Revista de la
Unión Matemática Argentina
Volumen 38, 1993.

ABOUT THE L^p-BOUNDEDNESS OF SOME INTEGRAL OPERATORS

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ABSTRACT. In this note we prove the L^p-boundedness, $1<p<\infty$, and the weak type 1-1 of integral operators with kernels of the form $\Omega(x,y)|x-y|^{-\alpha}|x+y|^{-n+\alpha}$, with Ω in $L^\infty(\mathbb{R}^{2n})$ and $0<\alpha<n$.

1. INTRODUCTION.

In [R-S], authors show the boundedness in $L^2(\mathbb{R})$ of the operator

$$Tf(x) = \int |x-y|^{-\alpha} |x+y|^{\alpha-1} f(y)\, dy \qquad (1.1)$$

for $0<\alpha<1$.

The fundamental tool for the proof of this result is the following generalization of Schur's boundedness criterion.

LEMMA I. Let $K\geq 0$ *be a measurable function on* R^{2n}. *Assume that there exist a function* $g \in L^1_{loc}(R^n)$ *with* $g>0$ *a.e.such that*

$$Tg(x) = \int K(x,y)\, g(y)\, dy \leq c\, g(x) \qquad a.e.$$

and

$$T^* g(x) = \int K(y,x)\, g(y)\, dy \leq c\, g(x) \qquad a.e.$$

Then the operator T defined by the kernel K is bounded on $L^2(R^n)$.

[1] Partially supported by CONICOR and SECYT (U.N.C.)

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For the proof of lemma 1 see [B-H-S]. As this is an n-dimensional result, we can expect that operators of the form

$$Tf(x) = \int \Omega(x,y)|x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ f(y)\ dy \tag{1.2}$$

with Ω in $L^{\infty}(\mathbb{R}^{2n})$ and $0<\alpha<n$, be bounded on $L^2(\mathbb{R}^n)$. Indeed, we obtain the boundedness of (1.2) on $L^p(\mathbb{R}^n)$, $1<p<\infty$.

2.THE MAIN RESULT.

THEOREM I. Let Ω be afunction of $L^{\infty}(R^{2n})$. Then for $0<\alpha<n$, the operator given by (1.2) is bounded on $L^p(R^n)$, $1<p<\infty$.

PROOF. $|Tf(x)| \le \|\Omega\|_{\infty} \int |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |f(y)|\ dy$. We will show that the operator T_1 with kernel $|x-y|^{-\alpha}\ |x+y|^{-n+\alpha}$ is bounded on $L^2(\mathbb{R}^n)$ and of weak type 1-1. The Marcinkiewicz interpolation theorem will then provide the boundedness of T_1 on $L^p(\mathbb{R}^n)$ for $1<p<2$. As it is self adjoint, it is also bounded on $L^p(\mathbb{R}^n)$ for $2<p<\infty$, and the theorem follows. See ([S]) for details.

To study T_1, we set $k(x,y) = |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}$. This kernel satisfies the hypothesis of lemma 1. We take $g(x) = |x|^{-\beta}$, for some $0<\beta<n$. It is enough to obtain, for some constant $c>0$, $\int k(x,y)\ g(y)\ dy \le c\ g(x)$, for almost every x. For $x \neq 0$.

$$\int |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |y|^{-\beta}\ dy = \int_{A_1} + \int_{A_2} + \int_{A_3} + \int_{A_4}$$

with $A_1 = \{\ y: |y-x| \le |x|/2\ \}$, $A_2 = \{\ y: |y+x| \le |x|/2\ \}$, $A_3 = \{\ y: |y| \le |x|/2\}$ and $A_4 = (A_1 \cup A_2 \cup A_3)^c$.

Now, if y belongs to A_1, $|y+x| = |y+2x-x| \ge 2|x|-|y-x| \ge |x|$ and $|y| \ge |x|-|x-y| \ge |x|/2$. Then

$$\int_{A_1} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |y|^{-\beta}\ dy \le 2^{\beta}|x|^{-n+\alpha-\beta} \int_{\Sigma} \int_{0<r<|x|/2} r^{-\alpha+n-1}\ dr\ d\sigma =$$

$$=2^{\beta+\alpha-n}(n-\alpha)^{-1}|\Sigma|\,|x|^{-\beta} = c_1\ g(x).$$

Similarly, $\int_{A_2} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |y|^{-\beta}\ dy \le c_2\ g(x)$.

For $y \in A_3$, $|x-y| \ge |x|-|y| \ge |x|/2$, and also, $|x+y| \ge |x|/2$. So

$$\int_{A_3} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |y|^{-\beta}\, dy \le 4^n |x|^{-n} \int_{A_3} |y|^{-\beta} dy =$$

$$= 4^n |x|^{-n} |\Sigma| \int_{0<r<|x|/2} r^{-\beta+n-1}\, dr = c_3\ g(x).$$

To estimate the fourth integral, we define $B = \{\ y:|y|<3|x|\ \}$ and we observe that for $y\in B^c$, $|x-y| \ge |y|-|x| \ge 2|y|/3$; also $|x+y| \ge 2|y|/3$. So

$$\int_{A_4} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |y|^{-\beta} dy = \int_{A_4\cap B} + \int_{A_4 \cap B^c} \le$$

$$2^{n+\beta} |x|^{-n-\beta} \int_B dy\ + (3/2)^n \int_{B^c} |y|^{-n-\beta} dy = c_4\ g(x).$$

Then lemma 1 implies the boundedness of T_1 on $L^2(\mathbb{R}^n)$. It remains to check the weak type 1-1 of T_1.We set $E_\lambda = \{\ x:|T_1 f(x)| > \lambda\}$. Then $E_\lambda \subseteq E^1_\lambda \cup E^2_\lambda \cup E^3_\lambda$, where

$$E^1_\lambda = \{\ x: \int_{A_1} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |f(y)|\ dy > \lambda/3\ \},$$

$$E^2_\lambda = \{\ x: \int_{A_2} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |f(y)|\ dy > \lambda/3\ \} \text{ and}$$

$$E^3_\lambda = \{\ x: \int_{(A_1\cup A_2)^c} |x-y|^{-\alpha}\ |x+y|^{-n+\alpha}\ |f(y)|\ dy > \lambda/3\ \}$$

Now for $x\in E^1_\lambda$ we have $\lambda/3 \le |x|^{\alpha-n} \int_{A_1} |x-y|^{-\alpha}\ |f(y)|\ dy =$

$$= |x|^{\alpha-n} \sum_{j\in N} \int_{2^{-j-1}|x|<|x-y|<2^{-j}|x|} |x-y|^{-\alpha}\ |f(y)|\ dy \le$$

$$|x|^{-n} \sum_{j\in N} 2^{(j+1)\alpha} \int_{|x-y|<2^{-j}|x|} |f(y)|\ dy \le 2^{\alpha} Mf(x) \sum_{j\in N} 2^{j(\alpha-n)}$$

where Mf denotes the Hardy Littlewood maximal function of f. As it is of weak type 1-1, we get $|E^1_\lambda| \le a_1 \|f\|_1/\lambda$, for some positive constant a_1.

In a similar way we obtain, for $x\in E^2_\lambda$,
$\lambda/3 \le 2^{n-\alpha} Mf(-x) \sum_{j\in N} 2^{-j\alpha}$, and so $|E^2_\lambda| \le a_2 \|f\|_1/\lambda$, for another positive constant a_2.

If $x\in E^3_\lambda$, $\lambda/3 \le 2^n |x|^{-n} \|f\|_1$ and so $x\in B(0,2(3\|f\|_1/\lambda)^{1/n})$, then $|E^3_\lambda| \le a_3 \|f\|_1/\lambda$, for some $a_3>0$.

So T_1 is of weak type 1-1 and the theorem follows. ■

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Recibido en noviembre de 1992.