) \leqslant H^1(K_n)/n$ *if* $n \geqslant 6$ *and* n *is even*

b) $\mu(\sin(j\pi/n)-2/n,n) \leqslant H^1(K_n).j/n$ *if* $n \geqslant 6$, n *is even and* $2 \leqslant j \leqslant n/2$.

c) $\mu(2(1-1/n).\sin(\pi/2n),n) =$
$= \mu(\sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n),n) \leqslant H^1(K_n)/n$ *if* $n \geqslant 5$ *and* n *is odd.*

d) $\mu(\sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n)-2/n,n) \leqslant H^1(K_n).j/n$ *if* $n \geqslant 5$, n *is odd and* $2 \leqslant j \leqslant (n-1)/2$.

Proof. Let $n \geqslant 5$. Recall that $C(K_n) = C(P_n) = T^n$, $Y_i^n(C(K_n)) =$

$= T_i^n$, $Y_j^n(Y_i^n(C(K_n))) = T_{ji}^n$, etc. We call C_e^n the center of P_n ie. $C_e^n = \sum_i V_i^n/n$. Thus it is easy to check that (recall $|P_n| = 1$)

$$d(V_i^n, C_e^n) = \begin{cases} 1/2 & \text{if n is even,} \\ 1/2\cos(\pi/2n) = 1/\sqrt{2.(1+\cos(\pi/n))} & \text{if n is odd.} \end{cases}$$

Since T_i^n contains V_i^n, $|T_i^n| = 1/n$ and

$$d(V_1^n, V_{j+1}^n) = d(V_1^n, V_{n-j+1}^n) = \begin{cases} \sin(j\pi/n) & \text{if n is even, } 1 \leqslant j \leqslant n/2 \\ \sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n) & \text{if n is odd, } 1 \leqslant j \leqslant (n-1)/2 \end{cases}$$

we get

(3)

$$d(T_1^n, T_{j+1}^n) = d(T_1^n, T_{n-j+1}^n) \geqslant \begin{cases} \sin(j\pi/n)-(2/n) & \text{if n is even, } 1 \leqslant j \leqslant n/2 \\ \sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n)-(2/n) & \text{if n is odd, } 1 \leqslant j \leqslant (n-1)/2 \end{cases}$$

Set center $T_i^n = Y_i^n(C_e^n)$. Then

(4) $d(\text{center } T_1^n, \text{center } T_2^n) = d(\text{center } T_1^n, \text{center } T_n^n) =$

$$= \begin{cases} (1-1/n).\sin(\pi/n) & \text{if n is even,} \\ \sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n) & \text{if n is odd.} \end{cases}$$

The above formulae imply:

(3')

$$\min_{i \neq j} d(T_i^n, T_j^n) \geqslant \begin{cases} (1-1/n).\sin(\pi/n)-1/n & \text{if n is even, } n \geqslant 6, \\ \sqrt{2/(1+\cos(\pi/n))}.[(1-1/n).\sin(\pi/n)-1/n] & \text{if n is odd, } n \geqslant 5. \end{cases}$$

Also using d) of lemma 3 we get, for $n \geqslant 6$,

(5) $$(1-1/n).\sin(\pi/n) < \sin(2\pi/n)-(2/n) < \sin(3\pi/n)-(2/n) < \ldots < \sin(\pi/2)-(2/n)$$

And for n odd, $n \geqslant 5$,

(6) $$\sqrt{2/(1+\cos(\pi/n))}\ .\ (1-1/n).\sin(\pi/n) <$$
$$<\sqrt{2/(1+\cos(\pi/n))}\ .\ \sin(2\pi/n) - (2/n) <$$
$$<\sqrt{2/(1+\cos(\pi/n))}\ .\ \sin(3\pi/n) - (2/n) < \ldots <$$
$$<\sqrt{2/(1+\cos(\pi/n))}\ .\ \sin((n-1)\pi/2n) - (2/n).$$

The first inequality may be verified directly for n=5 and is a consequence of (5) for $n \geq 6$.

a) Let C be a compact convex set of diameter $(1-1/n).\sin(\pi/n)$. Suppose $T_1^n \cap C \neq \{\emptyset\}$. From (3) and (5) we have that $C \cap T_j^n = \{\emptyset\}$ if $j \neq 1,2,n$. Thus from symmetry C can only intersect two elements of $\{T_1^n, T_2^n, T_n^n\}$. We assume C intersects T_1^n and T_2^n. Observe that $H^1(T_i^n \cap K_n) = H^1(K_n)/n$. By Theorem 1' we get that if L is any line in R^2 then

(7) $$H^1(L \cap K_n) = 0.$$

Let L_1, L_2 be two parallel lines at a distance $(1-1/n).\sin(\pi/n)$, perpendicular to the segment joining the centers of T_1^n, T_2^n and such that $C \subset W$ where W is (see figure 1) the strip : $W = C(L_1 \cup L_2)$. Recall that d (center T_1^n, center T_2^n) = $= (1-1/n).\sin(\pi/n)$ and observe that the set $(K_n \cap T_1^n)$ is a translation of the set $(K_n \cap T_2^n)$. Then from symmetry and (7) we obtain: $H^1((K_n \cap (T_1^n \cup T_2^n))-W) \geq H^1(K_n)/n$. Thus a) follows. This last argument will be used quite often. Case c) is proved in an analogous way using (3), (4) and (6).

b) Let n and j be as in b). Let C be a compact convex set of diameter $\sin(j\pi/n)-(2/n)$. Assume $C \cap T_1^n \neq \{\emptyset\}$. Then from (3) and (7) we get

$$0 = H^1(K_n \cap T_{j+1}^n \cap C) = H^1(K_n \cap T_{j+2}^n \cap C) = \ldots =$$
$$= H^1(K_n \cap T_{n-j+1}^n \cap C).$$

Thus we could assume that C intersects in a non-trivial way at most the sets: $T^n_{n-j+2}, T^n_{n-j+3}, \ldots, T^n_n, T^n_1, T^n_2, \ldots, T^n_{j-1}, T^n_j$. By symmetry and using this last argument repeatedly we obtain that C intersects in a non-trivial way at most j elements of $\{T_i\}$ and b) follows.

Case d) is proved in a similar way using (3) and (7). ■

LEMMA 5. *Let* n *and* i *be integers. Then*

a) $\mu(1-(1/n^i),n) \leqslant (1-(3/n^i)).H^1(K_n)$ *if* $n \geqslant 6,\ i \geqslant 1$

b) $\mu(1-(3/n^i),n) \leqslant (1-(1/n^{i-1})).H^1(K_n)$ *if* $n \geqslant 6,\ i \geqslant 2$

c) $\mu(1-(1/5^i),5) \leqslant (1-(2/5^i)).H^1(K_5)$ *if* $i \geqslant 1$

d) $\mu(1-(2/5^i),5) \leqslant (1-(1/5^{i-1})).H^1(K_5)$ *if* $i \geqslant 2$

e) $\mu(1-(3/n),n) \leqslant H^1(K_n)/2$ *if* $n \geqslant 6$

f) $\mu(1-(2/5),5) \leqslant H^1(K_5).2/5$

Proof. Let $n \geqslant 5$. It is clear that $H^1(K_n \cap T^n_{j_1\ldots j_i}) = H^1(K_n)/n^i$ and $|T^n_{j_1\ldots j_i}| = 1/n^i$. Call $n_o := [n/2]$. Then $V^n_1 \in T^n_{\underbrace{1\ldots 1}_{i}}, V^n_{n_o+1} \in T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}$, $d(V^n_1, V^n_{n_o+1}) = 1$.

Let C be a compact convex set of diameter $1-(1/n^i)$. Assume $C \cap T^n_{\underbrace{1\ldots 1}_{i}} \neq \emptyset$. Let L_1, L_2 be two lines perpendicular to the line that joins V^n_1 and $V^n_{n_o+1}$ and such that $d(L_1,L_2) = 1-(1/n^i)$ and $C \subset W$, where W is the strip between L_1 and L_2.

Then since $K_n \cap T^n_{\underbrace{1\ldots 1}_{i}}$ is a translation of $K_n \cap T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}$ we have, using (7), that

$$(8) \qquad H^1(K_n \cap (T^n_{\underbrace{1\ldots 1}_{i}} \cup T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}) \cap C) \leqslant H^1(K_n)/n^i$$

But a similar expression holds for pairs $(T^n_{\underbrace{j\ldots j}_{i}}, T^j_{\underbrace{n_o+j,\ldots,n_o+j}_{i}})$ $j = 2,\ldots,n_o$. If we assume $n \geqslant 6$ then there are at least 3 such pairs and a) is proved. If n=5 there are 2 such pairs and c) is proved.

e) Let $n \geqslant 6$. Observe that $d(T^n_j, T^n_{n_o+j}) \geqslant 1-(2/n) > 1-(3/n)$. Thus if C is a convex compact set of diameter $1-(3/n)$ and $C \cap T^n_j \neq \{\emptyset\}$ then $C \cap T^n_{n_o+j} = \{\emptyset\}$ and e) follows easily.

f) It is easy to check that if C is a convex compact set of diameter $1-(2/5)$ and $C \cap T^5_1 \neq \{\emptyset\}$ then $H^1(C \cap (T^5_3 \cup T^5_4) \cap K_5) = 0$. Using symmetry f) follows.

b) Let $i \geqslant 2$, $n \geqslant 5$ and let Q^n_i be the intersection (see fig.2) of the line L joining V^n_1 and $V^n_{n_o+1}$ and the line L' perpendicular to L such that L' contains the point $\underbrace{Y^n_1(Y^n_1\ldots Y^n_1}_{i}(Y^n_n(V^n_1))\ldots) \in T^n_{\underbrace{11\ldots 1n}_{i}}$.

It is easy to check that

$$d(V^n_1, Q^n_i) = \begin{cases} n^{1-i}.(1-1/n).\sin^2(\pi/n) & \text{if } n \text{ is even} \\ \sqrt{2/(1+\cos(\pi/n))}\ .\ n^{1-i}.(1-1/n).\sin(\pi/n).\sin(\pi/2n) = \\ = 2.n^{1-i}.(1-1/n).\sin^2(\pi/2n) & \text{if } n \text{ is odd} \end{cases}$$

Let, for $n \geqslant 6$, C be a compact convex set of diameter $1-(3/n^i)$. Assume $C \cap T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}} \neq \{\emptyset\}$. Then, from the fact that

$$d(T^n_{\underbrace{1\ldots 1}_{i}}, T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}) = 1-(2/n^i) > 1-(3/n^i)$$

we get $C \cap T^n_{\underbrace{1\ldots 1}_{i}} = \{\emptyset\}$. Also by e) (or f) if n is odd) of lemma 3 $d(V^n_1, Q^n_i) \leqslant 2/n^i$.

Since $T^n_{\underbrace{11\ldots1n}_{i}}$ and $T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}$ are translations one of the other, we can use an argument similar to the one used in a) and get

$$H^1(K_n \cap (T^n_{\underbrace{11\ldots1n}_{i}} \cup T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}) \cap C) \leqslant H^1(K_n)/n^i$$

which combined with the fact that $C \cap T^n_{\underbrace{1\ldots1}_{i}} = \{\emptyset\}$ gives

$$(9) \quad H^1(K_n \cap (T^n_{\underbrace{1\ldots1}_{i}} \cup T^n_{\underbrace{1\ldots1n}_{i}} \cup T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}) \cap C) \leqslant H^1(K_n)/n^i$$

Note that if C does not intersect $T^n_{\underbrace{1\ldots1}_{i}}$ nor $T^n_{\underbrace{n_o+1,\ldots,n_o+1}_{i}}$ then (9) holds. b) follows from (9) and the fact that the same argument can be repeated for all the triples $(T^n_{\underbrace{j\ldots j}_{i}}, T^n_{\underbrace{j\ldots j(j-1)}_{i}}, T^n_{\underbrace{n_o+j,\ldots,n_o+j}_{i}})$, $2 \leqslant j \leqslant n_o$ (for n odd observe that C can only intersect n_o elements of the form $\{T_{\underbrace{j\ldots j}_{i}}\}$ $j = 1,\ldots,n$).

Case d) (n=5) is proved in a similar way using $d(V^5_1,Q^5_i) \leqslant 2/5^i$. ∎

LEMMA 6. *Let* i *be an integer. Then*

a) $\mu(1-(2/3^{i+1}),3) \leqslant (1-(1/3^i)).H^1(K_3)$ *if* $i \geqslant 1$

b) $\mu(1-(1/3^i),3) \leqslant (1-(5/3^{i+1})).H^1(K_3)$ *if* $i \geqslant 1$

c) $\mu(1-(5/3^{i+1}),3) \leqslant (1-(2/3^i)).H^1(K_3)$ *if* $i \geqslant 1$

Proof. Recall that $T^3_{j_1\ldots j_i}$ is an equilateral triangle of base equal to $1/3^i$.

a) Let $i \geqslant 1$ and let C be a convex compact set of diameter

$1-(2/3^{i+1})$. Then if $C \cap (T^3_{\underbrace{1\ldots1}_{i+1}} \cup T^3_{\underbrace{2\ldots2}_{i+1}} \cup T^3_{\underbrace{3\ldots3}_{i+1}}) = \{\emptyset\}$ we have

$$H^1(K_3 \cap C) \leqslant (1-(1/3^i)).H^1(K_3)$$

Therefore we may assume $C \cap T^3_{\underbrace{1\ldots1}_{i+1}} \neq \{\emptyset\}$.

Since $d(T^3_{\underbrace{1\ldots1}_{i+1}}, T^3_{\underbrace{2\ldots2}_{i+1}}) = d(T^3_{\underbrace{1\ldots1}_{i+1}}, T^3_{\underbrace{3\ldots3}_{i+1}}) = 1-(2/3^{i+1})$ (see fig.3) we have

$$H^1(K_3 \cap (T^3_{\underbrace{2\ldots2}_{i+1}} \cup T^3_{\underbrace{3\ldots3}_{i+1}}) \cap C) = 0$$

It is not difficult to check that the segment $[P_{i+1}, Q_{i+1}]$ is perpendicular to $[V^3_1, V^3_2]$. Thus $d(P_{i+1}, V^3_1) > 1-(2/3^{i+1})$ and by an argument similar to that given in lemma 5 a) we have that

$$H^1(K_3 \cap (T^3_{\underbrace{2\ldots23}_{i+1}} \cup T^3_{\underbrace{11\ldots1}_{i+1}}) \cap C) \leqslant H^1(K_3)/3^{i+1}$$

and a) follows.

b) Let $i \geqslant 1$ and C be a convex compact set of diameter $1-(1/3^i)$. Then if $C \cap (T^3_{\underbrace{1\ldots1}_{i}} \cup T^3_{\underbrace{2\ldots2}_{i}} \cup T^3_{\underbrace{3\ldots3}_{i}}) = \{\emptyset\}$ we have

$$H^1(K_3 \cap C) \leqslant (1-1/3^{i-1}).H^1(K_3)$$

Let us suppose that $C \cap T^3_{\underbrace{1\ldots1}_{i}} \neq \{\emptyset\}$. We have, by symmetry, only three subcases:

b1) $C \cap T^3_{\underbrace{1\ldots1}_{i+1}} \neq \{\emptyset\}$

b2) $C \cap T^3_{\underbrace{j\ldots j}_{i+1}} = \{\emptyset\}$, $j = 1,2,3$; $C \cap T^3_{\underbrace{1\ldots12}_{i+1}} = \{\emptyset\}$,

$$C \cap T^3_{\underbrace{1\ldots1}_{i+1}3} \neq \{\emptyset\}$$

b3) $$C \cap T^3_{\underbrace{j\ldots j}_{i+1}} = \{\emptyset\},\ j = 1,2,3\ ;\ C \cap T^3_{\underbrace{1\ldots1}_{i+1}2} \neq \{\emptyset\},$$

$$C \cap T^3_{\underbrace{1\ldots1}_{i+1}3} \neq \{\emptyset\}.$$

b1) It is easy to see that (see fig.3)

$d(T^3_{\underbrace{1\ldots1}_{i+1}}, T^3_{\underbrace{2\ldots2}_{i+1}})$ and $d(T^3_{\underbrace{1\ldots1}_{i+1}}, T^3_{\underbrace{2\ldots2}_{i+1}3}) \geq d(T_{i+1}, Q_{i+1}) = 1-(1/3^i)$

Thus $H^1(K_3 \cap (T^3_{\underbrace{2\ldots2}_{i+1}} \cup T^3_{\underbrace{2\ldots2}_{i+1}3}) \cap C) = 0$ and by symmetry

$H^1(K_3 \cap (T^3_{\underbrace{3\ldots3}_{i+1}} \cup T^3_{\underbrace{3\ldots3}_{i+1}2}) \cap C) = 0$. Also as $d(V^3_1, R^{i+1}) =$

$= 1-(1/3^i)$ we have that $H^1(K_3 \cap (T^3_{\underbrace{1\ldots1}_{i+1}} \cup T^3_{\underbrace{2\ldots2}_{i+1}1}) \cap C) \leq$

$\leq H^1(K_3)/3^{i+1}$.

b2) Since $d(S_{i+1}, P_{i+1}) = 1-(1/3^i)$, it follows that

(10) $$H^1(K_3 \cap (T^3_{\underbrace{1\ldots1}_{i+1}3} \cup T^3_{\underbrace{2\ldots2}_{i+1}3}) \cap C) \leq H^1(K_3)/3^{i+1}$$

b3) From b2) one gets (10) again and by symmetry

$$H^1(K_3 \cap (T^3_{\underbrace{1\ldots1}_{i+1}2} \cup T^3_{\underbrace{3\ldots3}_{i+1}2}) \cap C) \leq H^1(K_3)/3^{i+1}$$

c) Let $i \geq 1$ and let C be a compact convex set of diameter $1-(5/3^{i+1})$. We assume $C \cap T^3_{\underbrace{1\ldots1}_{i}} \neq \{\emptyset\}$ (if

$C \cap (T^3_{\underbrace{1\ldots1}_{i}} \cup T^3_{\underbrace{2\ldots2}_{i}} \cup T^3_{\underbrace{3\ldots3}_{i}}) = \{\emptyset\}$ then $H^1(K_3 \cap C) \leq$

$\leq (1-(1/3^{i-1})).H^1(K_3))$.

Then, by symmetry, only two choices are possible:

c1) $C \cap T^3_{\underbrace{1\ldots1}_{i+1}} \neq \{\emptyset\}$. Consequently, $C \cap (T^3_{\underbrace{2\ldots2}_{i}} \cup T^3_{\underbrace{3\ldots3}_{i}}) = \{\emptyset\}$.

c2) $C \cap T^3_{\underbrace{j\ldots j}_{i+1}} = \{\emptyset\}$, $j = 1,2,3$; $C \cap T^3_{\underbrace{1\ldots12}_{i+1}} \neq \{\emptyset\}$. Then,

$$\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{3\ldots3}_{i}} \cup T^3_{\underbrace{2\ldots2}_{i+1}} \cup T^3_{\underbrace{2\ldots23}_{i+1}} \cup T^3_{\underbrace{1\ldots1}_{i+1}}) \cap C) = 0 \qquad \blacksquare$$

2.2 PROOF OF THEOREM 5

Recall that property Z holds for K_n $n \geq 3$. Thus $\mu(\delta,n)$ is continuous on $(0,\infty)$. Let $f(\delta,n) = \mu(\delta,n)/\delta$. Then, if $1 < \delta$, $f(\delta,n) = \mathcal{H}^1(K_n)/\delta < \mathcal{H}^1(K_n) = f(1,n) \leq 1$ (th.3). Therefore to prove the theorem we must show $\mathcal{H}^1(K_n) \geq 1$. Observe that any number $0 < \Delta_n < \min_{i\neq j} d(T_i,T_j)$ could be used as Δ in property A. Therefore from theorem 3 and 4 we get

i') $f(\delta,n) \leq 1$ on $[\Delta_n, 1]$

ii') $f(\delta_o,n) = 1$ on for some $\delta_o \in [\Delta_n,1]$

From the continuity of $\mu(\delta,n)$ one gets i') and ii') for $\Delta_n = \min_{i\neq j} d(T^n_i,T^n_j)$ ie.

i) $f(\delta,n) \leq 1$ on $[\min_{i\neq j} d(T^n_i,T^n_j),1]$

ii) $f(\delta_o,n) = 1$ for some $\delta_o \in [\min_{i\neq j} d(T^n_i,T^n_j),1]$

We recall formulae (3') of lemma 4

(3')

$$\min_{i\neq j} d(T^n_i,T^n_j) \geq \begin{cases} (1-1/n).\sin(\pi/n)-1/n & \text{if } n \text{ is even, } n \geq 6 \\ \sqrt{2/(1+\cos(\pi/n))}.[(1-1/n).\sin(\pi/n)-1/n] & \text{if } n \text{ is odd, } n \geq 5 \end{cases}$$

and $\min_{i\neq j} d(T^3_i,T^3_j) = 1/3$. Let n be even, $n \geq 8$. Define the functions $g(\delta,n)$ and $h(\delta,n)$ as follows:

$$g(\delta,n) = \begin{cases} 1/n & \text{if } \delta \in [(1-1/n).\sin(\pi/n)-(1/n),(1-1/n).\sin(\pi/n)) \\ 2/n & \text{if } \delta \in [(1-1/n).\sin(\pi/n),\sin(2\pi/n)-(2/n)) \\ (j+1)/n & \text{if } \delta \in [\sin(j\pi/n)-(2/n),\sin((j+1)\pi/n)-(2/n)) \\ & \text{and } 2 \leqslant j \leqslant (n/2)-1 \end{cases}$$

(11)

$$h(\delta,n) = \begin{cases} 1-1/n^i & \text{if } \delta \in [1-1/n^i,1-3/n^{i+1}) \text{ , } i = 1,2,\ldots \\ 1-3/n^{i+1} & \text{if } \delta \in [1-3/n^{i+1},1-1/n^{i+1}) \text{ , } i = 0,1,2,\ldots \\ 1/2 & \text{if } \delta \in [1/2,1-3/n) \end{cases}$$

Then $h(\delta,n)$ is defined on $[1/2,1)$ and $g(\delta,n)$ on $[(1-1/n).\sin(\pi/n)-(1/n),1-2/n)$. Also $h(\delta,n)/\delta \leqslant 1$ and by lemma 3 a,b,c) we get $g(\delta,n)/\delta \leqslant 1$. By lemmas 5 a,b,e), 4 a,b) and from the fact that $\mu(\delta,n)$ is non decreasing we get

(12) $$f(\delta,n)/H^1(K_n) \leqslant h(\delta,n)/\delta \leqslant 1 \quad \text{if} \quad \delta \in [1/2,1)$$

and

$$f(\delta,n)/H^1(K_n) \leqslant g(\delta,n)/\delta \leqslant 1 \quad \text{if}$$

$$\delta \in [(1-1/n).\sin(\pi/n)-(1/n),1-2/n)$$

and using the continuity of $\mu(\delta,n)$

(13) $$f(\delta,n)/H^1(K_n) \leqslant 1 \quad \text{if} \quad \delta \in [\min_{i\neq j} d(T_i^n,T_j^n),1]$$

Using property ii) above we get $H^1(K_n) \geqslant 1$.

Thus $H^1(K_n) = 1$ if n is even, $n \geqslant 8$.

The proof of the other cases are similar.

Let n be odd, $n \geqslant 7$. Define $h(\delta,n)$ as in (11) and

$$g(\delta,n) = \begin{cases} 1/n & \text{if } \delta \in [\sqrt{2/(1+\cos(\pi/n))}.((1-1/n).\sin(\pi/n)-(1/n)), \\ & \quad \sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n)) \\ 2/n & \text{if } \delta \in [\sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n), \\ & \quad \sqrt{2/(1+\cos(\pi/n))}.\sin(2\pi/n)-(2/n)) \\ (j+1)/n & \text{if } \delta \in [\sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n)-(2/n), \\ & \quad \sqrt{2/(1+\cos(\pi/n))}.\sin((j+1)\pi/n)-(2/n)) \\ & \quad 2 \leqslant j \leqslant (n-1)/2 - 1 \end{cases}$$

$g(\delta,n)$ is defined on $[\sqrt{2/(1+\cos(\pi/n))}.((1-1/n).\sin(\pi/n)-(1/n)),$

$\sqrt{2/(1+\cos(\pi/n))}.\sin((n-1)\pi/2n)-(2/n)).$

Using lemma 3 a,b,c,d) we get $g(\delta,n)/\delta \leqslant 1$. By lemma 4 c,d) it follows that $f(\delta,n)/H^1(K_n) \leqslant g(\delta,n)/\delta \leqslant 1$. As we have seen, lemma 5 a,b,e) implies (12). Thus (13) holds and the proof ends as in the previous case.

For n=6, $h(\delta,6)$ is defined as in (11) and

$$g(\delta,6) = \begin{cases} 1/6 & \text{if } \delta \in [(1-1/6).\sin(\pi/6)-1/6,(1-1/6).\sin(\pi/6)) \\ 1/3 & \text{if } \delta \in [(1-1/6).\sin(\pi/6),\sin(\pi/3)-1/3] \end{cases}$$

and the proof runs in a similar way using lemma 3 a),b), lemma 5 a),b),e), and lemma 4 a),b).

For n=5 let

$$h(\delta,5) = \begin{cases} 1-2/5^i & \text{if} \quad \delta \in [1-2/5^i,1-1/5^i) \quad i = 1,2,\ldots \\ 1-1/5^{i-1} & \text{if} \quad \delta \in [1-1/5^{i-1},1-2/5^i) \quad i = 2,3,\ldots \\ 2/5 & \text{if} \quad \delta \in [2/5,1-2/5) \end{cases}$$

$g(\delta,5) = 1/5$, if $\delta \in [\sqrt{2/(1+\cos(\pi/5))}.[(1-1/5).\sin(\pi/5)-1/5],$ $\sqrt{2/(1+\cos(\pi/5))}.(1-1/5).\sin(\pi/5)]$ and use lemmas 5 c,d,f), 4c),3a).

For n=3 we define only one function $g(\delta,3)$ in the following way

$$g(\delta,3) = \begin{cases} 1-2/3^i & \text{if} \quad \delta \in [1-2/3^i,1-5/3^{i+1}) \quad i \geqslant 1 \\ 1-5/3^{i+1} & \text{if} \quad \delta \in [1-5/3^{i+1},1-1/3^i) \quad i \geqslant 1 \\ 1-1/3^i & \text{if} \quad \delta \in [1-1/3^i,1-2/3^{i+1}) \quad i \geqslant 1 \end{cases}$$

Thus $g(\delta,3)$ is defined on [1/3,1) and this case follows from lemma 6.

Case n=3 is considered in [Mn]. Case n=4 may be found in [F] and [Mn]. The proofs given there are different.

2.3. EXAMPLE 2.

The unique compact set K such that

$$K = \bigcup_{i=1}^{4} Y_i(K)$$

where Y_i are similitudes of the complex plane defined by $Y_1(z) = z/3$; $Y_2(z) = z.(1/2+i\sqrt{3}/2)/3+1/3$; $Y_3(z) = z.(1/2-i\sqrt{3}/2)/3+(1/2+i/2\sqrt{3})$; $Y_4(z) = z/3+2/3$, is the well known Koch curve.

It is not difficult to see that $C(K) = C(\{0,1,1/2+i/2\sqrt{3}\})$ and therefore using int $C(K)$ one can prove that an "open set condition" holds for K. Therefore K is self similar (see [F]). Moreover $s = \log 4/\log 3$.

Alternatively K can be defined with only two similitudes ie.

$$K = \bigcup_{i=1}^{2} Y_i'(K)$$

where $Y_1'(z) = z.(-\sqrt{3}/2-i/2)/\sqrt{3}+(1/2+i/2\sqrt{3})$;
$Y_2'(z) = z.(-\sqrt{3}/2+i/2)/\sqrt{3}+1$ (primes will be used to describe elements that arise from this definition).

Property Z holds for K and therefore $\mu(\delta)$ and $f(\delta)$ are continuous. Figure 4 shows how K looks like.

Let C be a compact set of diameter $\delta < 1/3\sqrt{3}$ such that (by theorem 1) $\mu(\delta) = H^s(C \cap K)$. If C intersects T_1' or T_2' but not both then using $Y_1'^{-1}$ (or $Y_2'^{-1}$) one can prove that

$$(1) \qquad \mu(\delta.\sqrt{3}) = (\sqrt{3})^s.H^s(C \cap K)$$

If C intersects both T_1' and T_2' then C can intersect at most the set $\{T_{23},T_{24},T_{31},T_{32}\}$ (fig.5). But

$Y(K \cap (T_{23} \cup T_{24} \cup T_{31} \cup T_{32})) = K \cap \{T_{11} \cup T_{12} \cup T_{13} \cup T_{14}\}$ where Y is a similitude with contraction ratio 1. Therefore one could assume that C only intersects T_1' and (1) holds. Thus we have proved that if $\delta < 1/3\sqrt{3}$ then $f(\delta) = f(\delta.\sqrt{3})$. Therefore theorem 4 holds with $\varepsilon_1 = \Delta$, Δ any number less than $1/3\sqrt{3}$ and $\varepsilon_2 = \Delta.\sqrt{3}$. In fact, in its proof we have only used the thesis of lemma 2. From this lemma and Th.3 we obtain:

i) $f(\delta) \leqslant 1 \qquad \delta \in [1/3\sqrt{3}, 1/3]$

ii) $f(\delta_o) = 1 \qquad$ for some $\delta_o \in [1/3\sqrt{3}, 1/3]$

We note that property A holds for K for some $\Delta << 1/3\sqrt{3}$.

Upper and lower bounds for K had been given in [B]

$$0.026 \approx 2^{-s-4} \leqslant H^s(K) \leqslant 2^{s-2} \approx 0.5995.$$

In [M 2] an alternative proof of the upper bound was given and it was conjectured that $H^s(K) = 2^{s-2}$. But we shall see that indeed $H^s(K) < 2^{s-2}$.

Now to get a lower bound for $H^s(K)$ we need to compute h_r. The following is a table of a function $\tilde{h}_2$ which is an

$\tilde{h}_2(6/16) = 1/3\sqrt{3}$	$\tilde{h}_2(9/16) \approx 0.29397$
$\tilde{h}_2(7/16) = 2/9$	$\tilde{h}_2(10/16) = 1/3$
$\tilde{h}_2(8/16) = 2/9$	$\tilde{h}_2(11/16) = 4/9$

approximation of h_2. We recall the definition of h_2:

$$h_2(\alpha) = \min_{\beta\varepsilon G_2^\alpha} (\max_{\Gamma,\Gamma'\varepsilon\beta} d(\Gamma,\Gamma'))$$

where d(.,.) is the usual distance between sets.

Let $p_1 = 0$, $p_2 = 1$, $p_3 = 1/2+i/2\sqrt{3}$, $p_4 = 1/3$, $p_5 = 2/3$, $p_6 = 1/6+i/6\sqrt{3}$, $p_7 = 1/3+i/3\sqrt{3}$, $p_8 = 2/3+i/3\sqrt{3}$,

$p_9 = 5/6+i/6\sqrt{3}$ and $\Gamma = T_{i_1 i_2}$, $\Gamma' = T_{j_1 j_2}$. Then set

$$\tilde{d}(\Gamma,\Gamma') = \min_{1\le k,m\le 9} d(Y_{i_1}\circ Y_{i_2}(p_k),\ Y_{j_1}\circ Y_{j_2}(p_m))$$

and define $\tilde{h}_2$ in the following way:

$$\tilde{h}_2(\alpha) = \min_{\beta\varepsilon G_2^\alpha} \left(\max_{\Gamma,\Gamma'\varepsilon\beta} \tilde{d}(\Gamma,\Gamma')\right)$$

Notice that $\tilde{d}(\Gamma,\Gamma') - 1/54 \leqslant d(\Gamma,\Gamma') \leqslant \tilde{d}(\Gamma,\Gamma')$. Therefore

$$\tilde{\tilde{h}}_2 := \tilde{h}_2 - 1/54 \leqslant h_2 \leqslant \tilde{h}_2$$

and if we define $\tilde{\tilde{U}}_2(\delta) := \max\{\alpha\colon \tilde{\tilde{h}}_2(\alpha) \leqslant \delta\}$ then $U_2 \leqslant \tilde{\tilde{U}}_2$. h_2 and $\tilde{h}_2$ are non decreasing. (This is a general fact: H_r and h_r are non decreasing functions if $k_i = k_1$ for all i). To compute the supremum of $\tilde{\tilde{U}}_2$ on $[1/3\sqrt{3},1/3]$ we do not need all the values of $\tilde{h}_2$ but only those displayed in the table above.

From i,ii) above and theorem 2 a) we get

$$1/\tilde{\tilde{B}}_2 \leqslant 1/B_2 \leqslant H^s(K)$$

where $\tilde{\tilde{B}}_2 = \sup_{\delta\varepsilon[1/3\sqrt{3},1/3]} \tilde{\tilde{U}}_2(\delta)/\delta^s$; $B_2 = \sup_{\delta\varepsilon[1/3\sqrt{3},1/3]} U_2(\delta)/\delta^s$. Using the table it is easy to compute $\tilde{\tilde{B}}_2$. We have $\tilde{\tilde{B}}_2 \approx 3.723$ and

$$0.26 \leqslant H^s(K)$$

We compute now an upper bound. Observe that $Q = \{T_{212} \cup T_{213} \cup T_{214} \cup T_{221} \cup T_{222} \cup T_{223} \cup T_{224} \cup T_{231} \cup T_{232} \cup T_{233} \cup T_{234} \cup T_{241} \cup T_{242} \cup T_{243} \cup T_{244} \cup T_{311} \cup T_{312} \cup T_{313} \cup T_{314} \cup T_{321} \cup T_{322} \cup T_{323} \cup T_{324} \cup T_{331} \cup T_{332} \cup T_{333} \cup T_{334} \cup T_{341} \cup T_{342} \cup T_{343}\}$ has diameter $\delta' = \sqrt{(292/243)}/3 \approx 0.36539$ (see fig.6) and that $H^s(Q \cap K) = (30/64)H^s(K)$.

Therefore

$$30.\mathcal{H}^s(K)/(64.\delta'^s) \leqslant \mu(\delta')/\delta'^s \leqslant 1$$

ie. $\mathcal{H}^s(K) < 0.5989 < 0.5995 \approx 2^{s-2}$.

The numbers displayed in example 2 are all exact up to the last digit.

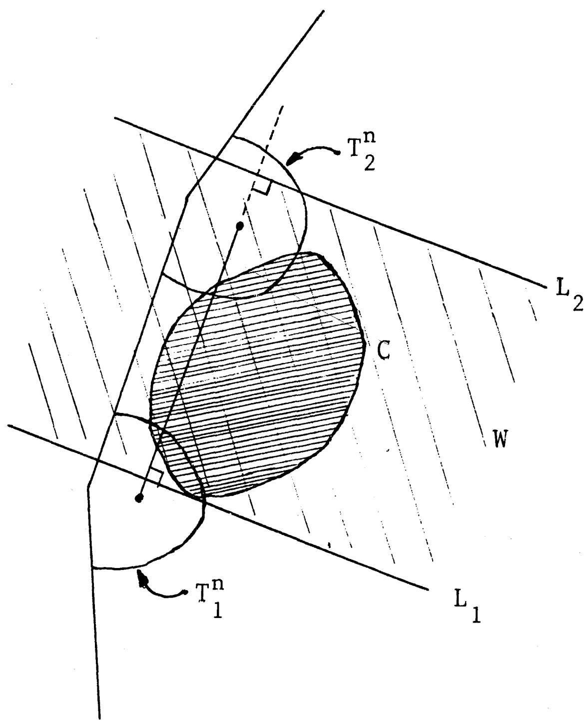
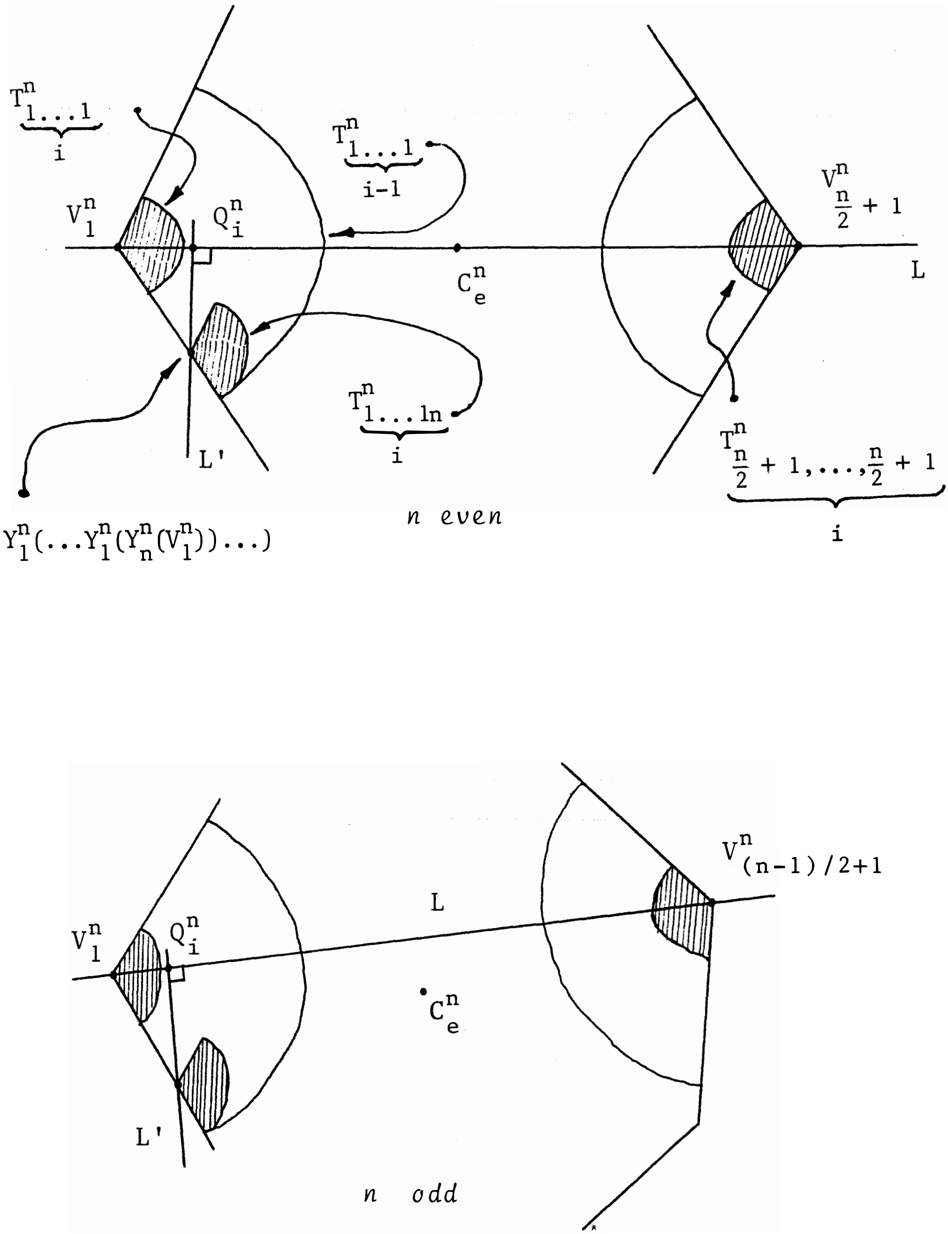


Figure 1



$T^n_{\underbrace{1\ldots1}_{i}}$
$T^n_{\underbrace{1\ldots1}_{i-1}}$
V^n_1
Q^n_i
C^n_e
$V^n_{\frac{n}{2}+1}$
L
$T^n_{\underbrace{1\ldots1n}_{i}}$
L'
$T^n_{\underbrace{\frac{n}{2}+1,\ldots,\frac{n}{2}+1}_{i}}$
$Y^n_1(\ldots Y^n_1(Y^n_n(V^n_1))\ldots)$
n even
$V^n_{(n-1)/2+1}$
L
V^n_1
Q^n_i
C^n_e
L'
n odd

Figure 2

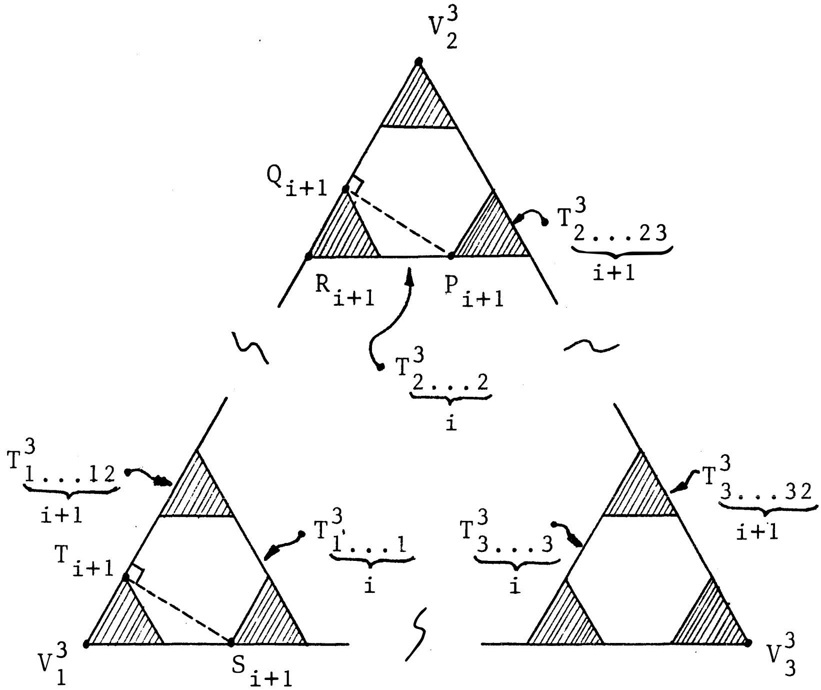


Figure 3

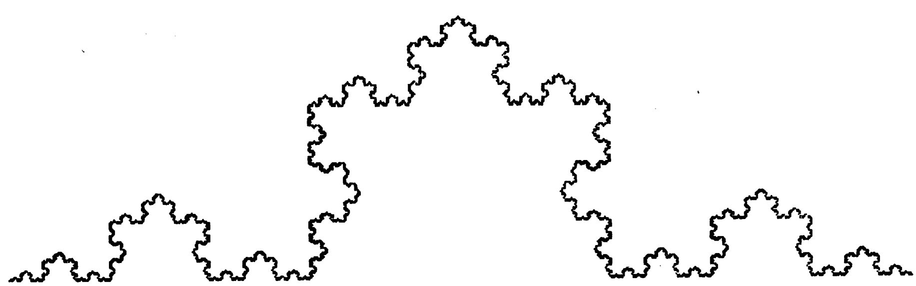


Figure 4

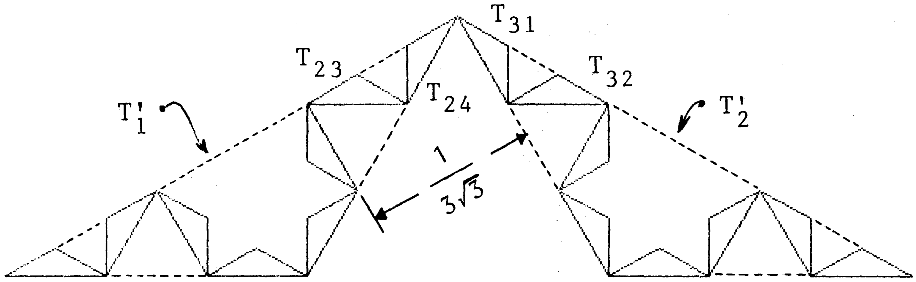
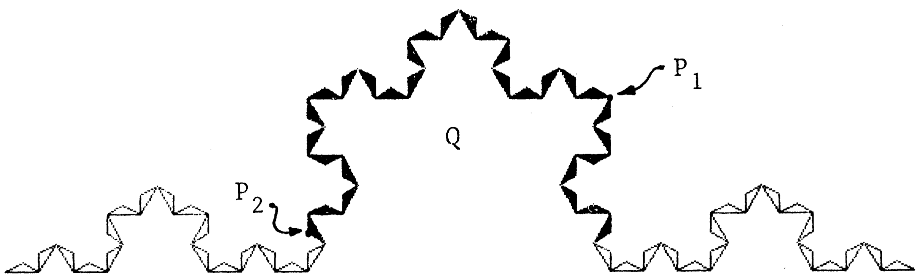
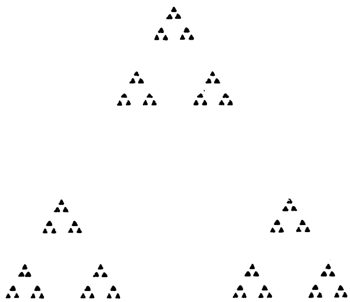


Figure 5



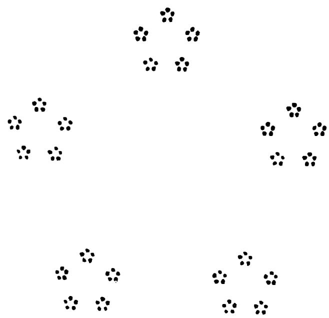
$\delta' = d(P_1, P_2)$

Figure 6



K_3

Figure 7



K_5

Figure 8

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Departamento e Instituto de Matemática

Universidad Nacional del Sur.
Av. Alem 1253. (8.000) Bahía Blanca.
Argentina.

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