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A CLASS OF MINIMAL SUBMANIFOLDS IN A 2-STEP NILPOTENT LIE GROUP

Carlos J. Ferraris

ABSTRACT

We study the existence of minimal submanifolds in a simply connected 2-step nilpotent Lie group N with a left invariant metric, the main result being that every one-parameter subgroup of N lies in at least one 2-dimensional connected subgroup which is a minimal submanifold of N. A similar result is also obtained for a class of solvable Lie groups with a left invariant metric.

INTRODUCTION

This article deals with the existence of minimal submanifolds in a 2-step nilpotent Lie group N with a left invariant metric.

In Section 1, we introduce the notation and recall some of the concepts to be used in the sequel.

In Section 2, we derive the existence of minimal submanifolds naturally associated to one-parameter subgroups of N.

In Section 3, we obtain a similiar result for a class of solvable Lie groups (semi-direct product of $\mathbf{R}$ and a 2-step nilpotent Lie group) with a left invariant metric.

1. MINIMAL SUBMANIFOLDS IN A LIE GROUP

Let $(G, \langle\ ,\ \rangle)$ be a connected Lie group with a left invariant metric $\langle\ ,\ \rangle$ and let $(\mathcal{G}, \langle\ ,\ \rangle)$ denote its Lie algebra with the inner product $\langle\ ,\ \rangle$. The Levi-Civita connection for $(G, \langle\ ,\ \rangle)$ can be expressed in terms of $\mathcal{G}$ by the formula

$$\nabla_X Y = 1/2([X, Y] - \mathrm{ad}_X^* - ad_Y^* X), \ X, Y \in \mathcal{G}, \tag{1}$$

where $*$ stands for the adjoint relative to the inner product $\langle\ ,\ \rangle$ on $\mathcal{G}$, cf. [2].

DEFINITION 1.1. A submanifold M of $(G, \langle\ ,\ \rangle)$ is called a *minimal* submanifold if trace $\Pi_M \equiv 0$, where Π_M stands for the second fundamental form of M in $(G, \langle\ ,\ \rangle)$.

Let H be a connected Lie subgroup of G and let $\mathcal{H}$ be its Lie subalgebra.

PROPOSITION 1.2. A connected Lie subgroup H of G is a minimal submanifold of $(G, \langle\ ,\ \rangle)$ if and only if $\sum_i \nabla_{X_i} X_i$ belongs to $\mathcal{H}$, for an orthonormal basis (X_i) of $(\mathcal{H}, \langle\ ,\ \rangle)$. ($\sum_i \nabla_{X_i} X_i$ is independent of the orthonormal basis (X_i)).

PROOF: It follows from the formula

$$\nabla_X Y = \nabla_X^{\mathcal{H}} Y + \Pi_{H(X,Y)}, \qquad X, Y \in \mathcal{G}$$

(where $\nabla^{\mathcal{H}}$ indicates the covariant derivative in H) and the fact that

$$\text{trace } \Pi_H = \sum_i \Pi_H(X_i, X_i)$$

for an orthonormal basis (X_i) of $(\mathcal{G}, \langle\ ,\ \rangle)$.

The remark that $\sum_i \nabla_{X_i} X_i$ is independent of the orthonormal basis (X_i) follows from the properties of the covariant derivative and the fact that if (Y_j) is another orthonormal basis, then $Y_j = \sum_i a_{ij} X_i$, where (a_{ij}) is an orthogonal matrix, and hence

$$\sum_j \nabla_{Y_j} Y_j = \sum_j \left(\sum_{i,k} a_{ij} a_{kj} \nabla_{X_i} X_k \right) = \sum_{i,k} \left(\sum_j a_{ij} a_{kj} \right) \nabla_{X_i} X_k =$$

$$= \sum_{i,k} \delta_{ik} \nabla_{X_i} X_k = \sum_i \nabla_{X_i} X_i.$$

2. 2-STEP NILPOTENT LIE GROUPS

Let $\mathcal{N}$ be a real finite dimensional 2-step nilpotent Lie algebra, i.e. $0 \neq [\mathcal{N}, \mathcal{N}] \subset \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{N}$. If $\langle\ ,\ \rangle$ is a positive definite inner product on $\mathcal{N}$, then

$$\mathcal{N} = \mathcal{V} \oplus \mathcal{Z},$$

where $\mathcal{V} = \mathcal{Z}^{\perp}$ = orthogonal complement of $\mathcal{Z}$.

For every $Z \in \mathcal{Z}$, there is a skew-symmetric operator $j(Z)$ on $\mathcal{V}$ defined by

$$\langle j(Z)V, W\rangle = \langle [V, W], Z\rangle, \quad Z \in \mathcal{Z},\ V, W \in \mathcal{V}.$$

Let $(N, \langle\ ,\ \rangle)$ denote the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$ end left invariant metric $\langle\ ,\ \rangle$.

From formula (1) follows:

i) $\nabla_V W = 1/2\,[V, W],\ V, W \in \mathcal{V}$,

ii) $\nabla_Z Z' = 0,\ Z, Z' \in \mathcal{Z}$,

iii) $\nabla_V Z = \nabla_Z V = -1/2\, j(Z)V,\ Z \in \mathcal{Z}, V \in \mathcal{V}$.

THEOREM 2.1. Let $(N, \langle\ ,\ \rangle)$ be a simply connected 2-step nilpotent Lie group with a left invariant metric $\langle\ ,\ \rangle$. Every one-parameter subgroup of N lies in at least one 2-dimensional connected (abelian) subgroup H which is a minimal submanifold of $(N, \langle\ ,\ \rangle)$.

PROOF: We consider first the generic case, i.e., let $X \in \mathcal{N}$, with $X = V + Z$, $0 \neq V \in \mathcal{V}$ and $0 \neq Z \in \mathcal{Z}$. In this case, we take $\mathcal{H} = \operatorname{span}\{V, Z\}$ and we prove that the connected subgroup $H = \exp(\mathcal{H})$, which contains the one-parameter, subgroup $\exp tX$, is a minimal submanifold of $(N, \langle\ ,\ \rangle)$. In fact, by applying formulas (i) and (ii) to the orthonormal basis $\{V/_{||V||}, Z/_{||Z||}\}$ of $(\mathcal{H}, \langle\ ,\ \rangle)$ we obtain

$$\nabla_{V/_{||V||}} V/_{||V||} + \nabla_{Z/_{||Z||}} Z/_{||Z||} = 0.$$

Hence $H = \exp(\mathcal{H})$ is a minimal submanifold of $(N, \langle\ ,\ \rangle)$ by Proposition 1.2. In the case $V = 0$ or $Z = 0$ we just take a non zero vector either in $\mathcal{V}$ or $\mathcal{Z}$, to obtain a 2-dimensional minimal subgroup H of $(N, \langle\ ,\ \rangle)$. This concludes the proof of the theorem.

REMARK. We observe that if $H = \exp(\mathcal{H})$ for $\mathcal{H} = \text{span}\{V, Z\}$, $0 \neq V \in \mathcal{V}$ and $0 \neq Z \in \mathcal{Z}$ and $(N, \langle\ ,\ \rangle)$ is of *Heisenberg type*, i.e., $j^2(Z) = -||Z||^2$ Id., for every $Z \in \mathcal{Z}$, cf. [3] and [5], or more generally *nonsingular* $\text{ad}_X\colon \mathcal{N} \to \mathcal{Z}$ is surjective for every $X \in \mathcal{N} - \mathcal{Z}$) then the subgroup H is a minimal submanifold which is neither *totally geodesic* ($\nabla_X Y \in \mathcal{H}$, for every $X, Y \in \mathcal{H}$) nor *flat* (the sectional curvature $K(V, Z) = 0$). In fact,

$$0 \neq \nabla_V Z = -1/2\, j(Z)V \notin \mathcal{H}$$

and a simple calculation shows that

$$K(V, Z) = \frac{1}{4}\, ||j(Z)V||^2 \quad \text{if } V \text{ and } Z \text{ are unit vectors.}$$

The property of Theorem 2.1 is not shared for every 3-step nilpotent Lie group with a left invariant metric as it is shown by the following example. Let $\mathcal{N}$ be the Lie algebra of matrices

$$\mathcal{N} = \left\{ \begin{bmatrix} 0 & x_1 & y_1 & z \\ 0 & 0 & x_2 & y_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad x_1, x_2, x_3, y_1, y_2, z \in \mathbf{R} \right\}$$

$\mathcal{N}$ is a 3-step nilpotent Lie algebra and if we set

$$X_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and } Z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$[X_1, X_2] = Y_1, \qquad [X_1, Y_2] = Z, \qquad [Y_1, X_3] = Z,$$

$$[X_2, X_3] = Y_2, \quad \text{the rest of the brackets being } 0.$$

We give to $\mathcal{N}$ the inner product $\langle\ ,\ \rangle$ such that the vectors $\{X_1, X_2, X_3, Y_1, Y_2, Z\}$ form an orthonormal basis. Let $(N, \langle\ ,\ \rangle)$ denote the simply connected 3-step nilpotent Lie group with Lie algebra $\mathcal{N}$ and left invariant metric $\langle\ ,\ \rangle$

We show there is a vector in $\mathcal{N}$ for which the corresponding one-parameter subgroup cannot lie in any 2-dimensional minimal subgroup of $(N, \langle\ ,\ \rangle)$.

We consider $V = 1/\sqrt{3}\,(X_3+Y_2+Z)$ and we shall prove that the one-parameter subgroup $\exp tV$ cannot lie in any 2-dimensional minimal subgroup H of $(N, \langle\ ,\ \rangle)$.

In fact, let us assume that $H = \exp(\mathcal{H})$ is a 2-dimensional subgroup containing the one-parameter subgroup $\exp tV$, then $\mathcal{H}$ must contain a unit vector U such that $\langle U, V\rangle = 0$ and $[U, V] = 0$, the latter is due to the fact that $\mathcal{H}$ must be abelian since it is a 2-dimensional nilpotent Lie algebra. We shall show that H cannot be minimal.

Let

$$U = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \beta_1 Y_1 + \beta_2 Y_2 + \gamma Z,$$

since

$$0 = [U, V] = -\alpha_2 Y_2 - (\beta_1 + \alpha_1)Z,$$

this implies $\alpha_2 = 0$ and $\beta_1 + \alpha_1 = 0$.

On the other hand

$$0 = \langle U, V\rangle = \alpha_3 + \beta_2 + \gamma$$

and hence

$$\gamma = -(\alpha_3 + \beta_2).$$

Therefore

$$U = \alpha_1(X_1 - Y_1) + \alpha_3 X_3 + \beta_2 X_2 - (\alpha_3 + \beta_2)Z$$

and

$$2(\alpha_1^2 + \alpha_3^2 + \beta_2^2 + \alpha_3\beta_2) = 1.$$

From now on we indicate $\alpha = \alpha_3$ and $\beta = \beta_2$, with this notation a computation using formula (1) gives us

$$\begin{aligned}\nabla_U U + \nabla_V V = [1/3 - \beta(\alpha+\beta)]X_1 + [1/3 + (\alpha_1^2 + \alpha\beta)]X_2 - \\ - \alpha_1(\alpha+\beta)X_3 + [1/3 - \alpha(\alpha+\beta)]Y_1 + \\ + \alpha_1(\alpha+\beta)Y_2.\end{aligned}$$

We have two cases to be considered, namely.

CASE 1: $\alpha_1 = 0$. We claim that $\nabla_U U + \nabla_V V \neq 0$. In fact, if it were equal to zero, then

$$1/3 - \beta(\alpha+\beta) = 1/3 + \alpha\beta = 1/3 - \alpha(\alpha+\beta) = 0.$$

From this follows easily that $\alpha^2 = \beta^2 = 2/3$.

On the other hand

$$||U||^2 = 2\alpha^2 + 2\beta^2 + 2\alpha\beta$$

and by substituting the values of α^2 and β^2, we would have that $||U||^2 = 2$, contradicting the fact that U is a unit vector. It is also clear that $\nabla_U U + \nabla_V V \notin \mathcal{H}$.

CASE 2: $\alpha_1 \neq 0$. We have two subcases:

i) $\alpha + \beta = 0$, and then it follows immediately that $\nabla_U U + \nabla_V V \notin \mathcal{H}$.

ii) $\alpha + \beta \neq 0$, then $\nabla_U U + \nabla_V V \neq 0$. We claim that $\nabla_U U + \nabla_V V \notin \mathcal{H}$, if it were not so, then there would exist constants A and B such that

$$\nabla_U U + \nabla_V V = AU + BV.$$

Now by looking at the components in the X_3, Y_2 and Z directions we would have $A = B = 0$, contradicting the fact that $\nabla_U U + \nabla_V V \neq 0$. This concludes our assertion.

REMARK. It seems quite reasonable to conjecture that the property of Theorem 2.1 characterizes the 2-step nilpotent Lie groups in the class of nilpotent Lie groups with a left invariant metric.

3. SOLVABLE LIE GROUPS

Let $(\mathcal{N}, \langle\ ,\ \rangle)$ be a 2-step nilpotent Lie algebra with an inner product $\langle\ ,\ \rangle$ as in Section 2.

We define

$$\begin{aligned} &\mathcal{N} = \mathcal{V} \oplus \mathcal{Z} \\ &\mathcal{S} = \mathbf{R}\,A \oplus \mathcal{N} && \text{by} \\ &\mathrm{ad}_A V = \lambda V, && V \in \mathcal{V}, \\ &\mathrm{ad}_A Z = 2\lambda Z, && Z \in \mathcal{Z}, \end{aligned}$$

$\lambda \in \mathbf{R}$, $\lambda \neq 0$.

$\mathcal{S}$ becomes a solvable Lie algebra and we define an inner product on $\mathcal{S}$ by assuming that $||A|| = 1$ and A is orthogonal do $\mathcal{N}$.

Let $(S, \langle\ ,\ \rangle)$ denote the simply connected solvable Lie group with Lie algebra $\mathcal{S}$ and left invariant metric $\langle\ ,\ \rangle$.

We have the following formulas for the covariant derivative

$$\begin{cases} \nabla_A B = 0. \qquad \nabla_A V = 0; \qquad \nabla_A Z = 0; \\ \nabla_V A = -\lambda V; \qquad \nabla_Z A = -2\lambda Z; \\ \nabla_V W = 1/2\,[V, W] + \lambda \langle V, W \rangle A; \\ \nabla_Z Z' = 2\lambda \langle Z, Z' \rangle A; \\ \nabla_V Z = \nabla_Z V = -1/2\, j(Z) V \end{cases}$$

where $B \in \mathbf{R}\,A$, $V, W \in \mathcal{V}$, $Z, Z' \in \mathcal{Z}$.

For more details on the geometry of $(S, \langle\ ,\ \rangle)$, see [1] and [4].

THEOREM 3.1. Let $(S, \langle\ ,\ \rangle)$ be as above. Every one-parameter subgroup of S lies in at least one 3-dimensional connected subgroup H which is a minimal submanifold of $(S, \langle\ ,\ \rangle)$.

PROOF: We just consider the generic case, that is, $X = a\,A + V + Z$, $a \neq 0$, $0 \neq V \in \mathcal{V}$ and $0 \neq Z \in \mathcal{Z}$. We take $\mathcal{H} = \mathrm{span}\,\{A, V, Z\}$ and $H = \exp(\mathcal{H})$. By appling formulas (2) to the orthonormal basis $\{A, V/||V||, Z/||Z||\}$ of $(\mathcal{H}, \langle\ ,\ \rangle)$ we get

$$\begin{aligned} &\nabla_A A + \nabla_{V/||V||} V/||V|| + \nabla_{Z/||Z||} Z/||Z|| = \\ &= \lambda \langle V/||V||, V/||V||\rangle\, A + 2\lambda \langle Z/||Z||, Z/||Z||\rangle\, A = \\ &= 3\lambda A \in \mathcal{H}. \end{aligned}$$

This shows that $H = \exp(\mathcal{H})$ is a minimal submanifold of $(S, \langle\ ,\ \rangle)$ by Proposition 1.2.

REMARK. If $(N, \langle\ ,\ \rangle)$ is of Heisenberg type or nonsingular then the subgroup $H = \exp(\mathcal{H})$ of Theorem 3.1 is minimal without being totally geodesic, in fact, $\nabla_Z V = -1/2\, j(Z)V \notin \mathcal{H}$.

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Carlos J. Ferraris
CFM - Depto. de Matemática
Universidade Federal de Santa Catarina
Campus Universitário – Trindade
CEP 88049 Florianópolis, SC
Brasil

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