Revista de la
Unión Matemática Argentina
Volumen 38, 1993.



# DYNAMIC BEHAVIOR OF POSITIVE SOLUTIONS TO REACTION–DIFFUSION PROBLEMS WITH NONLINEAR ABSORPTION THROUGH THE BOUNDARY

by
JULIÁN LÓPEZ GÓMEZ
VIVIANA MÁRQUEZ
and
NOEMÍ WOLANSKI*

## 1. Introduction

In this paper we study a reaction–diffusion problem with nonlinear absorption through the boundary. We are interested in the existence of positive global solutions.

We find out how the existence of such a solution depends on the relation of the initial datum with the nonnegative solution/solutions to the associated stationary problem.

We give a precise description of the set of positive stationary solutions and the long time behavior of positive global solutions to the evolutionary problem. This description depends on some parameters appearing in the problem.

The problem we consider is

$$\begin{aligned} &u_t = u_{xx} - \lambda u^p \quad 0 < t < T\,, \quad 0 < x < R\,,\\ &u_x(0,t) = 0\,, \quad u_x(R,t) = u^q(R,t) \quad t > 0\,,\\ &u(x,0) = u_0(x) \quad 0 < x < R\,,\end{aligned} \tag{1.1}$$

where $\lambda > 0$, $p, q > 1$ are given constants and $u_0 > 0$, $u_0 \in C^{2+\alpha}[0,R]$ satisfies the compatibility condition,

$$u_0'(0) = 0 \quad , \qquad u_0'(R) = u_0^q(R)\,.$$

Under these conditions there exists a unique maximal solution ([Al], [Am], [An]). From the compatibility condition the smoothness of $u(x,t)$ up to the boundaries $x = 0$ and $x = R$ is deduced (see [Al] for the smoothness of $u$ in $[0,R] \times (0,T)$ and [LMW] for an argument that applied to (1.1) gives the smoothness up to $t = 0$). So that $u \in C^{2+\alpha,1+\frac{\alpha}{2}}([0,R] \times [0,T))$.

When $\lambda = 0$ it was shown in [LMW] (see also [F]) that no global solution exists. In this paper we show that when $\lambda > 0$ a completely different picture arises since positive stationary solutions exist in most cases.

Thus as a first step towards the understanding of the dynamic behavior of the solutions to (1.1) for different initial values we analyze in Section 2 the stationary problem,

$$\begin{aligned} &u_{xx} = \lambda u^p \qquad 0 < x < R,\\ &u_x(0) = 0, \quad u_x(R) = u^q(R),\end{aligned} \tag{1.2}$$

*Member of Consejo Nacional de Investigaciones Científicas y Técnicas of Argentina.

This work was partially supported by the University of Buenos Aires under grant EX-117.

and give a complete bifurcation diagram.

In Section 3 we analize the stability of the stationary solutions and the long time behavior of the solutions to (1.1). We also prove that when $p \leq q$ blow up may occur only at $x = R$.

There have been several papers dealing with the balance between different processes. For instance the balance between reaction and a nonlinear diffusion both when the reaction takes place in the interior or at the boundary as well as reaction against dispersion (see among others [Fi], [ChLS], [LPSSt], [L]). In this paper we are considering two different reactions one in the substance that consumes energy and the other with the surrounding medium that produces energy.

## 2. The stationary problem

In this section we analyze problem (1.2) and give a complete bifurcation diagram.

Let us call $u_0 = u(0)$ and $u_R = u(R)$. Then if $u$ is a positive solution of (1.2), $u_0$ and $u_R$ satisfy

$$R = \sqrt{\frac{p+1}{2\lambda}} u_0^{-\frac{p-1}{2}} \int_1^{\frac{u_R}{u_0}} \frac{dt}{\sqrt{t^{p+1}-1}} \tag{2.1}$$

and

$$u_R^{2q} = \frac{2\lambda}{p+1}(u_R^{p+1} - u_0^{p+1}) \tag{2.2}$$

Conversely if $u_0$ and $u_R$ are related by (2.1) and (2.2) and $u$ is defined on $[0, R]$ by

$$x = \sqrt{\frac{p+1}{2\lambda}} \int_{u_0}^{u(x)} \frac{ds}{\sqrt{s^{p+1} - u_0^{p+1}}} \tag{2.3}$$

$u$ is a solution of (1.2).

Assume first that $2q \neq p+1$. Let us call $\theta = \dfrac{u_R}{u_0}$. From (2.1) we see that $\theta > 1$. Let us rewrite (2.1) in terms of $\theta$. We get,

$$R = \left(\frac{p+1}{2\lambda}\right)^{\alpha} \frac{\theta^{q\beta}}{(\sqrt{\theta^{p+1}-1})^{\beta}} \int_1^{\theta} \frac{dt}{\sqrt{t^{p+1}-1}}$$

where $\alpha = \dfrac{q-1}{2q-p-1}$, $\quad \beta = \dfrac{p-1}{2q-p-1}$.

Let $f(\theta) = \dfrac{\theta^{q\beta}}{(\sqrt{\theta^{p+1}-1})^{\beta}} \displaystyle\int_1^{\theta} \frac{dt}{\sqrt{t^{p+1}-1}}$. Then (2.1) is equivalent to

$$\lambda^{\alpha} = c_{pq} R^{-1} f(\theta) \tag{2.4}$$

where $c_{pq} = \left(\dfrac{p+1}{2}\right)^{\alpha}$. It is easy to verify that $f'(\theta) > 0$ if

$$\theta^{-(p+1)} \leq 1 - \frac{p+1}{2q} \tag{2.5}$$

This is the case for instance when $2q > p+1$ and $\theta \geq \left(\frac{2q}{2q-p-1}\right)^{\frac{1}{p+1}}$. If (2.5) does not hold we can show that

$$f'(\theta) > \theta^{q\beta-1}\left(\frac{1}{\sqrt{\theta^{p+1}-1}}\right)^{\beta} \frac{2}{\sqrt{\theta^{p+1}-1}} \frac{q-p}{2q-p-1}$$

by applying the inequality

$$\int_1^\theta \frac{dt}{\sqrt{t^{p+1}-1}} < \frac{2}{p+1}\sqrt{\theta^{p+1}-1}$$

Thus we conclude that $f'(\theta) > 0$ for all $\theta > 1$ if $q \geq p$ or $2q < p+1$. Also $f'(\theta) > 0$ for $\theta$ sufficiently large when $2q > p+1$.

Let us first consider the case $q \geq p$. We have $\alpha > 0$ and $f$ increasing, thus there exists a unique $\theta_\lambda$ for each $\lambda$ such that,

$$\left(c_{pq}R^{-1}f_m\right)^{1/\alpha} < \lambda < \left(c_{pq}R^{-1}f_M\right)^{1/\alpha}$$

where $f_m = \inf_{\theta>1} f(\theta)$, $f_M = \sup_{\theta>1} f(\theta)$. It is easy to see from the definition of $f$ and the fact that $\beta > 0$, $q > \frac{p+1}{2}$ that $f_M = +\infty$. In order to compute $f_m$ we apply L'Hospital and get

$$f_m = \begin{cases} 0 & p < q \\ \dfrac{2}{p+1} & p = q \end{cases}$$

Thus if $p < q$ there exists a unique $\theta_\lambda$ for each $\lambda > 0$ whereas if $p = q$ there exists a unique $\theta_\lambda$ if $\lambda > R^{-1}$. In both cases $\frac{\partial}{\partial\lambda}\theta_\lambda > 0$. From the relation

$$u_R = \left[\frac{2\lambda}{p+1}\left(1 - \frac{1}{\theta^{p+1}}\right)\right]^{\frac{1}{2q-p-1}} \tag{2.6}$$

we get the existence of a unique $u_\lambda(R)$ for each $\lambda$ in the ranges above with $\frac{\partial}{\partial\lambda}u_\lambda(R) > 0$.

It is clear from the considerations above that no positive solution exists if $p = q$ and $\lambda \leq R^{-1}$. From (2.6) we see that

$$\begin{array}{ll} \lim_{\lambda\to\infty} u_\lambda(R) = \infty & \text{for} \quad p \leq q \\ \lim_{\lambda\to 0} u_\lambda(R) = 0 & \text{for} \quad p < q \\ \lim_{\lambda\to R^{-1}+} u_\lambda(R) = 0 & \text{for} \quad p = q \end{array}$$

Let us analyze the case $2q < p+1$. We know that $f'(\theta) > 0$. Since $\alpha < 0$ there exists a unique $\theta_\lambda$ for each $\lambda$ in the range

$$\left(c_{pq}R^{-1}f_M\right)^{1/\alpha} < \lambda < \left(c_{pq}R^{-1}f_m\right)^{1/\alpha}$$

and $\frac{\partial}{\partial\lambda}\theta_\lambda < 0$. From the fact that $\beta < 0$, $q < \frac{p+1}{2}$, it is easy to see that $f_M = +\infty$ and $f_m = 0$. Therefore, there exists a solution for each $\lambda > 0$. From (2.6),

$$\lim_{\lambda\to 0} u_\lambda(R) = \lim_{\substack{\theta\to\infty\\ \lambda\to 0}} \left[\frac{2\lambda}{p+1}\left(1 - \frac{1}{\theta^{p+1}}\right)\right]^{\frac{1}{2q-p-1}} = \infty$$

From the relation $u_\lambda(R) = u_\lambda(0)\theta_\lambda$ we can see that $\frac{\partial}{\partial\lambda}u_\lambda(R) < 0$ and $\lim_{\lambda\to\infty} u_\lambda(R) = 0$. In fact, $\frac{\partial}{\partial\lambda}u_\lambda(0) < 0$ and $\lim_{\lambda\to\infty} u_\lambda(0) = 0$ as can be seen from (2.1), and by (2.4) $\lim_{\lambda\to\infty}\theta_\lambda = 1$.

Assume now that $2q > p+1$ and $p > q$. This is $q < p < 2q-1$. We know that $f'(\theta) > 0$ for $\theta$ large enough. It is easy to see that there exists exactly one value $1 < \theta_0 < \infty$ such that $f'(\theta) < 0$ for $1 < \theta < \theta_0$ and $f'(\theta) > 0$ for $\theta_0 < \theta < \infty$. We see that this is true by showing that $\lim_{\theta\to 1} f(\theta) = +\infty$ and that $f'(\theta)$ cannot vanish more than once. To this end we write

$$f'(\theta) = \theta^{q\beta-1}\left(\frac{1}{\sqrt{\theta^{p+1}-1}}\right)^\beta g(\theta)$$

and show that $g$ is strictly increasing when $q < p < 2q-1$ by using the inequality

$$\int_1^\theta \frac{dt}{\sqrt{t^{p+1}-1}} > \frac{2}{p+1}\,\frac{\sqrt{\theta^{p+1}-1}}{\theta^p}$$

and the fact that $\beta - 1 > 0$. Let $\Lambda = \left(c_{pq}R^{-1}f(\theta_0)\right)^{1/\alpha}$. If $\lambda < \Lambda$ there is no positive solution. If $\lambda = \Lambda$ there is exactly one solution that corresponds to $\theta = \theta_0$. When $\lambda > \Lambda$ there are two different values of $\theta$, $\theta_\lambda^1 > \theta_\lambda^2$ for which $\frac{\partial}{\partial\lambda}\theta_\lambda^1 > 0$, $\frac{\partial}{\partial\lambda}\theta_\lambda^2 < 0$.

Associated to each of these $\theta$'s there is a solution $u_\lambda^i$. (2.6) immediately gives that $\frac{\partial}{\partial\lambda}u_\lambda^1(R) > 0$. From (2.1) we obtain $\frac{\partial}{\partial\lambda}u_\lambda^2(0) < 0$ and therefore also $\frac{\partial}{\partial\lambda}u_\lambda^2(R) < 0$. Finally, $\lim_{\lambda\to\infty} u_\lambda^1(R) = \infty$ and $\lim_{\lambda\to\infty} u_\lambda^2(R) = 0$ since this is true for $u_\lambda^2(0)$ and $\lim_{\lambda\to\infty}\theta_\lambda^2 = 1$. It is easy to see that $u_\lambda^i \to u_\Lambda$ uniformly as $\lambda \downarrow \Lambda$ since $\theta_\lambda^i \to \theta_\Lambda$ as $\lambda \downarrow \Lambda$.

So it only remains to analyze the case $2q = p+1$. From (2.2),

$$u_0^{2q} = u_R^{2q}\left(1 - \frac{q}{\lambda}\right)$$

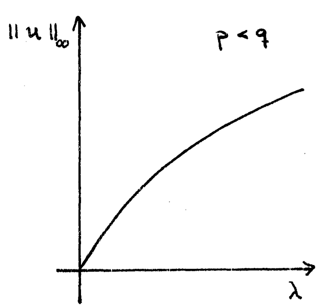
Thus it is clear that we must have $\lambda > q$. When this is the case $\frac{u_R}{u_0} = \left(\frac{\lambda}{\lambda-q}\right)^{\frac{1}{2q}}$ so that (2.1) gives an explicit formula for the value $u_\lambda(0)$,

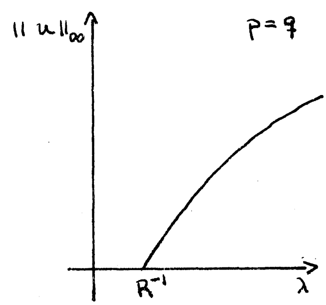
$$u_\lambda(0) = \left(R^{-1}\sqrt{\frac{q}{\lambda}}\int_1^{\left(\frac{\lambda}{\lambda-q}\right)^{1/2q}} \frac{dt}{\sqrt{t^{p+1}-1}}\right)^{\frac{1}{q-1}}$$

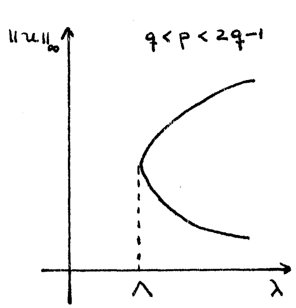
From this formula it is clear that $\lim_{\lambda\to\infty} u_\lambda(0) = 0$. Since $\lim_{\lambda\to\infty} \frac{u_\lambda(R)}{u_\lambda(0)} = 1$ we find that $\lim_{\lambda\to\infty} u_\lambda(R) = 0$.

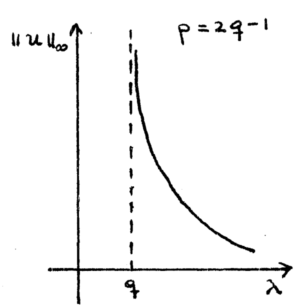
On the other hand $\lim_{\lambda\to q^+} \theta_\lambda = +\infty$. Since $\lim_{\lambda\to q^+} u_\lambda(0) > 0$ we deduce that $\lim_{\lambda\to q^+} u_\lambda(R) = +\infty$. It is easy to see by analyzing the behavior of $\theta_\lambda$ and $u_\lambda(0)$ that in this case $\frac{\partial}{\partial\lambda} u_\lambda(R) < 0$.

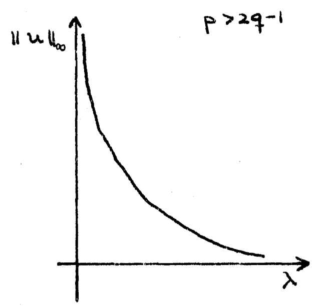
We may summarize these results in the following bifurcation diagrams. The stability or instability of the branches is a consequence of the results in Section 3.




It can be seen from the diagrams that when $q \le p \le 2q-1$ there are some values of $\lambda$ for which there is no positive solution. In all other cases there is exactly one positive solution for each $\lambda$.

We write these results in detail in the following theorem,

THEOREM 2.1. *Let $\lambda > 0$, $p, q > 1$. Then*

(1) $p < q$

*For each $\lambda > 0$ there exists a unique positive solution $u_\lambda$ to (1.2). $u_\lambda(R)$ is strictly increasing as a function of $\lambda$, $\lim_{\lambda\to 0} u_\lambda(R) = 0$ and $\lim_{\lambda\to\infty} u_\lambda(R) = \infty$.*

(2) $p = q$

i) *If $\lambda \le R^{-1}$ (1.2) has no positive solutions.*

ii) *If $\lambda > R^{-1}$ there exists a unique positive solution $u_\lambda$ of (1.2), $u_\lambda(R)$ is strictly increasing as a function of $\lambda$, $\lim_{\lambda\to R^{-1}+} u_\lambda(R) = 0$ and $\lim_{\lambda\to\infty} u_\lambda(R) = \infty$.*

(3) $q < p < 2q - 1$

*There exists $\Lambda > 0$ depending on $p, q$ and $R$ such that*

i) *If $\lambda < \Lambda$ (1.2) has no positive solutions.*

ii) *If $\lambda = \Lambda$ there exists a unique positive solution to (1.2).*

iii) *If $\lambda > \Lambda$ there exist exactly two positive solutions $u_\lambda^1 > u_\lambda^2$ and they satisfy $\frac{\partial}{\partial\lambda}u_\lambda^1(R) > 0$, $\frac{\partial}{\partial\lambda}u_\lambda^2(R) < 0$, $\lim_{\lambda\to\Lambda} u_\lambda^i = u_\Lambda$ uniformly on $[0, R]$, $\lim_{\lambda\to\infty} u_\lambda^1(R) = \infty$ and $\lim_{\lambda\to\infty} u_\lambda^2(R) = 0$.*

(4) $p = 2q - 1$

i) *If $\lambda \le q$ (1.2) has no positive solutions.*

ii) *If $\lambda > q$ there exists a unique positive solution $u_\lambda$ to (1.2), $u_\lambda(R)$ is strictly decreasing as a function of $\lambda$, $\lim_{\lambda\to q+} u_\lambda(R) = \infty$ and $\lim_{\lambda\to\infty} u_\lambda(R) = 0$.*

(5) $p > 2q - 1$

*For each $\lambda > 0$ there exists a unique positive solution $u_\lambda$ to (1.2), $u_\lambda(R)$ is strictly decreasing as a function of $\lambda$, $\lim_{\lambda\to 0} u_\lambda(0) = \infty$ and $\lim_{\lambda\to\infty} u_\lambda(R) = 0$.*

OBSERVATION. Let $u_R = \lim_{\lambda\to\lambda_0} u_\lambda(R)$ and $u_0$ given by

$$u_0^{p+1} = u_R^{p+1}\left(1 - \frac{p+1}{2\lambda_0}u_R^{2q-p-1}\right)$$

Let $u(x)$ be given by

$$x = \sqrt{\frac{p+1}{2\lambda_0}}u_0^{-\frac{p-1}{2}} \int_1^{\frac{u(x)}{u_0}} \frac{dt}{\sqrt{t^{p+1} - 1}}$$

then $u$ is a solution of

$$\begin{cases} u_{xx} = \lambda_0 u^p & 0 < x < R \\ u_x(0) = 0 \\ u_x(R) = u^q(R) \end{cases}$$

and $u_\lambda$ converges to $u$ as $\lambda \to \lambda_0$ uniformly on $[0, R]$.

## 3. The evolutionary problem

In this section we study the global behavior of the solution to (1.1) depending on the relation of the initial datum $u_0$ with respect to the stationary solution/solutions found in Section 2.

We also show that every solution remains bounded in time at each point of the interval $[0,R)$ when $p \leq q$.

Local existence of weak solutions to (1.1) was proved in [Al], [Am], [An]. [Al] and [Am] also show the solution to be classical for $t > 0$ when $u_0$ satisfies the compatibility condition

$$u_0'(0) = 0\,, \qquad u_0'(R) = u_0^q(R)$$

On the other hand argueying as in Section 1 in [LMW] it is possible to show that $u$ is smooth up to $t = 0$. This is,

Proposition 3.1. *Let $0 < \alpha < 1$ and $u_0 \in C^{2+\alpha}[0,R]$ be such that $u_0'(0) = 0$, $u_0'(R) = u_0^q(R)$. There exists a maximal $T = T(u_0) > 0$ (it could be infinity) such that (1.1) has a unique solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}([0,R]\times[0,T))$. Moreover if $T < \infty$, $\limsup_{t\uparrow T}|u(R,t)| = \infty$. Also if $u_0 \geq 0$, $u(x,t) \geq 0$ for $t > 0$.*

In the next proposition we characterize the $\omega$–limit set of a global solution. As one expects, if $v \in \omega$–limit set of $u_0$, $v$ is a solution of (1.2). In fact,

Proposition 3.2. *Let $u_0 \in H^1(0,R)$ be such that the weak solution to (1.1) exists for all times ($u \in L^2_{\text{loc}}(0,\infty;\, H^1(0,R))$. Let $\omega(u_0)$ be the $\omega$–limit set of $u_0$. If $v$ belongs to $\omega(u_0)$, $v$ is a solution to (1.2).*

*Proof.* As has already been observed in [Al] problem (1.1) has the following Liapunov functional,

$$V(u) = \frac{1}{2}\int_0^R u_x^2(x)dx + \frac{\lambda}{p+1}\int_0^R u^{p+1}(x)dx - \frac{1}{q+1}u^{q+1}(R)$$

In fact it is immediate to see that any solution satisfies,

(3.1)
$$\int_s^t\int_0^R u_t^2(x,\tau)dx\,d\tau + \frac{1}{2}\int_0^R u_x^2(x,t)dx + \frac{\lambda}{p+1}\int_0^R u^{p+1}(x,t)dx -$$
$$-\frac{1}{q+1}u^{q+1}(R,t) == \frac{1}{2}\int_0^R u_x^2(x,s)dx + \frac{\lambda}{p+1}\int_0^R u^{p+1}(x,s)dx - \frac{1}{q+1}u^{q+1}(R,s)$$

General results on dynamic behavior of semilinear parabolic problems (see for instance [H]) imply that $\omega(u_0)$ is an invariant subset of $\{\dot V = 0\}$ where $\dot V$ is the derivative of $V$ along trayectories. Simple computations show that $\dot V(u) = \int_0^R (u_{xx} - \lambda u^p)^2 dx$. Therefore any function $v \in \omega(u_0)$ satisfies

$$v_{xx} = \lambda v^p$$

In order to see that $v$ also satisfies the boundary conditions it is necessary to use the invariance of $\omega(u_0)$. In fact, let $w(\cdot,t) = S(t)v$ where $S(t)$ is the semigroup associated to problem (1.1). Since $\omega(u_0)$ is invariant $w(\cdot,t)$ satisfies

$$w_{xx} - \lambda w^p = 0$$

therefore $w_t \equiv 0$. This implies that $w = v$ and thus $v$ satisfies the boundary conditions.

REMARK. Identity (3.1) with $s = 0$ shows that bounded global solutions have no empty $\omega$–limit set since they belong to $L^\infty(0,\infty; H^1(0,R))$. Moreover, for every $t_n \to \infty$ there exists a subsequence $t_{n'} \to \infty$ such that $u(\cdot,t_{n'}) \rightharpoonup v \in \omega(u_0)$ in $H^1(0,R)$ as $n \to \infty$. In particular, if $\omega(u_0) = \{v\}$ and $u$ is a bounded solution, $u(\cdot,t) \rightharpoonup v$ in $H^1(0,R)$ $(t \to \infty)$ and also uniformly on $[0,R]$.

We state here a comparison result that can be found in [An].

PROPOSITION 3.3. *Let $u^1$ and $u^2$ be weak solutions of (1.1) with initial values $u_0^1 \le u_0^2$. The $u^1 \le u^2$ for $t < \min\left(T(u_0^1), T(u_0^2)\right)$.*

COROLLARY 3.1. *Let $u$ be a classical solution of (1.1) such that*

$$u_{0_{xx}} - \lambda u_0^p < (\text{resp. } >)\ 0 \quad \text{on } [0,R]$$

*then $u_t \le$ (resp. $\ge$) $0$ for $t < T(u_0)$.*

*Proof.* It follows by comparison between the two solutions of (1.1), $u$ and $u_\varepsilon$ where

$$u_\varepsilon(x,t) = u(x,t+\varepsilon)$$

and $\varepsilon < \varepsilon_0$.

COROLLARY 3.2. *Let $u$ be a solution of (1.1) with $u_0 = u_\mu$ where $u_\mu$ is a positive stationary solution of*

$$\begin{aligned} u_{xx} &= \mu u^p \qquad & 0 < x < R \\ u_x(0) &= 0\,, \qquad & u_x(R) = u^q(R) \end{aligned}$$

*Then if $\mu > \lambda$, $u_t \ge 0$ and if $\mu < \lambda$, $u_t \le 0$.*

We are now able to analyze the global behavior of the solutions to (1.1) for different initial values.

We prove the following result,

THEOREM 3.1. *Let $u$ be the maximal solution to (1.1) with $u_0 \in C^{2+\alpha}[0,R]$, $u_0 \geq 0$ and compatible and $T(u_0) \leq \infty$ its existence time. Then,*

(1) *Let $p < q$. Let $u_\lambda$ be the unique positive stationary solution. Then*
  (i) *If $u_0 > u_\lambda$, $T(u_0) < \infty$ and $\limsup_{t \to T(u_0)} u(R,t) = +\infty$.*
  (ii) *If $u_0 < u_\lambda$, $T(u_0) = +\infty$ and $u(\cdot,t) \to 0$ $(t \to \infty)$.*

(2) *Let $p = q$*
  (i) *If $\lambda \leq R^{-1}$, $\limsup_{t \to T(u_0)} u(R,t) = +\infty$ for every $u_0 > 0$. Moreover if $\lambda < R^{-1}$, $T(u_0) < \infty$.*
  (ii) *If $\lambda > R^{-1}$, let $u_\lambda$ be the unique positive stationary solution. Then,*
  (a) *If $u_0 > u_\lambda$, $\limsup_{t \to T(u_0)} u(R,t) = +\infty$.*
  (b) *If $u_0 < u_\lambda$, $T(u_0) = +\infty$ and $u(\cdot,t) \to 0$ $(t \to \infty)$.*

(3) *Let $q < p < 2q-1$. Let $\Lambda \in \mathbf{R}$ be such that for $\lambda > \Lambda$ there are two branches of positive stationary solutions. With the notation of Section 2,*
  (i) *If $\lambda < \Lambda$, $\limsup_{t \to T(u_0)} u(R,t) = +\infty$ for every $u_0$.*
  (ii) *If $\lambda = \Lambda$ there is a unique positive stationary solution $u_\lambda$ and,*
  (a) *$u_0 \leq u_\lambda$, $T(u_0) = +\infty$ and $u(\cdot,t) \to u_\lambda$ $(t \to \infty)$.*
  (b) *$u_0 > u_\lambda$, $\limsup_{t \to T(u_0)} u(R,t) = +\infty$.*
  (iii) *If $\lambda > \Lambda$ and,*
  (a) *$u_0 > u_\lambda^1$, $\limsup_{t \to T(u_0)} u(R,t) = +\infty$.*
  (b) *$u_0 < u_\lambda^1$, $T(u_0) = +\infty$ and $u(\cdot,t) \to u_\lambda^2$ $(t \to \infty)$.*

(4) *Let $p = 2q-1$ and*
  (i) *$\lambda \leq q$. Then $\limsup_{t \to T(u_0)} u(R,t) = +\infty$ for every $u_0$.*
  (ii) *$\lambda > q$. Let $u_\lambda$ be the unique positive stationary solution. Then $T(u_0) = +\infty$ and $u(\cdot,t) \to u_\lambda$ $(t \to \infty)$.*

(5) *Let $p > 2q-1$ and $u_\lambda$ be the unique positive stationary solution. Then $T(u_0) = +\infty$ and $u(\cdot,t) \to u_\lambda$ $(t \to \infty)$.*

*Proof.* (1) Let $p < q$ and $u_0 < u_\lambda$ on $[0,R]$. As a consequence of the uniform convergence on $[0,R]$ of $u_\mu$ to $u_\lambda$ as $\mu \to \lambda$, there exists $\mu < \lambda$ such that $u_0 < u_\mu$. Let $v$ be the solution of (1.1) with initial value $u_\mu$, then, by Corollary 3.2, $v_t \leq 0$ and $u \leq v$ by comparison. This estimate implies that $u$ is a global solution and

$$u(R,t) \leq v(R,t) \leq u_\mu(R) < u_\lambda(R)$$

for every $t > 0$. This in turn implies that $u_\lambda \notin \omega(u_0)$. Thus

$$u(\cdot,t) \to 0 \qquad (t \to \infty)$$

Let $u_0 > u_\lambda$. The argument above shows that there exists $\mu > \lambda$ such that $u_0 > u_\mu$. If $v$ is the solution of (1.1) with initial value $u_\mu$, $v_t \geq 0$ and $u \geq v$. This in turn implies that $u(R,t) \geq v(R,t)$. It is clear that $v(R,t) \to \infty$ as $t \to T(u_\mu)$ since $v$ cannot be bounded and at the same time

$$v(R,t) \geq u_\mu(R) > u_\lambda(R) > 0$$

Let us see that $v$ actually blows up in finite time. Assume $v$ exists globally; we claim that there exists a time $t_0$ such that

$$\frac{1}{R}\int_0^R v(x,t_0)dx > (\lambda R)^{\frac{1}{q-p}} \tag{3.2}$$

We prove it by contradiction. Let $m(t) = \frac{1}{R}\int_0^R v(x,t)dx$. Then,

$$\begin{aligned} Rm'(t) &= \int_0^R v_t(x,t)dx = \int_0^R v_{xx}(x,t)dx - \lambda\int_0^R v^p(x,t)dx \\ &= v^q(R,t) - \lambda\int_0^R v^p(x,t)dx \end{aligned} \tag{3.3}$$

From (3.3) and the fact that $v_t \geq 0$ we see that there exists $\lim_{t\to\infty} m'(t)$. Assume (3.2) is not true, then $m(t)$ is bounded from above by $(\lambda R)^{\frac{1}{q-p}}$ and nonnegative. By the considerations above $m'(t) \to 0$ as $t \to \infty$, so that there exists $t_1$ such that $t > t_1$ implies $Rm'(t) < 1$.

From (3.3) we see that for $t > t_1$, $v^q(R,t) < 1 + \lambda R v^p(R,t)$ since $v_0$ increasing implies that $v$ is increasing in space. This last inequality is impossible since $p < q$ and $v(R,t) \to \infty$. So let $t_0$ be such that (3.2) holds. Proceeding as before we get from (3.3)

$$\begin{aligned} m'(t) &\geq \frac{1}{R}\big(v^q(R,t) - \lambda R v^p(R,t)\big) = \frac{1}{R}v^p(R,t)\big(v^{q-p}(R,t) - \lambda R\big) \\ &\geq \frac{1}{R}v^p(R,t)\big(m(t)^{q-p} - \lambda R\big) \end{aligned} \tag{3.4}$$

thus $m'(t_0) > 0$ which implies that

$$m(t) > m(t_0) > (\lambda R)^{\frac{1}{q-p}} \tag{3.5}$$

for $t > t_0$, $t$ close to $t_0$. And therefore (3.5) holds for every $t$. We have

$$m'(t) \geq \frac{1}{R}v^p(R,t)\big(m(t_0)^{q-p} - \lambda R\big) \geq \frac{1}{R}\big(m(t_0)^{q-p} - \lambda R\big)\left[m(t)\right]^p$$

so $m(t)$ cannot exist for all $t$. Therefore $u$ does not exist globally and we have

$$\limsup_{t\to T(u_0)} u(R,t) = +\infty$$

(2) $p = q$

(i) Let $\lambda \leq R^{-1}$. Let $\mu > R^{-1}$ such that $u_\mu < u_0$. Since $\mu > \lambda$ if $v$ is the solution of (1.1) with $v(x,0) = u_\mu$, $v_t \geq 0$. Therefore $u(x,t) \geq u_\mu(x) \geq u_\mu(0) > 0$. Thus $u$ cannot remain bounded since there are no positive stationary solutions in this case. Moreover if $\lambda < R^{-1}$, $u$ blows up in finite time. In fact from (3.4)

$$m'(t) \geq \frac{1}{R}(1-\lambda R)v^p(R,t) \geq \frac{1}{R}(1-\lambda R)\big[m(t)\big]^p$$

and therefore $v$ cannot exist for all time.

(ii) If $\lambda > R^{-1}$, the same arguments as in the case $p < q$ show that for $u_0 > u_\lambda$, $\limsup_{t\to T(u_0)} u(R,t) = \infty$ and for $u_0 < u_\lambda$, $\lim_{t\to\infty} u(\cdot,t) = 0$.

(3) $q < p < 2q-1$

(i) Let $\lambda < \Lambda$. Let $u_0 > 0$ and let $\mu$ large enough so as to have $\mu > \Lambda$, $u_0 > u_\mu^2$. Let $v$ be the solution with initial datum $u_\mu^2$. Then

$$u(x,t) \geq v(x,t) \geq u_\mu^2(0) > 0$$

As there is no positive stationary solution we deduce that

$$\limsup_{t\to T(u_0)} u(R,t) = +\infty$$

(ii) $\lambda = \Lambda$

(a) Let $u_\Lambda \geq u_0 > 0$ on $[0,R]$. Let $\mu$ large enough so as to have $\mu > \Lambda$ and $u_0 > u_\mu^2$. Then

$$u_\Lambda(R) \geq u(x,t) \geq u_\mu^2(0) > 0$$

and we conclude that $u$ converges to $u_\Lambda$ as $t$ goes to infinity.

(b) Let $u_0 > u_\Lambda$ and let $\mu > \Lambda$ small enough as to have $u_0 > u_\mu^1$. We have

$$u(R,t) \geq u_\mu^1(R) > u_\Lambda(R)$$

as long as it exists. This implies that $u(x,t_n)$ cannot converge to $u_\Lambda$ or 0 as $n$ goes to infinity. Therefore

$$\limsup_{t\to T(u_0)} u(R,t) = +\infty$$

(iii) $\lambda > \Lambda$

(a) If $u_0 > u_\lambda^1$ we proceed as in (3.ii.b) to conclude that

$$\limsup_{t\to T(u_0)} u(R,t) = +\infty$$

(b) Let $0 < u_0 < u_\lambda^1$ on $[0,R]$. Let $\mu$ large enough as to have $\mu > \lambda$ and $u_0 > u_\mu^2$; and $\lambda > \nu > \Lambda$ close enough to $\lambda$ as to have $u_0 < u_\nu^1$. Then

$$0 < u_\mu^2(0) \leq u(x,t) \leq u_\nu^1(R) < u_\lambda^1(R)$$

and we conclude that $u$ converges to $u_\lambda^2$ as $t$ goes to infinity.

(4) $p = 2q - 1$

(i) If $\lambda \leq q$ we proceed as in the case (2.i) by taking $\mu > q$ sufficiently large and comparing $u$ with $v$. We deduce that

$$\limsup_{t \to T(u_0)} u(R,t) = +\infty$$

(ii) Let $\lambda > q$. We want to show that $u$ exists globally even if $u_0 > u_\lambda$. For this purpose we construct a supersolution $v$ with $v(x,0) = v_0 \geq u_0$. Let us see that there exist $\lambda > \mu > q$ and $c > 0$ such that the function

$$v_0(x) = cu_\mu(x) + \|u_0\|_{L^\infty}$$

satisfies

$$v_0'(0) = 0\,, \quad v_0'(R) = v_0^q(R) \quad \text{and} \quad v_0'' - \lambda v_0^p < 0$$

In fact the condition at the origin is immediate. We choose $q < \mu < \nu < \lambda$ such that $\nu$ is large enough so that $h(c) = c^{1/q} - c$ is a decreasing function of $c$ for $\left(\frac{\nu}{\lambda}\right)^{\frac{1}{p-1}} < c < 1$ and $\mu$ is sufficiently close to $q$ so as to have

$$\frac{\|u_0\|_{L^\infty}}{u_\mu(R)} < \left[\left(\frac{\nu}{\lambda}\right)^{\frac{1}{p-1}}\right]^{\frac{1}{q}} - \left(\frac{\nu}{\lambda}\right)^{\frac{1}{p-1}}$$

Here we use the fact that $\lim\limits_{\mu \to q^+} u_\mu(R) = +\infty$. Now let $\left(\frac{\nu}{\lambda}\right)^{\frac{1}{p-1}} < c < 1$ be such that

$$\frac{\|u_0\|_{L^\infty}}{u_\mu(R)} = c^{1/q} - c \tag{3.6}$$

Let us see that with this choice of $c$ and $\mu$, $v_0$ satisfies the conditions above. In fact,

$$\begin{aligned}(cu_\mu + \|u_0\|_{L^\infty})_{xx} - \lambda(cu_\mu + \|u_0\|_{L^\infty})^p = c\mu u_\mu^p - \lambda(cu_\mu + \|u_0\|_{L^\infty})^p = \\ = c^{1-p}\mu(cu_\mu)^p - \lambda(cu_\mu + \|u_0\|_{L^\infty})^p < 0\end{aligned}$$

because $c^{1-p}\mu < \lambda$ by construction. Let us check the boundary condition,

$$(cu_\mu + \|u_0\|_{L^\infty})_x(R) = cu_\mu^q(R) = (cu_\mu(R) + \|u_0\|_{L^\infty})^q$$

if and only if

$$c^{1/q}u_\mu(R) = cu_\mu(R) + \|u_0\|_{L^\infty}$$

and this is equivalent to (3.6).

Since $v_0 > u_0$ we have $v(x,t) > u(x,t)$ if $v$ is the solution of (1.1) with initial datum $v_0$. Also $v_t \leq 0$. Therefore $u$ is bounded. On the other hand we may choose $\sigma$ large enough as to have $\sigma > \lambda$ and $u_\sigma < u_0$, since $u_\sigma(R) \to 0$ ($\sigma \to \infty$) in this case. Therefore

$$u(x,t) \geq u_\sigma(0) > 0$$

for every $t$. This implies that $u$ converges to $u_\lambda$ as $t$ goes to $\infty$.

(5) $p > 2q - 1$

From Theorem 2.1 we know that $u_\mu(R) \to 0$ ($\mu \to \infty$) and $u_\nu(0) \to \infty$ ($\nu \to 0$).

Let $u_0 > 0$ on $[0, R]$, let $\mu$ large enough so as to have $\mu > \lambda$ and $u_\mu < u_0$ and $\nu$ small enough so as to have $\nu < \lambda$ and $u_0 < u_\nu$. Let $v$ be the solution of (1.1) with initial datum $u_\mu$ and $w$ the solution with initial datum $u_\nu$. We have

$$0 < u_\mu(0) \le v(x,t) \le u(x,t) \le w(x,t) \le u_\nu(R)$$

Therefore $T(u_0) = +\infty$. Since $u(x, t_n)$ cannot converge to zero for any sequence $t_n \to \infty$ we deduce that $u$ converges to $u_\lambda$ as $t \to \infty$.

This concludes the proof of Theorem 3.1.

Finally we prove that when $p \le q$ blow up may occur only at the boundary $x = R$.

PROPOSITION 3.4. *Let $p \le q$ and let $u$ be a classical positive solution of (1.1), there exists a constant $c = c(u_0)$ such that*

$$u(x,t) \le \frac{c}{(R-x)^{\frac{1}{q-1}}}$$

*as long as it exists.*

*Proof.* The ideas in this proof have already been used in [LMW], also in [FMcL] for the semilinear equation. Assume first that $u_x \ge 0$. We show that there exists a nonnegative smooth function $g$ with $g(R) > 0$ such that

$$u_x(x,t) \ge g(x) u^q(x,t) \tag{3.7}$$

In fact let $v = u_x$ and $w = g(x)u^q$ where $g$ is chosen smooth convex with $g(0) = 0$, $g(R) > 0$ and such that

$$\frac{u_0'}{u_0^q} \ge g(x) \qquad \text{on} \quad [0, R]$$

then

$$\begin{aligned}(w-v)_t - (w-v)_{xx} = &-\lambda g q u^{p+q-1} - g'' u^q - 2g' q u^{q-1} u_x \\ &- q(q-1) g u^{q-2} u_x^2 + \lambda p u^{p-1} u_x\end{aligned}$$

also

$$\begin{aligned} v(x,0) &\ge w(x,0) && 0 < x < R \\ v(0,t) &= w(0,t) && t > 0 \\ v(R,t) &\ge w(R,t) && t > 0 \end{aligned}$$

Thus if $w - v$ were positive somewhere it would attain its positive maximum at an interior point $(x_0, t_0)$. At this point we would have $w = gu^q > u_x = v$. Also $(w - v)_t \geq 0$ and $(w - v)_{xx} \leq 0$. Therefore

$$0 \leq (w - v)_t - (w - v)_{xx} < \lambda g u^{p+q-1}(p - q) \leq 0$$

Therefore $v \geq w$ everywhere. This estimate leads to the inequality

$$u(x,t) \leq \frac{c}{(R-x)^{\frac{1}{q-1}}} \tag{3.8}$$

In fact after an integration of (3.7) we get to

$$\frac{1}{(q-1)u^{q-1}(x,t)} \geq \int_x^R g(s)ds$$

As $g(R)$ is positive an estimate from below of the integral gives (3.8).

When $u_x$ is not necessarily nonnegative it is possible to show that a nonpositive bound from below of $u_0'$ is a bound from below of $u_x(x,t)$. A slight modification of the arguments above leads to (3.8) (see [LMW] for the details).

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Julián López Gómez
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
Madrid, Spain.

Viviana Márquez and Noemí Wolanski
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
(1428) Buenos Aires, Argentina



Recibido en febrero de 1993.

Versión modificada en setiembre de 1993.