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BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS ON H_ω.

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Abstract: We study the boundedness of singular integral operators on Orlicz-Hardy spaces H_ω, in the setting of spaces of homogeneous type. As an application of this result, we obtain a characterization of $H_\omega \mathbb{R}^n$ in terms of the Riesz Transforms.

§ 1. NOTATION AND DEFINITIONS

Let X be a set. A function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ shall be called a quasi-distance on X if there exists a finite constant K such that

(1.1) $$d(x,y) = 0 \text{ if and only if } x = y$$

(1.2) $$d(x,y) = d(y,x)$$

and

(1.3) $$d(x,y) \leq K[d(x,z) + d(z,y)]$$

for every x, y and z in X.

In a set X, endowed with a quasi-distance $d(x,y)$, the balls

$$B(x,r) = \{y : d(x,y) < r\},\ r > 0,$$

form a basis for the neighbourhoods of x in the topology induced by the uniform structure on X.

We shall say that a set X, with a quasi-distance $d(x,y)$ and a non-negative measure μ defined on a σ-algebra of subsets of X containing the balls $B(x,r)$, is a normal space of homogeneous type if there exist four positive finite constants A_1, A_2, K_1 and $K_2, K_2 \leq 1 \leq K_1$, such that

$$A_1 r \leq \mu(B(x,r)) \text{ if } r \leq K_1\mu(X) \tag{1.4}$$

$$B(x,r) = X \qquad \text{if } r > K_1\mu(X) \tag{1.5}$$

$$A_2 r \geq \mu(B(x,r)) \quad \text{if } r \geq K_2\mu(\{x\}) \tag{1.6}$$

$$B(x,r) = \{x\} \qquad \text{if } r < K_2\mu(\{x\}). \tag{1.7}$$

We note that, under these conditions, there exists a finite constant A, such that

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) \tag{1.8}$$

holds for every $x \in X$ and $r > 0$.

We shall say that a normal space of homogeneous type (X, d, μ) is of order $\alpha, 0 < \alpha < \infty$, if there exists a finite constant K_3 satisfying

$$|d(x,z) - d(y,z)| \leq K_3 r^{1-\alpha} d(x,y)^\alpha \tag{1.9}$$

for every x, y and z in X, whenever $d(x,z) < r$ and $d(y,z) < r$ (See [MS]).

Throughout this paper $X = (X, d, \mu)$ shall denote a normal space of homogeneous type of order $\alpha, 0 < \alpha \leq 1$.

Let ρ be a positive function defined on $\mathbb{R}^+$. We shall say that ρ is of upper type m (respectively, lower type m) if there exists a positive constant c such that

$$\rho(st) \leq ct^m \rho(s), \tag{1.10}$$

for every $t \geq 1$ (respectively, $0 < t \leq 1$). A non-decreasing function ρ of finite upper type such that $\lim_{t \to 0^+} \rho(t) = 0$ is called a growth function.
For $\rho(t)$ a positive right-continuous non-decreasing function satisfying $\lim_{t \to 0^+} \rho(t) = 0$ and $\lim_{t \to \infty} \rho(t) = \infty$, the function

$$\Phi(t) = \int_0^t \rho(s)ds \tag{1.11}$$

will be called a *Young* function.

Given $\Phi(t)$ a *Young* function of finite upper type, we define the *Orlicz* space L_Φ by

$$L_\Phi = \{f : \int \Phi(|f(x)|)dx < \infty\},$$

and we denote by

$$\| f \|_{L_\Phi} = inf\{\lambda : \int \Phi\left(\frac{|f(x)|}{\lambda}\right) \leq 1\}$$

the Luxemburg norm.

Given a *Young* function, we consider the complementary *Young* function of Φ defined by

$$\psi(t) = \int_0^t q(s)ds \ , \text{ with } q(s) = \sup_{\rho(t) \leq s} t.$$

For $\Phi(x)$ a *Young* function, the *Hölder* inequality

$$| \int f(x)g(x)dx| \leq \| f \|_{L_\Phi} \| g \|_{L_\psi} \tag{1.12}$$

holds for every $f \in L_\Phi$ and $g \in L_\psi$.

We shall understand that two positive functions are equivalent if their ratio is bounded above and below by two positive constants.
Let ρ be a growth function. We shall say that a function $\psi(x)$ belongs to $Lip(\rho)$, if

$$\| \psi \|_{Lip(\rho)} = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{\rho(d(x,y))} < \infty$$

holds. When $\rho(t)$ is the function t^β, $0 < \beta < \infty$, we shall say that $\psi(t)$ is in $Lip(\beta)$ and, in this case, $\| \psi \|_\beta$ indicates its norm.

The space of distributions $(E^\alpha)'$, introduced by *Macías and Segovia* in [MS], is the dual space of E^α consisting of all function with bounded support belonging to Lip (β) for some $0 < \beta < \alpha$.

For $x \in X$ and $0 < \gamma < \alpha$, we consider the class $T_\gamma(x)$ of functions ψ belonging to E^α satisfying the following condition: there exists r such that $r \geq K_2\mu(\{x\})$, $supp\psi \subset B(x,r)$ and

(1.13) $$r \parallel \psi \parallel_\infty \leq 1 \quad \text{and} \quad r^{1+\gamma} \parallel \psi \parallel_\gamma \leq 1.$$

Given $\gamma, 0 < \gamma < \alpha$, we define the γ-maximal function $f^*_\gamma(x)$ of a distribution f on E^α by

(1.14) $$f^*_\gamma = sup\{|f(\psi)| : \psi \in T_\gamma(x)\}.$$

(1.15) **Definition**: *Let ρ be a growth function plus a non negative constant or $\rho \equiv 1$. A (ρ, q)-atom, $1 < q \leq \infty$, is a function $a(x)$ on X satisfying:*

(1.16) $$\int_X a(x) d\mu(x) = 0,$$

(1.17) the support of $a(x)$ is contained in a ball B and

(1.18) $$\left[\mu(B)^{-1} \int_B |a(x)|^q\right]^{1/q} \leq [\mu(B)\rho(\mu(B))]^{-1} \text{ if } q < \infty$$

or

$$\parallel a\parallel_\infty \leq [\mu(B)\rho(\mu(B))]^{-1}, \text{ if } q = \infty.$$

Clearly, when $\rho(t) = t^{1/p-1}, p \leq 1$, a (ρ, q)-atom is a (p, q)-atom in the sense of [M-S].

Let ω be a growth function of positive lower type l such that $l(1+\alpha) > 1$. For every γ with $0 < \gamma < \alpha$ and $l(1+\gamma) > 1$, we define

(1.19) $$H_\omega = H_\omega(X) = \left\{f \in (E^\alpha)' : \int \omega[f^*_\gamma(x)] d\mu(x) < \infty\right\}.$$

and we denote

(1.20) $$\parallel f\parallel_{H_\omega} = \parallel f^*_\gamma\parallel_{H_\omega} = inf\left\{\lambda > 0 : \int \omega\left[\frac{f^*_\gamma(x)}{\lambda^{1/l}}\right] d\mu(x) \leq 1\right\}.$$

Let ω be a growth function of positive lower type l. If $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$, we define the atomic Orlicz Space $H^{\rho,q}(X) = H^{\rho,q}, 1 < q \leq \infty$, as the space of all distributions f on E^{α} which can be represented by

$$f(\psi) = \sum_i b_i(\psi), \tag{1.21}$$

for every ψ in E^{α}, where $\{b_i\}_i$ is a sequence of multiples of (ρ, q)-atoms such that if $supp(b_i) \subset B_i$, then

$$\sum_i \mu(B_i)\omega\left(\| b_i\|_q \mu(B_i)^{-1/q}\right) < \infty. \tag{1.22}$$

Given a sequence of multiples of (ρ, q)-atoms, $\{b_i\}_i$ we set

$$\Lambda_q(\{b_i\}) = inf\left\{\lambda : \sum_i \mu(B_i)\omega\left(\frac{\| b_i\|_q \mu(B_i)^{-1/q}}{\lambda^{1/l}}\right) \leq 1\right\} \tag{1.23}$$

and we define

$$\| f\|_{H^{\rho,q}} = inf \Lambda_q(\{b_i\}), \tag{1.24}$$

where the infimum is taken over all possible representations of f of the form (1.21).

It has been shown in [V] that the spaces H_{ω} and $H^{\rho,q}$ are equivalent. More precisely, in that paper the following Theorem is proved

THEOREM A: *Let ω be a function of lower type l such that $l(1+\alpha) > 1$. Assume that $\omega(s)/s$ is non-increasing. Let $\rho(t)$ be the function defined by $t\rho(t) = 1/\omega^{-1}(1/t)$. Then $H_{\omega} \equiv H^{\rho,q}$ for every $1 < q \leq \infty$.*

We observe that the statement of the Theorem A implies in particular that the definition of H_{ω} is independent of $\gamma, 0 < \gamma < \alpha$ and $l(1+\gamma) > 1$. Furthermore, from proposition (3.1) in [V], we may assume without lost of generality, that ω is, in additon, continuous, strictly increasing and a subaditive function.

§ 2. BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS ON HARDY-ORLICZ SPACES

In this section (X, d, μ) shall mean a normal space of homogeneous type of order α, $0 < \alpha \leq 1$ and K shall denote the constant appearing in (1.3).

We assume that a singular kernel is a measurable function $k : X \times X \to I\!R$ satisfying the following conditions:

(2.1) $|k(x, y)| \leq cd(x, y)^{-1}$ for $x \neq y$

(2.2) There exist δ, $0 < \delta \leq \alpha$, such that

$$|k(x, y) - k(x', y)| + |k(y, x) - k(y, x')| \leq cd(x, x')^{\delta} d(x, y)^{-1-\delta},$$

provided $d(x, y) > 2d(x, x')$.

(2.3) Let $0 < r < R < \infty$, then

a) $\int_{r \leq d(x,y) < R} k(x, y) d\mu(y) = 0$, for every $x \in X$.

and

b) $\int_{r \leq d(x,y) < R} k(y, x) d\mu(y) = 0$, for every $x \in X$.

Given $\varepsilon > 0$, we define

$$T_{\varepsilon} f(x) = \int_{\varepsilon \leq d(x,y) < 1/\varepsilon} k(x, y) f(y) d\mu(y).$$

For singular integrals, in the context of spaces of homogeneous type, conditions for their boundedness on L^2 were given in [A], [D-J-S], [M-T] and [M-S-T].

In the sequel we shall assume that T is a bounded singular integral operator on $L^2(X)$ associated to a kernel $k(x, y)$ satisfying (2.1), (2.2) and (2.3). Under these assumptions we shall obtain, in Theorem 2.20, the boundedness of T on the spaces H_{ω}.

In order to prove the main theorem we shall need some previous results.

(2.4) LEMMA. *Let $k(x,y)$ be a kernel satisfying* (2.1) *and* (2.3). *Let $\Phi(t)$ be a Lipschitz function defined on $[0,\infty)$ such that $\Phi(t)=0$ for $t\geq 2$. Assume that $\Phi(t)$ satisfies one of the following two conditions:*

a) $\Phi(t)=1$ for $t\leq 1$, or

b) $\Phi(t)=0$ for $t\leq 1$.

Let $0<r<R<\infty$, then

$$\int_{r\leq d(x,y)<R} k(x,y)\Phi(d(x,y))d\mu(y)=0, \text{ for every } x\in X.$$

PROOF. We prove the lemma for Φ satisfying (a). The other case follows the same lines. Given $0<r<R$, we have three possibilities:

i) $2\leq r$,

ii) $0<r<2<R$

iii) $0<r<R\leq 2$.

If $r\geq 2$ the lemma follows inmediately. Supposse that (ii) holds. Since $k(x,y)$ satisfies (2.3) and $\Phi(t)=1$ for $t\leq 1$, it is enough to assume that $r\geq 1$ in this case. Given $\varepsilon>0$, let $P=\{t_0,t_1,\cdots t_N\}$ be a partition of the interval $[r,2]$, with $\Delta t_i=t_i-t_{i-1}<\delta$ and δ a constant depending on ε to be determined later. Then we have

$$\int_{r\leq d(x,y)<R} k(x,y)\Phi(d(x,y))d\mu(y)=\sum_{i=1}^{N}\int_{t_{i-1}\leq d(x,y)<t_i} k(x,y)[\Phi(d(x,y))-\Phi(t_i)]d\mu(y)$$
$$+\sum_{i=1}^{N}\Phi(t_i)\int_{t_{i-1}\leq d(x,y)<t_i} k(x,y)d\mu(y).$$

Using that Φ is a *Lipschitz* function and applying (2.1) and (2.3), we obtain

$$
\begin{aligned}
\Big|\int_{r\le d(x,y)<R} k(x,y)\Phi(d(x,y))d\mu(y)\Big| &\le c\delta\sum_{i=1}^{N}\int_{t_{i-1}\le d(x,y)<t_i} |k(x,y)|d\mu(y)\\
&\le c\delta \int_{1\le d(x,y)<2} |k(x,y)|d\mu(y)\\
&\le c\delta.
\end{aligned}
$$

Choosing δ such that $c\delta < \varepsilon$,we conclude the proof of (ii). The remaining case (iii) follows the same line.

(2.5) REMARK. *Let* Φ *be as in Lemma* (2.4). *For* $\varepsilon > 0$, *the kernel* $k(x,y)$ $\Phi(\frac{d(x,y)}{\varepsilon})$ *satisfies* (2.1) *and, from Lemma* (2.4), *also verifies* (2.3). *On other hand , since* X *is of order* α, (2.2) *holds with constant independent of* ε.

Let ψ_1 and ψ_2 in $C^\infty([0,\infty))$ satisfying the following conditions: $supp\psi_1 \subset [1/2,\infty)$ and $\psi_1(t) = 1$ if $t \ge 1$; $supp\psi_2 \subset [0,2]$ and $\psi_2(t) = 1$ for $t \le 1$. For $f \in L^p, 1 \le p < \infty$, we define

$$\widetilde{T}_\varepsilon f(x) = \int k(x,y)\psi_1(\frac{d(x,y)}{\varepsilon})\psi_2(\varepsilon d(x,y))f(y)d\mu(y).$$

(2.6) LEMMA. *Let* $k(x,y)$ *be a singular kernel satisfying* (2.1), (2.2) *and* (2.3). *Then,*

$$\| \widetilde{T}_\varepsilon f - Tf\|_{L^2} \to 0 \ , \ \text{as } \varepsilon \to 0.$$

PROOF. We have

$$
\begin{aligned}
\widetilde{T}_\varepsilon f(x) = &\int_{\varepsilon/2\le d(x,y)\le\varepsilon} k(x,y)\psi_1\left(\frac{d(x,y)}{\epsilon}\right) f(y)d\mu(y) + T_\varepsilon f(x)\\
&+ \int_{1/\varepsilon\le d(x,y)<2/\varepsilon} k(x,y)\psi_2(\varepsilon d(x,y))f(y)d\mu(y) = T^1_\varepsilon f(x) + T_\varepsilon f(x) + T^2_\varepsilon f(x).
\end{aligned}
$$

Since $T_\varepsilon f(x)$ converges to Tf in L^2, we only need to prove that $T^i_\varepsilon f$ converges to zero in L^2 for $i = 1,2$. Clearly from (2.1), we have

$$T^i_\varepsilon f(x) \le cMf(x) \ , \ \text{for } i = 1,2. \tag{2.7}$$

From (2.7) and by the density in L^2 of the *Lipschitz* γ functions with bounded support, it is enough to prove the convergence of $T^i_\varepsilon f$ for such functions. Let f be a function with bounded support belonging to $Lip(\gamma)$. Then by Lemma (2.4), we get

$$(2.8)\quad |T^1_\varepsilon f(x)| = |\int_{\varepsilon/2<d(x,y)<\varepsilon} k(x,y)\psi_1\left(\frac{d(x,y)}{\varepsilon}\right)[f(y)-f(x)]d\mu(y)| \leq c \parallel f \parallel_\gamma \varepsilon^\gamma.$$

On the other hand from (2.1), we obtain

$$\begin{aligned} |T^2_\varepsilon f(x)| &\leq \varepsilon \int_{1/\varepsilon\leq d(x,y)<2/\varepsilon} |\psi_2(\varepsilon d(x,y))||f(y)|d\mu(y) \\ &\leq \varepsilon \parallel f \parallel_{L^2} \left(\int_{1/\varepsilon\leq d(x,y)<2/\varepsilon} |\psi_2(\varepsilon d(x,y))|^2 d\mu(y) \right)^{1/2} \\ (2.9)\qquad &\leq c \parallel f \parallel_{L^2} \varepsilon^{1/2}. \end{aligned}$$

By (2.7), (2.8), (2.9) and the *Lebesgue* dominated convergence Theorem, the desired conclusion follows, ending the proof of the Lemma.

(2.10) LEMMA. (Partition of unity). *Let $x \in X$ and $r > 0$. Then, there exists a sequence $\{\Phi^r_j(x,y)\}_{j\geq 0}$ of non-negative functions satisfying:*

(2.11) *the support of Φ^r_j for $j \geq 1$ is contained in the ring $C(x,(2K)^j r,(2K)^{j+2}r)$,*

(2.12) *the support of Φ^r_0 is contained in $B(x,4Kr)$* and *$\Phi^r_0(x) = 1$ on $B(x,3Kr)$,*

(2.13) *there exists a constant c shuch that for every $j \geq 0, \Phi^r_j \in Lip(\alpha)$ as functions of y* with $\parallel \Phi^r_j \parallel_\alpha \leq c(2K)^{-j\alpha}r^{-\alpha}$,

(2.14) $\sum\limits_{j\geq 0}^{\infty} \Phi^r_j(x,y) = 1$ *for every $y \in X$.*

PROOF. Let $\eta(t)$ and $\gamma(t)$ in $C^\infty([0,\infty))$ satisfying: $0 \leq \eta(t) \leq 1$, *supp* $\eta \subset [0,4K]$, $\eta(t) = 1$ if $0 \leq t \leq 3K; 0 \leq \gamma(t) \leq 1$, *supp*$\gamma \subset [2K,8K^3]$ and $\gamma(t) = 1$ if $3K \leq t \leq 6K^2$.

Taking $\psi_0(x,y) = \eta\,(d(x,y)/r)$ and $\psi_j(x,y) = \gamma(\frac{d(x,y)}{r(2K)^{j-1}})$ for every $j \geq 1$, it follows easily that $\Phi_j^r(x,y) = \psi_j(x,y))/\sum_{k\geq 0}\psi_k(x,y)$ for $j \geq 0$, satisfy all the conditions in the lemma.

LEMMA (2.15). *Let $k(x,y)$ be a kernel satisfying* (2.1), (2.2) and (2.3). *Let $b(x)$ be a multiple of a (ρ,∞) atom with support contained in $B(x_0,r)$. Assume that $\{\Phi_j^r(x,y)\}_{j\geq 0}$ is as in Lemma* (2.10) *and T_j^r is the operator associated to the kernel $k_j^r = k(x,y)\Phi_j^r(x,y)$, for $j \geq 0$. Then*

(2.16) *the support of $T_j^r b$ is contained in $B(x_0,(2K)^{j+3}r)$ for $j \geq 0$,*

(2.17) $\quad \| T_j^r b\|_\infty \leq \frac{c\|b\|_\infty}{(2K)^{j(1+\delta)}}$ *for* $j \geq 1$, $\| T_0^r b\|_{L^2} \leq c \| b\|_\infty\, \mu(B(x_0,r))^{1/2}$, *and*

(2.18) $\quad \int T_j^r b(x) d\mu(x) = 0$ *for every* $j \geq 0$.

PROOF. Let us first note that if $C(x,(2K)^j r,(2K)^{j+2}r) \cap B(x_0,r) \neq \emptyset$ for $j \geq 1$, from (1.3),we have

$$(2K)^{j-1}r \leq d(x,x_0) \leq (2K)^{j+3}r. \tag{2.19}$$

Therefore if $x \notin C(x_0,(2K)^{j-1}r,(2K)^{j+3}r)$, then $T_j^r b(x) = 0$ for every $j \geq 1$. For $j = 0$, it is clear that supp $(T_0^r b) \subset B(x_0, 8K^2 r)$, and hence (2.16) follows. Next we shall prove (2.17). By remark (2.5), we get

$$\| T_0^r b\|_2 \leq c \| b\|_2 \leq c \| b\|_\infty\, \mu(B(x_0,r))^{1/2}.$$

On the other hand, since X is a normal space, from (2.5) and (2.19) we obtain, that for any $j \geq 1$.

$$\begin{aligned} |T_j^r b(x)| &= |\int [K(x,y)\Phi_j^r(x,y) - K(x,x_0)\Phi_j^r(x,x_0)]b(y)d\mu(y)| \\ &\leq c \| b\|_\infty \int_{d(y,x_0)<r} \frac{d(y,x_0)^\delta}{d(x_0,x)^{1+\delta}} d\mu(y) \\ &\leq \frac{c}{(2K)^{j(1+\delta)}} \| b\|_\infty . \end{aligned}$$

Finally, (2.18) is a consequence of Lemma (2.4).

Now we are in position to prove the main result.

THEOREM 2.20 *Let T be a singular integral operator associated to a kernel $k(x,y)$ satisfying* (2.1), (2.2) *with $\delta > 1/l - 1$ and* (2.3). *Assume that $l(1+\alpha) > 1$. Then, T is a bounded operator from H_ω into H_ω.*

PROOF: By the density of $L^2(X)$ in H_ω, it is enough to show the theorem for $f \in L^2(X) \bigcap H_\omega$. Given $\epsilon > 0$, from Theorem A and (1.24), there exists a sequence $\{b_k\}_k$ of multiples of (ρ, ∞) atoms with $supp(b_k) \subset B_k = B(x_k, r_k)$, such that $f = \sum_k b_k$ in $(E^\alpha)'$ and

$$\| f \|_{H_\omega} (1+\epsilon) \geq \Lambda_\infty(\{b_k\}). \tag{2.21}$$

If we are able to prove that

$$Tf = \sum_k Tb_k \quad \text{in} \quad (E^\alpha)' , \tag{2.22}$$

we will get $Tf \in H_\omega$ and $\| Tf \|_{H_\omega} \leq c \| f \|_{H_\omega}$. In fact, let $\{\Phi_j^k\}_j$ be a partition of the unity as in Lemma (2.10) associated to B_k, therefore

$$Tf = \sum_k \sum_{j \geq 1} T_j^k b_k + \sum_k T_0^k b_k \quad \text{in} \quad (E^\alpha)' . \tag{2.23}$$

Futhermore, Lemma (2.15) implies that $\{T_j^k b_k\}_{j,k}$ are multiples of a (ρ, ∞) atom. Hence, from (1.24) it follows that

$$\| Tf \|_{H_\omega} \leq \Lambda_2(\{T_j^k b_k\}_{j,k}) + \Lambda_2(\{T_0^k b_k\}_k). \tag{2.24}$$

Let $\eta \geq 1$ be a constant to be determined later, $\lambda = \eta \Lambda_\infty(\{b_k\}_k)$ and $B_k^j \supset supp(T_j^k b_k), j \geq 0$. We now estimate

$$\sum_k \sum_{j \geq 1} \mu(B_j^k) \omega \left(\frac{\| T_j^k b_k \|_2 \, \mu(B_j^k)^{-1/2}}{\lambda^{1/l}} \right) . \tag{2.25}$$

By (1.8), (2.16) and (2.17), the sum (2.25) is bounded by

$$c \sum_k \sum_{j \geq 1} (c2K)^j \mu(B_k) \omega \left(\frac{\| b_k \|_\infty}{(2K)^{j(1+\delta)} \lambda^{1/l}} \right)$$

since ω is of lower type $l > 1/1+\delta$), (2.25) is bounded by

$$c\sum_{j\geq 1}(c2K)^{j(1-(1+\delta)l)}\sum_k \mu(B_k)\omega\left(\frac{\| b_k \|_\infty}{\lambda^{1/l}}\right)$$
$$\leq c\sum_k \mu(B_k)\omega\left(\frac{\| b_k \|_\infty}{\lambda^{1/l}}\right).$$

Therefore, using again that ω is of lower type l and choosing $\eta = c$, the sum (2.25) is less than or equal to 1, which implies

$$\Lambda_2(\{T_j^k b_k\}_{j,k}) \leq c\Lambda_\infty(\{b_k\}). \tag{2.26}$$

On the other hand, by (2.5) T_0^k is a bounded operator on L^2, thus applying (1.8), (2.16), (2.17) and the fact that $\omega(s)/s$ is nonincreasing, we get

$$\sum_k \mu(B_0^k)\omega\left(\frac{\| T_0 b_k \|_2 \, \mu(B_k^0)^{-1/2}}{\lambda^{1/l}}\right) \tag{2.27}$$
$$\leq c\sum_k \mu(B_k)\omega\left(\frac{c \| b_k \|_\infty}{\lambda^{1/l}}\right)$$
$$\leq \sum_k \mu(B_k)\omega\left(c^{1/l}\frac{\| b_k \|_\infty}{\lambda^{1/l}}\right)$$

Taking $\eta = c$, and using (2.27), it follows that

$$\Lambda_2(\{T_0^k b_k\}_k) \leq c\Lambda_\infty(\{b_k\}_k). \tag{2.28}$$

Collecting the estimates (2.21), (2.24), (2.26) and (2.28), we obtain that

$$\| Tf \|_{H_\omega} \leq c \| f \|_{H_\omega}$$

In order to prove (2.22), let us first note that if $\widetilde{T}f$ is the operator of Lemma (2.6) associated to the kernel $\widetilde{k}_\varepsilon(x,y)$, then $\widetilde{k}_\varepsilon(x,.)$is a function of bounded support belonging to $Lip(\delta)$ for each $x \in X$. Therefore

$$\widetilde{T}_\varepsilon f = \sum_k \widetilde{T}_\varepsilon b_k, \text{ pointwise and in } (E^\alpha)' .$$

Moreover Lemma (2.6) implies that $\widetilde{T}_\varepsilon f$ converges to Tf in L^2. In consequence, if we are able to show

(2.29) $$\sum_k \widetilde{T}_\varepsilon b_k \underset{\varepsilon\to 0}{\longrightarrow} \sum_k Tb_k \text{ in } H_\omega,$$

then (2.22) holds inmediately, completing the proof of the Theorem. Now, in order to prove (2.29), we decompose both operators, $\widetilde{T}_\varepsilon$ and T, as in (2.23). Therefore, we have

(2.30) $$\begin{aligned}\sum_k(\widetilde{T}_\varepsilon b_k - Tb_k) &= \sum_k\sum_{j\geq 0}(\widetilde{T}^k_{\varepsilon,j}b_k - T^k_j b_k)\\ &= \sum_k\sum_{j\geq 0}\bar{T}^k_{\varepsilon,j}b_k\ ,\end{aligned}$$

where $\bar{T}^k_{\varepsilon,j}$ is the operator associated to the kernel

$$\bar{K}^k_{\varepsilon,j}(x,y) = K(x,y)[\psi_1(\frac{d(x,y)}{\varepsilon})\psi_2(d(x,y)\varepsilon) - 1]\Phi^k_j(x,y) =: \bar{K}_\varepsilon(x,y)\Phi^k_j(x,y).$$

Since by (2.5) $\bar{K}_\varepsilon(x,y)$ satisfies (2.1), (2.2) and (2.3) with a constant independent of ε, using Lemma (2.15) and proceeding as in estimates (2.25) and (2.27), we get that

$$\sum_k\sum_{j\geq 0}\mu(\bar{B}^k_j)\omega(\|\ \bar{T}^k_{\varepsilon,j}b_k\ \|_2\ \mu(\bar{B}^k_j)^{-1/2}) < \infty,$$

where $\bar{B}^k_j \supset supp(\bar{T}^k_{\varepsilon,j}b_k)$. Thus, given $0<\beta\leq 1$, there exists $N = N(\beta)$ such that

(2.31) $$\sum_{|k|>N}\sum_{j>N}\mu(\bar{B}^k_j)\omega(\|\ \bar{T}^k_{\varepsilon,j}b_k\ \|_2\ \mu\left(\bar{B}^k_j)^{-1/2}\right) < \beta/2.$$

This finishes the proof of the Theorem.

§ 3. CHARACTERIZATION OF THE ORLICZ-HARDY SPACES H_ω

In this section we shall work, as before, on a normal space $X = (X, d, \mu)$ of order α.

Let $\{b_i\}_i$ a sequence of multiples of (ρ, q) atoms, $1 < q \leq \infty$, such that $\Lambda_q(\{b_i\}) < \infty$ and $\alpha_i = \| b_i \|_q \ \mu(Bi)^{-1/q}/\omega^{-1}(\mu(B_i)^{-1})$, where $B_i \supset supp(b_i)$. Let $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$ and $\psi(x) \in Lip(\rho)$. Then

$$(3.1) \qquad |\sum_i b_i(\psi)| \leq \| \psi \|_{Lip(\rho)} \sum_i \rho(r_i)\mu(B_i)^{1/q'} \ \| b_i \|_q$$
$$\leq c \ \| \psi \|_{Lip(\rho)} \sum_i \alpha_i.$$

In order to estimate the sum $\sum_i \alpha_i$ we shall need the following lemma whose proof can be found in [V], p. 410.

(3.2) LEMMA: *Assume that $\rho(t)$, $\{b_i\}_i$ and α_i are as above. Then there exists a constant c independent of $\{b_i\}$, such that*

$$\sum_i \alpha_i \leq c(\Lambda_q(\{b_i\}) + 1)^{1/l^2}.$$

Using Lemma (3.2), by (3.1) it follows that the serie $\sum_i b_i(\psi)$ is absolutely convergent for every $\psi \in Lip(\rho)$. Thus, if we define

$$(3.3) \qquad f(\psi) = \sum_i b_i(\psi) \ ,$$

we obtain a linear funtional on $Lip(\rho)$ satisfying

$$(3.4) \qquad |f(\psi)| \leq c \ \| \psi \|_{Lip(\rho)} \ [\Lambda_q(\{b_i\} + 1)]^{1/l^2}$$

(3.5) DEFINITION: *Let ω be a growth function of positive lower type l. If $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$, we define $\widetilde{H}^{\rho,q}(X) = \widetilde{H}^{\rho,q}$, $1 < q \leq \infty$, as the linear space of all bounded linear functionals f on $Lip(\rho)$ which can be represented as in* (3.3), *where $\{b_i\}$ is a sequence of multiples of (ρ, q) atoms such that $\Lambda_q(\{b_i\}) < \infty$. For $f \in \widetilde{H}^{\rho,q}$, we define*

$$\| f \|_{\widetilde{H}^{\rho,q}} = inf \, \{\Lambda_q(\{bi\})\} \ ,$$

where the infimum is taken over all possible representations of f of the form (4.3).

We now observe that, since every ψ in E^α belongs to $Lip(\rho)$, we can define the linear transformation R from $\widetilde{H}^{\rho,q}$ into H_ω given by

$$(3.6) \qquad R\,(f) \ = \ \widetilde{f} \ ,$$

where $\widetilde{f}$ is the restriction of f to E^α.

The next result states that R is an isomorphism onto H_ω. Its proof makes use of the atomic decomposition of H_ω and Lemma (5.5) in [V], and it follows the lines of (5.9) in [MS].

(3.7) THEOREM: *Let R be as in* (3.6). *Then R defines a one to one linear mapping from $\widetilde{H}^{\rho,q}$ onto H^ω. Moreover, there exist two positive constants c_1 and c_2 such that*

$$c_1 \parallel f \parallel_{\widetilde{H}^{\rho,q}} \leq \parallel Rf \parallel_{H_\omega} \leq c_2 \parallel f \parallel_{\widetilde{H}^{\rho,q}} . \tag{3.8}$$

PROOF: Let $f = \sum_i b_i$ in $\widetilde{H}^{\rho,q}$. Theorem A implies that

$$R(\widetilde{H}^{\rho,q}) \subset H_\omega \text{ and } \parallel Rf \parallel_{H_\omega} \leq c \parallel f \parallel_{\widetilde{H}_{\rho,q}}$$

On the other hand, given $g \in H_\omega$, again by Theorem A, there exists a sequence $\{b_i\}$ of multiples of (ρ, q) atoms such that

$$g = \sum_i b_i \text{ in } (E^\alpha)' \text{ and } \Lambda_q(\{b_i\}) \leq c \parallel g \parallel_{H_\omega} .$$

By (3.4), the sum $\sum_i b_i$ defines an element f of $\widetilde{H}^{\rho,q}$ whose restriction to E^α coincides with g, that is $R(f) = g$. In order to show that R is one to one, we need to prove that $f(\psi) = 0$ for every $\psi \in E^\alpha$ implies $f(\psi) = 0$ for every ψ in $Lip(\rho)$. This result is obtained in Lemma (5.5) of [V] as a consequence of lemma (3.2).

In what follows we will restrict our attention to the case $X = I\!R^n$ and we shall study the connection of the *Hardy-Orlicz* spaces $H_\omega(I\!R^n)$ with *Riesz* transforms. Using the boundedness result established in section 2, we shall obtain in Theorem (3.38) a characterization of $H_\omega(I\!R^n)$ in terms of these operators

Let$P(x)$ be the *Poisson* kernel defined by $P(x) = c_n(1 + |x|^2)^{-\frac{n+1}{2}}$ and denote $P_t(x) = t^{-n}P(x/t)$. For $f \in L^2 \bigcap H_\omega(I\!R^n)$, we shall consider the $n+1$ harmonic functions in $I\!R^{n+1}_+ = \{(x,t) : x \in I\!R^n, t > 0\}$ defined by

$$u_1(t,x) = P_t{*}R_1 f(x), \cdots, u_n(t,x) = P_t{*}R_n f(x) \ , \ u_{n+1}(t,x) = P_t{*}f(x).$$

Let us denote by $F(x,t)$ the vector field associated to f given by

$$F(x,t) = (u_1(t,x),\dots,u_n(t,x),u_{n+1}(t,x)). \tag{3.9}$$

The vector field F satisfies the following generalized *Cauchy-Riemann* equations:

$$divF = \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j} = 0 \text{ and } \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \tag{3.10}$$

for every $j \neq k$; $j,k \in \{1,\dots,n+1\}$, where $x_{n+1} = t$.

Let $x \in I\!R^n$ and $\Gamma(x) = \{(y,t) \in I\!R^{n+1}_+ : |x-y| < t\}$ the cone of aperture one and vertex in x. We define the non-tangential maximal function $f^{**}(x)$ of f as

$$f^{**}(x) = \sup_{(y,t)\in\Gamma(x)} u(t,y) = \sup_{(y,t)\in\Gamma(x)} P_t * f(y).$$

We shall also consider the following maximal operator

$$f^*_M(x) \;=\; sup|f(\psi)|/A(\psi),$$

where $A(\psi) = \int |\psi(t)|dt + |supp\psi|^{M+1} \int |\psi^{(M+1)}(t)|dt$ and the supremum is taken over all the functions $\psi \in C^\infty$ with compact support such that $dist(x, supp\psi) < |supp\psi|$. For the case of H^p, $p \leq 1$, it is known that the norm $\| f^*_M \|_{L^p}$ is equivalent to that given by the atomic descomposition. On the other hand, in [V] (see Theorem A) the equivalence between the atomic *Orlicz* norm and the norm $\| f^*_\gamma \|_{L_\omega}$ is shown in the general context of spaces of homogeneous type.
For the case $I\!R^n$, following the same argument given in Theorem A it can also be established that the norm $\| f^*_M \|_{L_\omega}$ is equivalent to that defined in the atomic *Orlicz* space $H^{\rho,q}$. Therefore, in the following we shall make use of the maximal f^*_M instead of f^*_γ.

Moreover, following *García Cuerva - Rubio de Francia* ([GC-RF] pag. 247) it is easy to see that

$$\| f^*_M \|_{L_\omega} \leq c \, \| f^{**} \|_{L_\omega} \;\; \text{for} M \text{ such that} Ml > 1 \;.$$

On the other hand, the reverse inequality is a consequence of the following result whose proof is similar to that of Lemma (4.3) in [V].

(3.11) LEMMA: *Let ω a growth function of positive lower type $l > \frac{n}{n+1}$. Assume that $b(x)$ is a function belonging to be $L^q(I\!R^n)$, $1 < q \leq \infty$, with support cointained in*

$B = B(x_0, r_0)$ *and* $\int b(x)dx = 0$. *Then, there exists a constant* c, *independent of* $b(x)$, *such that*

$$\int \omega(b^{**}(x))dx \leq c|B|\omega(\| b \|_q |B|^{-1/q}) .$$

Therefore, in the following we shall assume that there exist two positive constants $0 < c_1 \leq c_2$, *satisfying*

$$c_1 \| f \|_{H_\omega} \leq \| f^{**} \|_{L_\omega} \leq c_2 \| f \|_{H_\omega} . \tag{3.12}$$

We shall need the following technical lemma concerning the equivalence between growth functions.

(3.13) LEMMA: *Let* $\gamma \geq 1$. *Let* $\psi(t)$ *be a continuous increasing function of lower type* α *and upper type* β *such that* $\beta \geq \alpha > \gamma$. Then, the function

$$\Phi(t) = t^\gamma \int_0^t \frac{\psi(s)}{s^{1+\gamma}} ds$$

is a continuous, increasing and convex function equivalent to $\psi(t)$.

PROOF: Since $\alpha > \gamma$, we get

$$\Phi(t) = \int_0^1 \frac{\psi(ts)}{s^{1+\gamma}} ds \leq c\psi(t) \int_0^1 \frac{s^\alpha}{s^{1+\gamma}} ds = \frac{c}{\alpha - \gamma} \psi(t).$$

On the other hand, using the fact that $\psi(t)$ is the upper type β, we have that

$$\psi(st) \geq cs^\beta \psi(t) \quad \text{if} \quad s \leq 1 .$$

Therefore, since $\beta > \gamma$, we obtain that

$$\Phi(t) = \int_0^1 \frac{\psi(ts)}{s^{1+\gamma}} ds \geq c\psi(t) \int_0^1 \frac{s^\beta}{s^{1+\gamma}} ds = \frac{c}{\beta - \gamma} \psi(t).$$

To prove that Φ is a convex function, it is enougth to see that $\Phi'(t)$ is increasing. Take $t_1 < t_2$. Since ψ is non-decreasing and $\gamma \geq 1$, it follows that

$$\begin{aligned}
\Phi'(t_2) - \Phi'(t_1) &= \gamma t_2^{\gamma-1} \int_{t_1}^{t_2} \frac{\psi(s)}{s^{1+\gamma}} ds + \gamma \left(t_2^{\gamma-1} - t_1^{\gamma-1}\right) \int_0^{t_1} \frac{\psi(s)}{s^{1+\gamma}} ds \\
&\quad + \frac{\psi(t_2)}{t_2} - \frac{\psi(t_1)}{t_1} \\
&\geq t_2^{\gamma-1} \psi(t_1) \left[t_1^{-\gamma} - t_2^{-\gamma}\right] + \frac{\psi(t_2)}{t_2} - \frac{\psi(t_1)}{t_1} \\
&\geq \frac{\psi(t_2) - \psi(t_1)}{t_2} \geq 0 ,
\end{aligned}$$

which ends the proof of the lemma.

In the sequel, we shall assume that $\Phi(t)$ is a continuous strictly increasing non negative function of lower type greater than one and of finite upper type, such that $\lim_{t\to 0+}\Phi(t)=0$ and $\lim_{t\to\infty}\Phi(t)=\infty$.

The following result, on harmonic majorization of subharmonic functions which are uniformly in an *Orlicz* space L_Φ, is an extension to that of Theorem 4.10 in [GC-RF].

(3.14) THEOREM: *Let $U(x,t)$ be a non-negative subarmonic function in $\mathbb{R}_+^{n+1}$ such that*

$$\sup_{t>0} \| U(.,t) \|_{L_\Phi} < \infty.$$

Then, $U(x,t)$ has a least harmonic majorant in $\mathbb{R}_+^{n+1}$. Moreover, this harmonic majorant is the Poisson integral of a function $h \in L_\Phi(\mathbb{R}^n)$, where h is obtained as the limit of $U(x,t_j)$ for any sequence $t_j \downarrow 0$ $(j \to \infty)$ in the weak - $*$ topology of L_Φ.

For the proof of Theorem (3.14) we shall need the next result.

(3.15) LEMMA: *Let $U(x,t)$ be a non-negative subarmonic function in $\mathbb{R}_+^{n+1}$ satisfying*

$$\sup_{t>0} \| U(.,t) \|_{L_\Phi} = M < \infty. \tag{3.16}$$

Then, there exists a constant c depending only on Φ and n, such that

$$U(x,t) \le cM\Phi^{-1}(1/t^n) \text{ ,for every } (x,t) \in \mathbb{R}_+^{n+1}. \tag{3.17}$$

Consequently, $U(x,t)$ is bounded in each proper sub-half-space $\{(x,t) \in \mathbb{R}_+^{n+1} : t \ge \bar{t} > 0\}$. Moreover, the following property holds:

$U(x,t) \to 0$ as $|(x,t)| \to \infty$ in each proper sub-half-space.

PROOF: Let $(x_0,t_0) \in \mathbb{R}_+^{n+1}$ and

$$\begin{aligned}\widetilde{B}_0 = B((x_0,t_0),t_0/2) &\subset B(x_0,t_0/2) \times (t_0 - t_0/2, t_0 + t_0/2)\\ &= B_0 \times (t_0/2, 3t_0/2).\end{aligned}$$

Since $U(x,t)$ is sub-harmonic, applying the *Hölder* inequality (1.12) with Ψ the complementary function of Φ,we have

$$\begin{aligned} U(x_0,t_0) &\le \frac{1}{|\widetilde{B}_0|}\int_{\widetilde{B}_0} U(x,t)dxdt \le \frac{c}{t_0^{n+1}}\int_{t_0/2}^{\frac{3}{2}t_0}\int_{\mathbb{R}^n} \chi_{B_0}(x)U(x,t)dxdt \\ &\le \frac{c}{t_0^{n+1}}\int_{t_0/2}^{\frac{3}{2}t_0} \| U(.,t) \|_{L_\Phi} \| \chi_{B_0} \|_{L_\Psi} dt. \end{aligned} \tag{3.18}$$

Taking $\| \chi_{B_0} \|_{L_\Psi} \equiv |B_0|\Phi^{-1}(1/|B_0|)$, from (3.16) and (3.18), we get (3.17). On the other hand, given $t_0 > 0$ fix and $\varepsilon > 0$, since $\lim_{s\to 0+} \Phi^{-1}(s) = 0$, there exists $t_1 > t_0$ such that $\Phi^{-1}(1/t_1^n) \le \varepsilon$. Thus, by (3.17) we obtain that

$$U(x,t) \le cM\varepsilon, \text{ for every } t \ge t_1 \text{ and } x \in I\!R^n .$$

It only remains to prove that $U(x,t) \le \varepsilon$, for every $t_0 \le t < t_1$ and $|x|$ big enough. Let $x \in I\!R^n$ and $|x| > t_1$. Take $\widetilde{B} = B((x,\tilde{t}), t_0/2)$ with $t_0 \le \tilde{t} < t_1$. Proceeding as in the first part of the proof, we get

$$U(x,\tilde{t}) \le \frac{c}{t_0^{n+1}} \| \chi_{B(x,t_0/2)} \|_{L_\Psi} \int_{t_0/2}^{\frac{3}{2}t_1} \| \chi_{B(x,t_0/2)}(.)U(.,t) \|_{L_\Phi} dt \tag{3.19}$$

Now, let us observe that, for each t, we have

$$\begin{aligned} \int \Phi\left[\chi_{B(x,t_0/2)}(y)U(y,t)\right] dy &= \int_{B(x,t_0/2)} \Phi(U(y,t))dy \\ &\le \int_{|y|\ge|x|-t_1/2} \Phi\left(U(y,t)\right) dy \end{aligned} \tag{3.20}$$

Since Φ is of finite upper type, from (3.16) and (3.20) it follows that

$$\| \chi_{B(x,t_0/2)}(.)U(.,t) \|_{L_\Phi} \to 0 \text{ as } |x| \to \infty \text{ for each } t .$$

Therefore, using in (3.19) the *Lebesgue* dominated convergence Theorem, we obtain that $U(x,\tilde{t}) \to 0$ as $|x| \to \infty$, uniformly for every $t_0 \le \tilde{t} < t$, completing the proof of the lemma.

PROOF OF THEOREM (3.14) : Let $\{t_j\}_j$ be a sequence such that $t_j \downarrow 0$ and denote $f_j(x) = U(x,t_j)$. Since $\| f_j \|_{L_\Phi} < \infty$ for every j, there exists a subsequence of $\{f_j\}$, that we also denote $\{f_j\}$, converging in the weak- $*$ topology of L_Φ (see Theorem 144 in [K]). That is, there exists a function $f \in L_\Phi$, such that for every $g \in L_\Psi$, Ψ being the complementary function of Φ, we have

$$\int f_j(x)g(x)dx \underset{j\to\infty}{\longrightarrow} \int f(x)g(x)dx. \tag{3.21}$$

If we are able to prove that

$$U(x,t+t_j) \le \int_{I\!R^n} P_t(x-y)f_j(y)dy \tag{3.22}$$

for every j, then using (3.21), the conclusion of the Theorem follows inmediately. Now, in order to prove (3.22) it is enough to see that the functions

$$G_j(x,t) = U(x,t+t_j) \text{ and } F_j(x,t) = P_t * f_j(x)$$

tend to zero when $|(x,t)| \to \infty$. In fact, if this happens, given $\varepsilon > 0$, there exists $R > 0$ big enough satisfying

$$D_j(x,t) = G_j(x,t) - F_j(x,t) \leq \varepsilon, \tag{3.23}$$

for every (x,t) such that $|(x,t)| \geq R$, and in particular, (3.23) holds for every (x,t) in the boundary of the region $K_R = \{(x,t) \in I\!R^{n+1}_{+} : |(x,t)| \leq R\}$. Since $D_j(x,t)$ is subharmonic, it follows that

$$D_j(x,t) \leq \varepsilon \ , \ \text{for every } (x,t) \in K_R \ ,$$

which together with (3.23) proves (3.22). Finally, let us prove the convergence of the functions G_j and F_j. Applying Lemma (3.15), we obtain,

$$G_j(x,t) \to 0 \text{ as } |(x,t)| \to \infty$$

and

$$f_j(x) \to 0 \text{ as } |x| \to \infty \ .$$

Using this fact and that $f_j \in L_\Phi$, by a standard argumente, we may conclude that

$$F_j(x,t) \to 0 \text{ as } |(x,t)| \to \infty \ ,$$

which completes the proof of the Theorem.

We also need the following lemma which gives a norm inequality between the vector field $F(x,t)$, defined in (3.9), and the function $f(x)$.

(3.24) LEMMA: *Let $F(x,t)$ be the function defined in* (3.9). *Then*

$$\sup_{t>0} \| F(.,t) \|_{L_\omega} \leq c \| f \|_{H_\omega} \ .$$

PROOF: Let $\eta = \eta_1 + \eta_2$ be a constant to be fixed later on. Let us estimate

$$\sup_{t>0} \int \omega \left[\frac{|F(x,t)|}{(\eta \| f \|_{H_\omega})^{1/l}} \right] dx \leq \int \omega \left[\sum_{j=1}^{n+1} \sup_t \frac{|u_j(t,x)|}{(\eta \| f \|_{H_\omega})^{1/l}} \right] dx \tag{3.25}$$

$$\leq \int \omega \left[\sup_{|y-x|<t} \frac{u(t,y)|}{(\eta \parallel f \parallel_{H_\omega})^{1/l}} \right] dx$$

$$+ \sum_{j=1}^{n} \int \omega \left[\sup_{|y-x|<t} \frac{|u_j(t,y)|}{(\eta \parallel f \parallel_{H_\omega})^{1/l}} \right] dx$$

$$\leq \int \omega \left[\frac{f^{**}(x)}{(\eta_1 \parallel f \parallel_{H_\omega})^{1/l}} (\frac{\eta_1}{\eta})^{1/l} \right] dx$$

$$+ \sum_{j=1}^{n} \int \omega \left[\frac{R_j f^{**}(x)}{(\eta_2 \parallel f \parallel_{H_\omega})^{1/l}} (\frac{\eta_2}{\eta})^{1/l} \right] dx$$

An application of Theorem 2.20, together with (3.25) and the fact that $\omega(s)$ is lower type l, imply

$$\sup_{t>0} \int \omega \left[\frac{|F(x,t)|}{(\eta \parallel f \parallel_{H_\omega})^{1/l}} \right] dx \leq \frac{\eta_1}{\eta} \int \omega \left[\frac{f^{**}(x)}{(\eta_1 \parallel f \parallel_{H_\omega})^{1/l}} \right] dx$$

$$+ \frac{\eta_2}{\eta} \sum_{j=1}^{n} \int \omega \left[\frac{R_j f^{**}(x)}{(\eta_2 \parallel R_j f \parallel_{H_\omega})^{1/l}} c^{1/l} \right] dx$$

$$\leq \frac{\eta_1}{\eta} + \frac{\eta_2}{\eta} = 1 ,$$

by choosing $\eta_1 = c_2$ and $\eta_2 = nc_2c$ with c_2 the constant appearing in (3.12). This finishes the proof of the lemma.

The next lemma provides the boundedness of the *Poisson* integral on H_ω.

(3.26) LEMMA: *Let $f \in H_\omega$. Then $u(t,x) = P_t * f(x)$ belongs to $L^q \bigcap H_\omega$, $1 < q \leq \infty$, and*

$$\parallel u(t,\cdot) \parallel_{H_\omega} \leq c \parallel f \parallel_{H_\omega} .$$

PROOF: In view of Theorem (3.7), we have that $f \in \widetilde{H}^{\rho,\infty}$ and there exists a sequence of multiples of (ρ, ∞) atoms such that

$$f(\Psi) = \sum_j b_j(\Psi) \text{ , for every } \Psi \in Lip(\rho) .$$

Since $P_t(x) \in Lip(\rho)$ with $\parallel P_t \parallel_{Lip(\rho)} \leq c(t)$, we get

$$(3.27) \quad |u(t,x)| = |f(P_t(x-.))| = |\sum_j b_j(P_t(x-.))| \leq c \parallel P_t(x-.) \parallel_{Lip(\rho)} \leq c(t) .$$

Therefore, $u(t,.)$ is an L^∞ function. Now, let us see that $u(t,.) \in L^q, 1 < q < \infty$. Given $g \in \mathcal{S}$, we have

$$u(t,\cdot)(g(\cdot)) = \int \lim_{N\to\infty} \sum_{j=1}^{N} b_j * P_t(x) g(x) dx$$

Using (3.27) and the *Lebesgue* dominated convergence Theorem, we obtain

$$\begin{aligned}|u(t,\cdot)(g(\cdot))| &= |\lim_{N\to\infty}\sum_{j=1}^{N}\int b_j{*}P_t(x)g(x)dx| \\ &= |\lim_{N\to\infty}\sum_{j=1}^{N}\int b_j(x)P_t{*}g(x)dx| \\ &= |\lim_{N\to\infty}\sum_{j=1}^{N} b_j(P_t{*}g)| \\ &= |f(P_t{*}g)| \le c \parallel P_t{*}g \parallel_{Lip(\rho)} .\end{aligned}$$

In order to prove that $u(t,.)\in L^q$, it is enough to show

$$\parallel P_t{*}g \parallel_{Lip(\rho)} \le c(t) \parallel g \parallel_{L^{q'}} . \tag{3.28}$$

Let $x, x' \in I\!R^n$ with $|x-x'|>t/2$. Then using the fact that ρ is of upper type $m<1$, we have

$$\begin{aligned}|P_t{*}g(x)-P_t{*}g(x')| &\le 2 \parallel P_t{*}g \parallel_\infty \le 2 \parallel P_t \parallel_{L^q} \parallel g \parallel_{L^{q'}} \\ &\le ct^{-n/q'}\rho\left(\frac{|x-x'|}{t}\right) \parallel g \parallel_{L^{q'}} \\ &\le ct^{-n/q'} max\{1/t,1\}^m \rho(|x-x'|) \parallel g \parallel_{L^{q'}} \\ &= c(t)\rho(|x-x'|) \parallel g \parallel_{L^{q'}}\end{aligned} \tag{3.29}$$

On the other hand, if $|x-x'|<t/2$, we obtain

$$\begin{aligned}|P_t{*}g(x)-P_t{*}g(x')| &\le \int |P_t(x-y)-P_t(x'-y)|\,|g(y)|dy \\ &\le \left(\int |P_t(x-y)-P_t(x'-y)|^q dy\right)^{1/q} \parallel g \parallel_{L^{q'}} \\ &\le |x-x'| \parallel g \parallel_{L^{q'}} \left(\int |\nabla_x P_t((x-y)+\theta(x-x'))|^q dy\right)^{1/q} \\ &\le c\frac{|x-x'|}{t^{n+1}} \parallel g \parallel_{L^{q'}} \left(\int_{|x-y|<t} dy + \int_{|x-y|>t} \frac{dy}{(t^1|x-y|)^{(n+2)q}}\right)^{1/q} \\ &\le c\left(\frac{|x-x'|}{t}\right)^m t^{-n/q'} \parallel g \parallel_{L^{q'}} \\ &\le c\rho\left(\frac{|x-x'|}{t}\right) t^{-n/q'} \parallel g \parallel_{L^{q'}} \le c(t)\rho(|x-x') \parallel g \parallel_{L^{q'}} ,\end{aligned} \tag{3.30}$$

because ρ is of upper type $m < 1$. Thus, from (3.29) and (3.30) we obtain (3.28). Next we prove that $u(t,\cdot) \in H_\omega$. In fact,

$$(3.31) \qquad u(t,\cdot)^{**}(x) = \sup_{|y-x|<s} |P_s * P_t * f(y)| \le \sup_{|y-x|<s+t} |P_{t+s} * f(x)| \le f^{**}(x) .$$

Therefore, we conclude that $P_t * f \in H_\omega$ with $\| P_t * f \|_{H_\omega} \le c \| f \|_{H_\omega}$.

(3.32) REMARK: *Let $f \in (Lip(\rho))'$, then $u(x,t) = f(P_t(x - \cdot))$ is a harmonic function in $I\!R^{n+1}_+$.* In fact, taking for example $\frac{1}{h}[u(x,t+h) - u(x)]$, it can be proved that this incremental quotient tends to $f(\frac{\partial}{\partial t}P_t(x - \cdot))$, by showing that for each (x,t) fixed

$$\| \frac{1}{h}[P_{t+h}(x - \cdot) - P_t(x - \cdot)] - \frac{\partial}{\partial t}P_t(x - \cdot) \|_{Lip(\rho)} \underset{h \to 0}{\longrightarrow 0} ,$$

This, in turn, is a consequence of the mean value Theorem, and the fact that ρ is upper type $m < 1$.

(3.33) LEMMA: *Let f be a distribution belonging to H_ω. Then*

$$\| u(t,.) - f \|_{H_\omega} \to 0, \text{ as } t \to 0.$$

PROOF: Let $\varepsilon > 0$. We first assume that $f \in H_\omega \bigcap L^q$, $1 < q \le \infty$. Thus, there exists a ball $B = B(x_0, R)$ such that

$$(3.34) \qquad \int_{CB} \omega(f^{**}(x))\, dx < \varepsilon/2.$$

Since by (3.31) $u(t,\cdot)^{**}(x) \le f^{**}(x)$, it follows that

$$(3.35) \qquad \int_{CB} \omega\,[(u(t,.) - f(.))^{**}(x)]\, dx \le 2 \int_{CB} \omega(f^{**}(x))\, dx < \varepsilon.$$

On the other hand, if $\lambda_t = \| u(t,\cdot) - f \|_{L^q} |B|^{-1/q}$, using that $\omega(s)/s$ is non increasing, we have

$$\begin{aligned} \omega((u(t,.) - f)^{**}(x)) &\le c\omega(M(u(t,.) - f)(x)) \\ &\le c\omega\,[(M(u(t,.) - f)(x) + \lambda_t] \\ &\le c\omega(\lambda_t)\left(\frac{M(u(t,.) - f)(x)}{\lambda_t} + 1\right). \end{aligned}$$

Integrating on B, we obtain

$$(3.36) \qquad \int_B \omega[(u(t,.)-f)^{**}(x)]dx$$

$$\leq C\omega(\lambda_t)[\lambda_t^{-1} \| M(u(t,.)-f) \|_q |B|^{1/q'} + |B|] \underset{t\to 0}{\longrightarrow 0}$$

$$\leq C\omega(\lambda_t)|B| \underset{t\to 0}{\longrightarrow 0} .$$

From (3.35) and (3.36), since ω is of finite upper type, we get

$$(3.37) \qquad \| u(t,\cdot) - f \|_{H_\omega} \underset{t\to 0}{\longrightarrow} 0,$$

which proves the lemma under the assumnption $f \in H_\omega \bigcap L^q$. Next, we shall remove that assumption. Let $f \in H_\omega$. Given $\varepsilon > 0$, by the density of L^q in H_ω (see Theorem (4.16) in [V]), there exists $g \in L^q$ such that $\| f - g \|_{H_\omega} < \varepsilon$. Hence, in view of Lemma (3.26), we have that there exists $t_0 = t_0(\varepsilon)$ such that

$$\begin{aligned} \| u(t,\cdot) - f \|_{H_\omega} &\leq \| P_t*(f-g) \|_{H_\omega} + \| P_t*g - g \|_{H_\omega} + \| f - g \|_{H_\omega} \\ &\leq \varepsilon + c \| f - g \|_{H_\omega} \leq c\varepsilon , \end{aligned}$$

for every $t \leq t_0$, as we wanted to prove.

Now we are in a position to prove the main theorem, which gives another characterization of the *Hardy-Orlicz* spaces.

(3.38) THEOREM: *Let ω be a function of lower type l such that $l > \frac{n}{n+1}$. Assume that $\omega(s)/s$ is non increasing. Then there exist two constants c_1 an c_2 satisfying*

$$(3.39) \qquad c_1 \| f \|_{H_\omega} \leq \| f \|_{L_\omega} + \sum_{j+1}^{n} \| R_j f \|_{L_\omega} \leq c_2 \| f \|_{H_\omega} ,$$

for every $f \in L^q \bigcap H_\omega(I\!R^n), 1 \leq q < \infty$, and

$$(3.40) \qquad c_1 \| f \|_{H_\omega} \leq \| \lim_{t\to 0} u(t,.) \|_{L_\omega} + \sum_{j=1}^{n} \| \lim_{t-0} R_j(u(t,.)) \|_{L_\omega}$$

$$\leq c_2 \| f \|_{H_\omega}, \text{ for every } f \in H_\omega.$$

PROOF: Let $f \in L^q \bigcap H_\omega(I\!R^n)$. Let us first check the right inequality on (3.39). Since P_t*f tends to f in L^q, we have that

$$|f(x)| \leq f^{**}(x) \text{ and } |R_j f(x)| \leq (R_j f)^{**}(x) \quad \text{for} \quad a.e. x \in I\!R^n.$$

Therefore,

$$\int \omega \left[\frac{|f(x)|}{(c \parallel f \parallel_{H_\omega})^{1/l}}\right] \leq \int \omega \left[\frac{|f^{**}(x)|}{(c \parallel f \parallel_{H_\omega})^{1/l}}\right] dx \leq 1 \ ,$$

and, applying Theorem (2.20),

$$\int \omega \left[\frac{|R_j f(x)|}{(c \parallel f \parallel_{H_\omega})^{1/l}}\right] dx \leq \int \omega \left[\frac{R_j f^{**}(x)}{(c \parallel f \parallel_{H_\omega})^{1/l}}\right] dx \leq 1 \ ,$$

for every $j = 1, \cdots n$, which implies that

$$\parallel f \parallel_{L_\omega} + \sum_{j=1}^{n} \parallel R_j f \parallel_{L_\omega} \leq c_2 \parallel f \parallel_{H_\omega} \ .$$

On the other hand, in order to prove the left inequality on (3.39), we shall consider the function

$$U(y,t) \ = \ |F(y,t)|^{l'} \ , \tag{3.41}$$

with $\frac{n-1}{n} < \frac{n}{n+1} < l' < l$, which is subharmonic in view of Lemma 4.14 in [GC, RF]. Now, we observe that Lemma (3.13) implies that the function $\psi(t) = \omega(t^{1/l'})$ is equivalent to a *Young* function $\Phi(t)$ of lowe type $l/l' > 1$ and of upper type $1/l'$. Then using Lemma (3.24), we get

$$\sup_{t>0} \int \Phi \left[\frac{U(y,t)}{(c \parallel f \parallel_{H_\omega})^{l'/l}}\right] dy \leq \sup_{t>0} \int \omega \left(\frac{|F(y,t)|}{(c \parallel f \parallel_{H_\omega})^{1/l}}\right) dy \leq 1 \ .$$

Therefore

$$\sup_{t>0} \parallel U(\cdot,t) \parallel_{L_\Phi} \leq c \parallel f \parallel_{H_\omega}^{l'/l} < \infty \ .$$

By Theorem (3.14), there exists a function $h \in L_\Phi$ such that

$$U(y,t) \ \leq \ P_t * h(y). \tag{3.42}$$

Moreover, for $t_j \downarrow 0$ $(j \to \infty)$ and $g \in L_\psi$, with ψ the *Young* complementary function of Φ, we have

$$\int h(x) g(x) dx = \lim_{j \to \infty} \int U(x, t_j) g(x) dx. \tag{3.43}$$

Now, if $G(x) = \sup_{(y,t) \in \Gamma(x)} |F(y,t)|$, by (3.41) and (3.42) we obtain that

$$\int \omega \left[G(x)/(c \parallel h \parallel_{L_\Phi})^{1/l'}\right] dx = \int \omega \left[\sup_{(y,t) \in \Gamma(x)} (U(y,t)/c \parallel h \parallel_{L_\Phi})^{1/l'}\right] dx$$

$$\leq \int \omega \left(\frac{h^{**}(x)}{c \parallel h \parallel_{L_\Phi}}\right)^{1/l} dx \leq \int \Phi \left(\frac{Mh(x)}{c \parallel h \parallel_{L_\Phi}}\right) dx \ ,$$

where $Mh(x)$ is the *Hardy-Littlewood* maximal function. From the maximal operator theory in *Orlicz* spaces, it is known that M is bounded on L_Φ. Therefore, it follows that

(3.44) $$\| G \|_{L_\omega} \leq c \| h \|_{L_\Phi}^{l/l'} ,$$

This implies, in particular, that F is non-tangentially bounded at almost every $x \in I\!R^n$. Consequently, by Theorem 4.21 in [GC, RF]), there exists a function $F_0(x)$ such that

(3.45) $$F_o(x) = \lim_{(y,t)\to x} \text{non tang} \quad F(y,t) \quad , \quad \text{for } a.e.x \in I\!R^n$$

In view of (3.43) and (3.45), we get

(3.46) $$h(x) = |F_0(x)|^{l'} \text{ for } a.e.x \in I\!R^n \text{ and } \| F_0 \|_{L_\omega} \approx \| h \|_{L_\Phi}^{l/l'}$$

Futhermore, since $P_t * f$ converges to f in L^q, we obtain

(3.47) $$|F_0(x)| = \left(f(x)^2 + \sum_{j=1}^{n} (R_j f(x))^2 \right)^{1/2} \quad \text{for } a.e.x \in I\!R^n \text{ and}$$

$$\| F_0 \|_{L_\omega} \leq \| f \|_{L_\omega} + \sum_{j=1}^{n} \| R_j f \|_{L_\omega} .$$

Then, from (3.44), (3.46) and (3.47), we have

$$\int \omega \left[f^{**}(x) / (c(\| f \|_{L_\omega} + \sum_{j=1}^{n} \| R_j f \|_{L_\omega}))^{1/l} \right] dx$$

$$\leq \int \omega \left[G(x) / (c(\| f \|_{L_\omega} + \sum_{j=1}^{n} \| R_j f \|_{L_\omega}))^{1/l} \right] dx$$

$$\leq \int \omega \left[\frac{G(x)}{(c \| F_0 \|_{L_\omega})^{1/l}} \right] dx \leq \int \omega \left[\frac{G(x)}{(c \| h \|_{L_\Phi})^{1/l'}} \right] dx$$

$$\leq 1 ,$$

which completes the proof of the Theorem for the case $f \in L^q \bigcap H_\omega$. Now, we assume that $f \in H_\omega$. Since Lemma (3.26) implies that $u(t,\cdot) \in L^q \bigcap H_\omega$, applying (3.39) it follows that

(3.48) $$c_1 \| u(t,.) \|_{H_\omega} \leq \| u(t,.) \|_{L_\omega} + \sum_{j=1}^{n} \| R_j(u(t,.)) \|_{H_\omega}$$

$$\leq c_2 \| u(t,\cdot) \|_{H_\omega}$$

From Lemma (3.26) and Remark (3.33), we may conclude that $u(t,x)$ is harmonic and non-tangentially bounded function. Hence, there exists $\lim_{t\to 0} u(t,x)$ for $a.e.x \in I\!R^n$. Therefore, taking limit in (3.48) and applying Lemma (3.33) and the *Lebesgue* dominated convergence Theorem, we obtain (3.40) ending the proof of the Theorem./ / /

REFERENCIAS

[A] Aimar, H., "Singular integrals and approximate identities on spaces on homogeneous type". Trans. Amer. Math. Soc. 292 (1985), 135-153.

[D-J-S] David, G., Jouneé, J.L. and Semmes, S., "Opérateurs de Calderón - Zygmund, fonctions para accrétives et interpolation". Rev. Mat. Iberoamericana, 1 (1985), 1-56.

[F-S] Fefferman, C. and Stein, E.M., "H^P Spaces of Several Variables". Acta Mathe-matica 129, p. 137-193, 1972.

[GC-RF] García-Cuevas, J. and Rubio de Francia, J.L. "Weighted Norm Inequalities an Related Topocs". North-Holland, Amsterdam, New York, Oxford. 1985.

[K] Krasnosel'skii, M.A. and Rutickii, Y.B. "Convex Functions and Orlicz Spa ces". Groningen, 1961.

[M-S] Macías, R.A. and Segovia. C. "A Decomposition into Atoms of Distribution on Spaces of Homogeneous Type". Advances in Math. 33 (1979), 271-309.

[M-S-T] Macías, R.A., Segovia, C. and Torrea, J.L. "Singular Integral Operator with non Necessarily Bounded Kernels on Spaces of Homogenous Type". Adv. in Math., V93, N° 1, 1992.

[M-T] Macías, R.A. and Torrea, J.L. "L^2 and L^P Boundedness of Singular Integrals on non Necessarily Normalized Spaces of Homogeneous Type". Revista de la Unión Matemática Argentina. Vol. 34, p. 97-114, 1988.

[V] Viviani, B. "An Atomic Decomposition of the Predual of BMO(ρ)". Rev. Mat. Iberoamericana. Vol. 3, N^{os} 3 y 4, p. 401-425, 1987.

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