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THE CORE-STABLE SETS-THE BARGAINING THEORY FROM A FUNCTIONAL AND MULTI CRITERION VIEWPOINT

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Abstract:

In this paper, it is introduced the concept of f-imputation from which the core is defined and a theorem of analogous characterization to that given in Owen(1982)is proved.

Also, the bargaining theory from the viewpoint analogous to that developed in Davis and Maschler(1963)and Peleg(1963)is exposed.

1. Introduction

In his excellent book [2], G.Owen provides a characterization of the core of a game as a subset of $\mathbb{R}^n$. There, he defines the usual notions of imputations and domination for cooperative n-person games. Davis-Maschler and Peleg in [1] and [3], introduce the notion of stable coalitions, bargaining sets and prove existence theorems for the bargaining set $\mathcal{M}_1^{(i)}$ in euclidean spaces.

In this paper we introduce the concept of f-imputation which generalizes the classical notion of imputation. We also extend the concept of core. In particular, we characterize the latter as a subset of a topological space. Besides, in the same framework, we study the bargaining set and prove an existence theorem only assuming the continuity of the function f.

X will indicate a compact connected subset of a topological space.
N will indicate a finite set of index, card(N)=n.
v will indicate a defined function on the subsets of N to nonnegative real values such that:

$$v(\phi)=0 \qquad (1\text{-}1)$$

$$v(S\cup T) \geq v(S)+v(T) \quad , \quad S\cap T=\phi \qquad (1\text{-}2)$$

For all $i \in N$ let $f_i : X \to [0,\infty)$ and we indicate $f : X \to [0,\infty)^n$ to the application defined by

$$f(x) = \{f_i(x)\}_{i \in N}$$

Definition 1-1: An element $x \in X$ is an *f-imputation* for a game v, if :

$$\begin{cases} i) \sum_{i \in N} f_i(x) = v(N) \\ ii) f_i(x) \geq v(\{i\}) \quad \text{for all } i \in N \end{cases} \qquad (1\text{-}3)$$

Definition 1-2: Let x and y be two f-imputations, $S \subset N$, then we say that *x dominates y through S* and we denote this by $x \underset{S}{>} y$, if

$$\begin{cases} i) f_i(x) > f_i(y) \quad \text{for all } i \in S \\ ii) \sum_{i \in S} f_i(x) \leq v(S) \end{cases} \qquad (1\text{-}4)$$

2 - **The Core**

Definition 2-1: The set of all undominated f-imputations for a game v, wil be called *core* and we will denote it by C(v)

Theorem 2.1: Let $f : X \to [0,\infty)^n$ be surjective, then the core for game v is the set of all $x \in X$ that satisfy:

$$\begin{cases} i) \sum_{i \in S} f_i(x) \geq v(S) \quad \text{for all } S \subset N \\ ii) \sum_{i \in N} f_i(x) = v(N) \end{cases} \qquad (2\text{-}1)$$

Proof:

Let x be (2-1) i) and ii)

If $S=\{i\}$ the condition i) means that $f_i(x) \geq v(\{i\})$ that together with the condition ii) means that x is an f-imputation.

x is undominated, in fact, let us suppose that there exists $y \in X$ and $S \subset N$ such that $f_i(y) > f_i(x)$ for all $i \in S$, but this together with (2-1) i) means

$$\sum_{i\in S} f_i(y) > v(S)$$

and this contradicts (1-4) ii).Hence $x\in C(v)$.

Conversely, suppose that y does not satisfy (2-1) i) or ii).

If ii) fails,y is not an f-imputation and hence $y\notin C(v)$.

If y is such that it does not verify i) then there exists $S\subset N$ such that $\sum_{i\in S} f_i(y) < v(S)$;this is $\sum_{i\in S} f_i(y) = v(S)-\varepsilon$ with $\varepsilon > 0$.

Let $\alpha = v(N)-v(S)-\sum_{i\in S} v(\{i\})$ and $\Delta = \text{card}(S)\ \alpha \geq 0$.

Let $t=\{t_i\}_{i\in N}\in [0,\infty)^n$ where

$$t_i = \begin{cases} f_i(y)+\frac{\varepsilon}{\Delta} & \text{if } i\in S \\ v(\{i\})+\frac{\alpha}{n-\Delta} & \text{if } i\notin S \end{cases}$$

then by the surjectivity of f,there exists $z\in X$ such that $f(z)=t$,then :

$$f_i(z) = \begin{cases} f_i(y)+\frac{\varepsilon}{\Delta} & \text{if } i\in S \\ v(\{i\})+\frac{\alpha}{n-\Delta} & \text{if } i\notin S \end{cases}$$

Clearly z is an f-imputation and $z \underset{S}{>} y$,then $y\notin C(v)$. ■

3-The Bargaining Theory

Hence forth,let us suppose that $v:\mathcal{P}(N) \to [0,1]$ is such that

$$\begin{cases} \text{i)} \quad v(\{i\}) = 0 \\ \text{ii)} \quad v(N) = 1 \end{cases} \qquad (3\text{-}1)$$

moreover properties (1-1) and (1-2).

For each $i\in N$, $f_i:X \to [0,1]$ is continuous and $f:X\to[0,1]^n$ is surjective.

<u>Definition</u> <u>3-1</u>:By an *f-coalition structure* (f.c.s.) for $N=\{1,2,..n\}$ we shall mean a partition

$\mathcal{T}=\{T_1,T_2,\ldots\ldots,T_m\}$ of N

Definition 3-2 : An *f-payoff configuration* (f.p.c.) for a game v is : $(x;\mathcal{T}) = (f_1(x),\ldots,f_n(x);T_1,\ldots,T_m)$, where $\mathcal{T}$ is an f-coalition structure (f.c.s.) and $x\in X$ is such that

$$\sum_{i\in T_k} f_i(x) = v(T_k) \qquad \text{for } k = 1,2,\ldots,m$$

Definition 3-3: Given a f-payoff configuration as in definition 3-2 , we say that it is *individually rational* (i.r.f.p.c.) for a game v if it verifies that

$$f_i(x) \geq v(\{i\}) = 0 \qquad \text{for all } i\in N$$

y is *coalitionally rational* (c.r.f.p.c.) for a game v if verifies that

$$\sum_{i\in S} f_i(x) \geq v(S) \qquad \text{for } S \subset T_k \in \mathcal{T}$$

Definition 3-4 : Let $(x;\mathcal{T})$ be a c.r.f.p.c. for a game v and let μ and λ ($\mu \neq \lambda$) be belonging to an f-coalition T_j of $\mathcal{T}$.

An *f-objection of λ against μ in* $(x;\mathcal{T})$ is a vector $f^C(y) = (f_k(y))_{k\in C}$ where C is an f-coalition containing λ but not μ, and where its coordinates satisfy :

$$f_\lambda(y) > f_\lambda(x)$$

and

$$f_k(y) \geq f_k(x) \qquad (k\neq\lambda;\, k\in C)$$

and

$$\sum_{k\in C} f_k(y) = v(C)$$

Definition 3-5: As in definition 3-4, an *f-counter objection* to this f-objection is a vector $f^D(z) = (f_k(z))_{k\in D}$, where D is an f-coalition containing μ but not λ and whose coordinates satisfy

$$f_k(z) \geq f_k(x) \qquad \text{for each } k\in D$$

and

$$f_k(z) \geq f_k(y) \qquad \text{for each } k\in D\cap C$$

and

$$\sum_{k\in D} f_k(D) = v(D)$$

Definition 3-6 : We say that $\imath$ *is stronger than* κ (or equivalently, that κ *is weaker than* $\imath$) in $(x;\mathcal{T})$ if $\imath$ has an f-objection against κ which cannot be f-countered.
We denote this by $\imath \gg \kappa$. We say that $\imath$ and κ are equal if neither $\imath \gg \kappa$ nor $\kappa \gg \imath$. We denote this by $\imath \sim \kappa$.
Remark : By definition $\imath \sim \kappa$ in $(x;\mathcal{T})$ if $\imath$ and κ belong to different f-coalitions.

Definition 3-7: An f-coalition T_j in $\mathcal{T}$ is called *f-stable* in $(x;\mathcal{T})$ if each two of its members are equal.

Definition 3-8 : The set of all f-stable individually rational f-payoff configurations is called the *f-bargaining set* and we denote it by $\mathcal{M}_1^{(\imath)}(f)$.
Given an f-coalition structure $\mathcal{T}$, we denote $X(\mathcal{T})$ the set of $x \in X$ such that $(x;\mathcal{T})$ is an i.r.f.p.c.

Lemma 3-1 : Let $c_1(x), c_2(x), \ldots, c_n(x)$ be continuous functions defined for $x \in X(\mathcal{T})$ to nonnegative real values.
If, for each $x \in X(\mathcal{T})$ and for each $T_j \in \mathcal{T}$ there exists $\imath \in T_j$ such that $c_\imath(x) \geq f_\imath(x)$ then, there exists $\xi \in X(\mathcal{T})$ such that $c_\imath(\xi) \geq f_\imath(\xi)$ for each $\imath \in N$.

Proof:

For $x \in X(\mathcal{T})$ and $\imath \in N$ we denote, using the surjectivity of f,

$$f_\imath(z) = \begin{cases} f_\imath(x) - c_\imath(x) & \text{if } f_\imath(x) \geq c_\imath(x) \\ 0 & \text{if } f_\imath(x) \geq c_\imath(x) \end{cases} \tag{3-2}$$

and if $\imath \in T_j$

$$f_\imath(y) = f_\imath(x) - f_\imath(z) + \frac{1}{\tau_j} \sum_{k \in T_j} f_k(z) \tag{3-3}$$

where $\tau_j = \mathrm{card}(T_j)$

It is clear that $f(y)$ is a continuous function of $f(x)$. Moreover, it can be see that $f_\imath(y) \geq 0$ and $\sum_{\imath \in T_j} f_\imath(y) = v(T_J)$ and as $0 = v\{(\imath)\} \leq f_\imath(y)$ then $y \in X(\mathcal{T})$.

Let us suppose now $f_\imath(x) > c_\imath(x)$. This means that $f_\imath(z) > 0$. Moreover, there exists $\kappa \in T_j$ such that $f_k(x) \leq c_k(x)$, then by (3-2), $f_k(z) = 0$. Hence

$$f_k(y) \geq f_k(x) + \frac{f_i(z)}{\tau_j} > f_k(x)$$

then $f(x)$ is not a fixed point by the application of $[0,1]^n$ in $[0,1]^n$ that to $f(x)$ it assigns $f(y)$ defined in (3-3). Then ,by Brouwer's fixed point theorem, there exists $\xi \in X(\mathcal{T})$ such that

$$f_i(\xi) = f_i(\xi) - f_i(z) + \frac{1}{\tau_j} \sum_{k \in T_j} f_k(z)$$

and clearly, this means by (3-2) that

$$f_i(\xi) \leq c_i(\xi) \qquad \text{for all } i \in N$$

■

Definition 3-9 : Let $(x;\mathcal{T})$ be an i.r.f.p.c., and let C be an f-coalition. Then the *f-excess of* C is

$$e(C) = v(C) - \sum_{i \in C} f_i(x)$$

Lemma 3-2 : If in $(x;\mathcal{T})$, λ has an f-objection $f^C(y)$ against μ and this f-objection cannot be f-countered , then each f-coalition D, for $\mu \in D$, and $e(D) \geq e(C)$, must contain λ.

proof:

Let us suppose that $e(D) \geq e(C)$ and $\lambda \notin D$ we shall see that there exists $z \in X$ such that $f^D(z)$ is an f-counter objection of μ against λ.

Let $z \in X$, such that

$$f_k(z) = \begin{cases} f_k(y) & \text{if } k \in C \cap D \\ f_k(x) + \varepsilon_k & \text{if } k \in D - C \end{cases} \qquad (3\text{-}4)$$

We compute $\varepsilon_k \geq 0$

In fact, by hypothesis:

$$v(D) - v(C) + \sum_{C-D} f_k(x) - \sum_{D-C} f_k(x) \geq 0 \qquad (3\text{-}5)$$

and

$$v(D) = v(C) - \sum_{C-D} f_k(x) + \sum_{D-C} f_k(x) + \sum_{D-C} \varepsilon_k$$

Then, by (3-5)

$$\sum_{D-C} \varepsilon_k = v(D) - v(C) + \sum_{C-D} f_k(x) - \sum_{D-C} f_k(x) \geq 0$$

Selecting

$$\varepsilon_k = \frac{v(D)-v(C)+\sum_{C-D} f_k(x)-\sum_{D-C} f_k(x)}{\text{card } (D-C)} \geq 0$$

there results that $f^D(z)$ is an f-counter objection.

■

Lemma 3-3: Let $(x;\mathcal{T})$ be an i.r.f.p.c. Then, the relation $\gg$ is acyclic.

proof:

It is clear that if i and k are in different f-coalitions, then $i \sim k$. Let us suppose that an f-coalition $T_i \in \mathcal{T}$ is such that $T_i=\{1,2,..,t\}$ and that $1 \gg 2 \gg 3 \gg ..... \gg t$.

Then each $i \in T_i$ has an f-objection through the f-coalition C against $i+1 (\text{mod} t)$, which cannot be f-counter objected.

Let C_{i_o} be f-coalition (among $C_1,.....,C_t$) which has maximal f-counter objected.

We claim that i_o can f-counter object against i_o-1 (mod t) through the f-coalition C_{i_o}. Clearly i_o-1(mod t) has only the amount $e(C_{i_o-1})$ at his disposal to from the f-objecting coalition ; having i_o the amount $e(C_{i_o}) \geq e(C_{i_o-1})$ at his disposal , can always f-counter object unless $i_o-1(\text{mod } t) \in C_{i_o}$.

Repeating this argument, we must have $i_o-2(\text{mod } t) \in C_{i_o}$, etc., and eventually $i_o+1(\text{mod } t) \in C_{i_o}$. But this is obviously impossible.

■

Theorem 3-1 : Given v as in (3-1) , and $\mathcal{T}$ any f-structure coalition. Then there exists at least $x \in X$ such that $(x;\mathcal{T}) \in \mathcal{M}_1^{(i)}(f)$.

proof:

Let $(x;\mathcal{T})$ be an i.r.f.p.c.

We denote by $(y^{T_j}, x^{N-T_j};\mathcal{T})$ the i.r.f.p.c. which is obtained by keeping $f_i(x)$ fixed for $i \in N-T_j$ and replacing $f_k(x)$ by $f_k(y)$ for $k \in T_j$ where $f_k(y) \geq 0$ and $\sum_{k \in T_j} f_k(y) = v(T_j)$.

Let $E^i_j(x)$ be the set of points y^{T_j} such that in the i.r.f.p.c. $(y^{T_j}, x^{N-T_j}, \mathcal{T})$, i $(i \in T_j)$ is not weaker than any other $j \in N$. The set $E^i_j(x)$ is closed and contains the set of y from the face $f_i(y)=0$ of simplex Δ_j (since, if $f_i(y)=0$, i can f-counter object with an f-coalition of only one element).

We define the function

$$c_i(x)=f_i(x)+ \max_{y^{T_j} \in E^i_j(x)} \min_{k \in T_j} (f_k(x)-f_k(y)) \qquad (3\text{-}6)$$

where T_j is the f-coalition in $\mathcal{T}$ that contains i. It can be easily seen that $c_i(x)$ is continuous as function of x ; since $E^i_j(x)$ is upper and lower semi-continuous.

$E^i_j(x)$ is upper semi-continuous since given $x_n \to x$; $y_n \to y$ with $y^{T_j}_n \in E^i_j(x_n)$.

For each $y^{T_j}_n \in E^i_j(x_n)$ in each i.r.f.p.c. $(y^{T_j}_n, x^{N-T_j}_n; \mathcal{T})$ i is not weaker than any other $j \in N$, i.e., i has an f-objection $f^C(z)$ against each $j \in N$ which cannot be f-counter objected. Then

$$f_i(z) > f_i(y_N)$$

$$f_k(z) \geq f_k(y_N) \qquad \text{for } k \in C \subset T_j$$

$$\sum_{k \in C} f_k(z) = v(C)$$

and for all $f^D(t)$ where D is any f-coalition such that $i \notin D$

$$f_k(t) < f_k(y_n) \qquad \text{for some } k \in D$$

or

$$f_k(t) < f_k(z) \qquad \text{for some } k \in D \cap C$$

or

$$\sum_{k \in C} f_k(t) \neq v(D)$$

Then considering the continuity of f_k, there results $y^{T_j} \in E^i_j(x)$ and E^i_j is upper semi-continuous.

$E^i_j(x)$ is lower semi-continuous. In fact, let us suppose $x_n \to x$, and for all sequence $y_n \to y$ there exists $\eta_\circ$ such that $y^{T_j}_{\eta_\circ} \notin E^i_j(x_{\eta_\circ})$. We shall prove that

$y^{T_j} \notin E_j^i(x)$.

By the assumption, there exists $\mu \notin T_j$ and $f^{T_j}(z)$ such that

$$f_\mu(z) > f_\mu(y_{\eta_\circ})$$

and

$$f_k(z) \geq f_k(y_{\eta_\circ}) \qquad \text{for } k \in T_j$$

then, by the continuity of f_k, there results $y_{\eta_\circ}^{T_j} \notin E_j^i(x_{\eta_\circ})$ and $E_j^i(x)$ is lower semi-continuous. Moreover, it can be seen that $c_i(x)$ is nonnegative. Then, by Lemma 3-3, for any $x \in X(\mathcal{T})$ and any $T_j \in \mathcal{T}$, there exists $i \in T_j$ such that i is not weaker than any $k \in T_j$; then

$$x^{T_j} \in E_j^i(x) \quad \text{and} \quad c_i(x) \geq f_i(x)$$

Then, by Lemma 3-1 there exists ξ such that $c_i(\xi) \geq f_i(\xi)$ for all $i \in N$. Moreover, it is clear that

$v(T_j) = \sum_{k \in T_j} f_k(\xi) = \sum_{k \in T_j} f_k(y)$, and $c_i(\xi) \leq f_i(\xi)$ for all i, since, if

there exists $i_\circ \in N$ such that $c_{i_\circ}(\xi) > f_{i_\circ}(\xi)$, then

$$\mathop{\mathrm{Max}}_{y^{T_j} \in E_j^i(\xi)} \; \mathop{\mathrm{Min}}_{k \in T_j} \; (f_k(\xi) - f_k(y)) > 0$$

and there exists $y^{T_j} \in E_j^i(\xi)$ such that for all $k \in T_j$, $f_k(\xi) > f_k(y)$, then

$$\sum_{k \in T_j} f_k(\xi) > \sum_{k \in T_j} f_k(y)$$

which contradicts (3-7). Hence, there results $c_i(\xi) = f_i(\xi)$ for all i.

But this means that there exists $y \in E_j^i(\xi)$ for all i, such that $f_k(y) = f_k(\xi)$ and therefore $\xi^{T_j} \in E_j^i(\xi)$ for each i and each j.

Then, in $(\xi;\mathcal{T})$ no member is stronger than another. This means that $(\xi;\mathcal{T}) \in \mathcal{M}_1^{(i)}$

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