Revista de la
Unión Matemática Argentina
Volumen 38, 1993.

CONVERGENCE IN L^1 OF SINGULAR INTEGRALS WITH NON-STANDARD TRUNCATIONS

LILIANA DE ROSA and CARLOS SEGOVIA

Presentado por Agnes Benedek

ABSTRACT. *Singular integrals with non standard truncations (intervals) are considered. Under the assumption that the limit of the truncations belongs to* $L^1(w)$, *it is shown that the truncated integrals converge to its limit in* $L^1(v)$ *where* (v,w) *satisfies an* A_1 *type condition.*

1. NOTATIONS AND DEFINITIONS. By $\mathbf{R}^n$ we denote the n-dimensional euclidean space. As usual, the norm of $x \in \mathbf{R}^n$ is the number $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ and for $t > 0$, the ball B_t centered at the origin is the set $\{x : |x| < t\}$. If A is a Lebesgue measurable set, we denote its measure by $|A|$ and the characteristic function of A by $\chi_A(x)$.

Let $f(x)$ be a measurable function. The Hardy-Littlewood maximal function $Mf(x)$ is defined as

$$Mf(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|dy ,$$

where Q is any cube in $\mathbf{R}^n$.

If $u(x) \geq 0$ is a locally integrable function and A is a measurable set, its measure with respect to the weight $u(x)$ is denoted by $u(A) = \int_A u(x)dx$. A measurable function f belongs to $L^1(u)$ if $\|f\|_{L^1(u)} = \int_{\mathbf{R}^n} |f(x)|u(x)dx$ is finite. If $u \equiv 1$ then we simply write $\|f\|_{L^1}$ and L^1, as usual.

We shall say that a pair of non-negative and measurable functions (v,w) belongs to A_1 if

$$(1.1) \qquad |Q|^{-1} \int_Q v(x)dx \leq c \operatorname*{ess.inf}_{x\in Q} w(x) ,$$

where Q is any cube in $\mathbf{R}^n$ and c is a constant not depending on Q. The least constant c such that (1.1) holds is called the constant of the pair (v,w) in the class A_1. We observe that condition (1.1) holds if and only if $Mv(x) \leq c\,w(x)$ a.e.. In the case $v \equiv w$, we simply write $w \in A_1$.

Let $S(x)$ be the even function defined on $\mathbf{R}^n$

(1.2) $$S(x) = (1+|x|)^{-n}\chi_{[0,1]}(|x_n|)$$

where $x = (x_1, \ldots, x_n)$ and $S_t(x) = t^{-n}S(x/t)$. We shall say that a pair of non negative and measurable functions (v,w) belongs to the class $\mathcal{A}_1$ if there exists a finite constant c such that

(1.3) $$\sup_{t>0}(S_t * v)(x) \leq c\,w(x)$$

holds almost everywhere in x. The least constant c satisfying (1.3) shall be called the constant of the pair (v,w) in the class $\mathcal{A}_1$. If $v \equiv w$, we simply write $w \in \mathcal{A}_1$.

Let $k(x) = \Omega(x)|x|^{-n}$ be a measurable function defined on $\mathbf{R}^n - \{0\}$. Let us assume that Ω satisfies

(i) for every $\lambda > 0$ and $x \neq 0$, $\Omega(\lambda x) = \Omega(x)$,

(ii) $\Omega(x)$ is bounded on $\mathbf{R}^n - \{0\}$,

(iii) $\int_\Sigma \Omega(x)d\sigma_x = 0$, where Σ is the unit sphere, $\Sigma = \{x : |x| = 1\}$ and $d\sigma_x$ is the element of surface area,

(iv) if $w(t) = \sup\limits_{|h|\leq t}\sup\limits_{x\in\Sigma}|\Omega(x+h) - \Omega(x)|$ then the Dini condition
$$\int_0^1 w(t)\,\frac{dt}{t} < \infty$$
holds.

We say that a function $k(x)$ satisfying conditions (i) to (iv) is a singular integral kernel and that $K_B f(x)$, where

(1.4) $$K_B f(x) = \lim_{t\to 0+} K_{B_t}f(x) = \lim_{t\to 0+}\int_{x-y\,\notin\, B_t} k(x-y)f(y)dy$$

is the corresponding singular integral of f at the point $x \in \mathbf{R}^n$, whenever it exists.

Let $\{R_t\}_{t>0}$ be a family of n-dimensional intervals centered at the origin with sides parallel to the axes. Let $R_t = \{x : |x_i| \leq \varphi_i(t),\ 1 \leq i \leq n\}$.

(1.5) DEFINITION *We shall say that the family of intervals $\{R_t\}_{t>0}$ is admissible if the functions $\varphi_i(t)$ satisfy*

(i) *there exists* i_0 *such that* $\varphi_{i_0}(t) = t$ *and for every* i, $i \neq i_0$, $\varphi_i(t) \geq t$ *holds,*

(ii) *for every* i, $\lim_{t \to 0+} \varphi_i(t) = 0$ *and*

(iii) *for each* i, $\lim_{t \to 0+} \varphi_i(t)/t$ *exists (this limit may be equal to infinity).*

We can assume that $i_0 = n$. Let

$$K_{R_t} f(x) = \int_{x-y \notin R_t} k(x-y) f(y) dy \,.$$

We shall call $K_{R_t} f$ the singular integral truncated by the interval R_t and

$$K_R f(x) = \lim_{t \to 0+} K_{R_t} f(x) \tag{1.6}$$

the singular integral associated to the family $\{R_t\}$, whenever this limit exists.

2. STATEMENTS OF THE RESULT. Singular integrals with non-standard truncations of the type (1.5) were considered by E.O. Harboure in [**3**]. In that paper, norm inequalities in L^p, $1 < p < \infty$, for singular integral operators $K_R f$ and the maximal singular integral operator $K_R^* f$ were obtained. For the case of $p = 1$, weak type estimates are shown to hold. Some years later, H. Aimar and E.O. Harboure introduced in [**1**] new techniques to deal with the weighted case, generalizing the results of [**3**].

In the present paper, we study the convergence in $L^1(v)$, of the truncated singular integrals with non standard truncations of the type considered by E.O. Harboure, under the assumption that both f and $K_R f$ belong to $L^1(w)$. The case of $K_B f$, that is to say, the case of ordinary truncations by a family of balls with weights was considered by O.N. Capri and C. Segovia in [**2**]. For rather general singular integral kernels and pairs of weights we refer to the paper of L. de Rosa and C. Segovia [**4**].

The main result of this paper is the following theorem:

(2.1) THEOREM. *Let* (v, w) *belong to* $\mathcal{A}_1$ *and let* $\{R_t\}_{t>0}$ *be an admissible family of n-dimensional intervals. Let* $k(x)$ *be a singular integral kernel. Then, if* $f \in L^1(w)$ *the function* $K_R f(x)$, *introduced in (1.6) is well defined almost everywhere. Moreover, if* $K_R f(x)$ *belongs to* $L^1(w)$, *we have:*

(i) $\|K_{R_t} f\|_{L^1(v)} \leq c\big(\|f\|_{L^1(w)} + \|K_R f\|_{L^1(w)}\big)$ *for every* $t > 0$. *The constant* c *does not depend on either* t *or* f.

(ii) $\lim_{t \to 0+} \|K_{R_t} f - K_R f\|_{L^1(v)} = 0$.

If $w \in A_1$, *then* (i) *and* (ii) *hold taking* $v = w$.

3. PROOFS. In order to prove the theorem we shall need some results.

We begin with the following proposition which gives a characterization of the class $\mathcal{A}_1$:

(3.1) PROPOSITION. *Let $S_t(x)$ be the function defined in (1.2). Then, the inequality*

$$\|S_t * f\|_{L^1(v)} \le c\,\|f\|_{L^1(w)} \tag{3.2}$$

holds for every $t > 0$ with a constant c not depending on t and f, if and only if (v, w) belongs to $\mathcal{A}_1$.

Proof. If $(v, w) \in \mathcal{A}_1$ and assuming as we can that $f \ge 0$, since $S_t(x)$ is an even function, we have

$$\begin{aligned}\int (S_t * f)(x)v(x)dx &= \iint f(y)S_t(x-y)v(x)dx\,dy \\ &= \int f(y)(S_t * v)(y)dy \le c\int f(y)w(y)dy\ ,\end{aligned}$$

where c is the constant of (v, w) in the class $\mathcal{A}_1$.

Let us assume that (3.2) holds, then

$$\begin{aligned}\int f(x)(S_t * v)(x)dx &= \int (S_t * f)(x)v(x)dx \\ &\le c\int f(x)w(x)dx\ .\end{aligned}$$

Thus, $(S_t * v)(x) \le c\,w(x)$ for almost every $x \in \mathbf{R}^n$. The set where the inequality does not hold, may depend on t. However, since for $p < q$, $S_p(x) \le (q/p)^n S_q(x)$, we get that

$$\sup_{t>0}(S_t * v)(x)$$

is equal to the supremum taken on the positive rational numbers. Then, for almost every $x \in \mathbf{R}^n$

$$\sup_{t>0}(S_t * v)(x) \le c\,w(x)\ .$$

(3.3) PROPOSITION. *If the pair (v, w) belongs to $\mathcal{A}_1$, then it also belongs to A_1.*

Proof. This follows from the fact that $S(x) \ge 2^{-n}\chi_B(x)$, where B is the unit ball in $\mathbf{R}^n$.

(3.4) LEMMA. *Let w belong to A_1. There exists r, $0 \le r < 1$, such that*

$$|R|^{-1}\int_R w(x)dx \le c_w\left(\frac{|Q|}{|R|}\right)^r \operatorname*{ess.inf}_{x\in Q}\, w(x) \tag{3.5}$$

holds for any cube Q *and any interval* R, *such that* $R \subset Q$ *with* c_w *a constant depending on the constant of* w *in* A_1 *and* r *only.*

Proof. Since $w \in A_1$, there exists $\eta > 0$ such that $w^{1+\eta} \in A_1$. Let $r = 1/(1+\eta)$. Then

$$|R|^{-1}\int_R w(x)dx \le \left(|R|^{-1}\int_R w(x)^{1+\eta}dx\right)^{1/1+\eta}$$
$$\le \left(\frac{|Q|}{|R|}\right)^{1/1+\eta}\left(|Q|^{-1}\int_Q w(x)^{1+\eta}dx\right)^{1/1+\eta}.$$

By the condition A_1 on $w(x)^{1+\eta}$, we get

$$|R|^{-1}\int_R w(x)dx \le c\left(\frac{|Q|}{|R|}\right)^{1/1+\eta} \operatorname*{ess.inf}_{x\in Q} w(x).$$

(3.6) PROPOSITION. *The weight* w *belongs to* A_1 *if and only if* w *belongs to* $\mathcal{A}_1$.

Proof. If $w \in \mathcal{A}_1$, Proposition (3.3) shows that $w \in A_1$. Now, let w belong to A_1. For every non negative integer k, let R_k be the interval centered at the origin, with sides parallel to the axes of length $2^{k+1},\dots,2^{k+1},2$, and $Q_k \supset R_k$ the cube with sides of length 2^{k+1}. It is easy to check that

$$S(x) \le 2^n \sum_{k=0}^{\infty} 2^{-nk}\chi_{R_k}(x).$$

Therefore,

$$S_t(x) \le 2^n t^{-n} \sum_{k=0}^{\infty} 2^{-nk}\chi_{tR_k}(x).$$

Thus, since $|R_k| = 2^n 2^{k(n-1)}$, we get

$$(S_t * w)(x) \le 2^n t^{-n} \sum_{k=0}^{\infty} 2^{-nk}\int_{x+tR_k} w(y)dy$$
$$= 4^n \sum_{k=0}^{\infty} 2^{-k}|x+tR_k|^{-1}\int_{x+tR_k} w(y)dy.$$

Then, taking $Q = x + tQ_k$ in Lemma (3.4), we obtain

$$(S_t * w)(x) \le c_w 4^n \sum_{k=0}^{\infty} 2^{-k}2^{kr} \operatorname*{ess.\,inf}_{y\in x+tQ_k} w(y)$$
$$\le c_w 4^n (2^{1-r}/(2^{1-r}-1))Mw(x) \le c\,w(x) \quad \text{a.e.}$$

Thus, we have shown that w belongs to $\mathcal{A}_1$.

We shall give an example showing that for pairs of weights, the class $\mathcal{A}_1$ is strictly contained in A_1. Let $v(x_1,x_2) = |x_1|\chi_{[-1,1]}(x_2)$, $(x_1,x_2) \in \mathbf{R}^2$. After some elementary computations it can be shown that the Hardy-Littlewood maximal function $Mv(x_1,x_2) = w(x_1,x_2)$ satisfies $w(x_1,x_2) \le |x_1| + 1$. As it is well known, the pair $(v,w) \in A_1$. Let $f(x_1,x_2) = \chi_Q(x_1,x_2)$, $Q = [-1,1]^2$. This function f belongs to $L^1(w)$, however $S*f$ does not belong to $L^1(v)$ and by Proposition (3.1) the pair (v,w) can not belong to $\mathcal{A}_1$. In fact, if $|x_2| < 1/2$, then

$$(S*f)(x_1,x_2) \ge \int \frac{\chi_{[-1,1]}(\xi_1)}{(2+|x_1-\xi_1|)^2}\,d\xi_1 \ge \frac{2}{(3+|x_1|)^2}\,.$$

Thus

$$\int (S*f)(x)v(x)dx_1dx_2 \ge \int \frac{2|x_1|}{(3+|x_1|)^2}\,dx_1 = +\infty\,.$$

We observe that Proposition (3.6) holds for a pair of weights (v,w) provided that there exists r, $0 \le r < 1$, such that

$$|R|^{-1}\int_R v(x)dx \le c_{v,w}\Big(\frac{|Q|}{|R|}\Big)^r \operatorname*{ess.inf}_{x\in Q} w(x) \tag{3.7}$$

holds for every cube Q and every interval R, $Q \supset R$. This is the two weight analogous of (3.5) in Lemma (3.4).

However, as the following example shows, the condition (3.7) is not necessary for a pair (v,w) of weights to belong to $\mathcal{A}_1$.

Let $v(x_1,x_2) = \chi_Q(x_1,x_2)$ where $Q = [-1,1]^2$ and $w(x_1,x_2) = Mv(x_1,x_2)$. It can be easily shown that $Mv(x_1,x_2) \approx (1+|x|)^{-2}$. On the other hand, if $R_N = [-N,N] \times [-1,1]$ and $Q_N = [-N,N]^2$, $N \ge 1$, we have

$$|R_N|^{-1}\int_{R_N} v(x_1,x_2)dx_1dx_2 = 1/N \quad \text{and}$$
$$\operatorname*{ess.inf}_{x\in Q_N} Mv(x) \le c(1+N)^{-2}\,.$$

Then, if we assume (3.7), we get

$$N^{-1} \le c\,N^r(1+N)^{-2} \le c\,N^{r-2}\,.$$

This inequality is false for any given c and N large enough. However, we shall show that (v,w) belongs to $\mathcal{A}_1$. In fact, since

$$(S_t*v)(x_1,x_2) \le 2\iint \frac{\chi_t(|x_2-y_2|)}{(t+|x_1-y_1|+|x_2-y_2|)^2}\chi_Q(y_1,y_2)dy_1dy_2\,,$$

for $|x_2| > 1+t$ we get $(S_t * v)(x_1, x_2) = 0$. If $|x_2| \leq 1+t$, for $|x_1| \geq 3$, we have

$$1 + |x_1| + |x_2| \leq (5/2)(t + |x_1 - y_1| + |x_2 - y_2|) .$$

Then

$$\begin{aligned}(S_t * v)(x_1, x_2) &\leq 2(5/2)^2(1 + |x_1| + |x_2|)^{-2} \iint \chi_t(|x_2 - y_2|)\chi_Q(y_1, y_2)dy_1\,dy_2 \\ &\leq 50(1 + |x_1| + |x_2|)^{-2} .\end{aligned}$$

If $|x_2| \leq 1+t$ and $|x_1| < 3$, we distinguish two cases: $t < 1$ and $t \geq 1$. In the first case we have $1 + |x_1| + |x_2| \leq 6$. Thus,

$$\begin{aligned}(S_t * v)(x_1, x_2) &\leq 2t^{-1} \int_{-\infty}^{\infty} \chi_t(|x_2 - y_2|)\Big(\int_{-\infty}^{\infty} \frac{t}{t^2 + |x_1 - y_1|^2}\, dy_1\Big)dy_2 \leq 4\pi \\ &\leq 144\pi(1 + |x_1| + |x_2|)^{-2} .\end{aligned}$$

In the second case, we have $1 + |x_1| + |x_2| \leq 3(t + |x_1 - y_1| + |x_2 - y_2|)$. Thus,

$$\begin{aligned}(S_t * v)(x_1, x_2) &\leq 18(1 + |x_1| + |x_2|)^{-2} \int \chi_Q(y_1, y_2)dy_1\,dy_2 \\ &= 72(1 + |x_1| + |x_2|)^{-2} .\end{aligned}$$

Therefore, we have shown that for every $t > 0$,

$$\begin{aligned}(S_t * v)(x_1, x_2) &\leq c(1 + |x_1| + |x_2|)^{-2} \\ &\leq c'\, M\,\chi_Q(x_1, x_2) = c'\, w(x_1, x_2)\end{aligned}$$

holds, proving that (v, w) belongs to $\mathcal{A}_1$.

(3.8) LEMMA. *If $f \in L^1$, then*

$$K_B f(x) = K_R f(x) + L\, f(x)$$

holds almost everywhere for $L = \lim_{t\to 0^+} \int_{R_t - B_t} k(y)dy$.

Proof. The proof of this lemma is contained in [**3**] and shall not be given here.

Let (v, w) belong to the class A_1. It is well known that for these weights there exists a constant $a > 0$ such that $a(1 + |x|)^{-n} \leq w(x)$, a.e.. In particular, if $f \in L^1(w)$, then $f \in L^1((1 + |x|)^{-n})$, which implies that f belongs locally to L^1.

(3.9) LEMMA. *Let* $\{R_t\}_{t>0}$ *be an admissible family of intervals and let* f *belong to* $L^1((1+|x|)^{-n})$. *Then,*

(i) *for every* $t>0$, $K_{R_t}f(x)$ *is well defined almost everywhere,*

(ii) *the limit* $K_R f(x) = \lim_{t\to 0+} K_{R_t}f(x)$ *exists everywhere,*

(iii) *for almost every point* x

$$K_B f(x) = K_R f(x) + Lf(x)$$

holds with $L = \lim_{t\to 0+} \int_{R_t - B_t} k(y)dy$.

Proof. Let $T>0$ and $|x|\le T$. We define $f_1(y)=f(y)$ if $|y|\le 2T$, $f_1(y)=0$ otherwise and $f_2(y)=f(y)-f_1(y)$. Then,

$$K_{R_t}f(x) = K_{R_t}f_1(x) + K_{R_t}f_2(x) .$$

The integral defining $K_{R_t}f_2(x)$ exists since

$$|K_{R_t}f_2(x)| \le c\int_{CB_t} \frac{|f_2(y)|}{|x-y|^n}\,dy \le C(T)\int \frac{|f(y)|}{(|y|+1)^n}\,dy .$$

We observe that $K_{R_t}f_2(x)$ does not depend on t for t small enough. Then by Lemma (3.8) the limit of $K_{R_t}f(x)$ exists almost everywhere. Moreover, since $K_{R_t}f_2(x) = K_B f_2(x)$ and by Lemma (3.8), we have

$$\begin{aligned} K_R f(x) &= K_R f_1(x) + K_B f_2(x) \\ &= K_B f_1(x) - Lf_1(x) + K_B f_2(x) \\ &= K_B f(x) - Lf(x) . \end{aligned}$$

(3.10) LEMMA. *Let* $\{R_t\}_{t>0}$ *be an admissible family of intervals. Let* $L = \lim_{t\to 0+} L_t$, $L_t = \int_{R_t-B_t} k(y)dy$, *and* (v,w) *in* $\mathcal{A}_1$. *If* $g(x)$ *is a bounded function with bounded support then*

$$\lim_{t\to 0+} \|Lg - (k\chi_{R_t-B_t}) * g\|_{L^1(v)} = 0 .$$

Proof. Since $|k(y)| \le c\,\dfrac{1}{|y|^n}$, then for $y\in R_t - B_t$ we have $|k(y)|\le cS_t(y)$. Thus,

$$\int_{R_t-B_t} |k(y)|dy \le c\int S_t(y)dy = c' . \tag{3.11}$$

On the other hand, since we assume that g is a bounded function with bounded support, given $\eta>0$ there exists $\delta>0$ such that

$$\int |g(x-y) - g(x)|v(x)dx < \eta \tag{3.12}$$

provided that $|y| < \delta$. Therefore by (3.11) and (3.12) we get

$$\|Lg - (k\chi_{R_t - B_t}) * g\|_{L^1(v)} \leq$$
$$|L - L_t| \, \|g\|_{L^1(v)} + \int_{R_t - B_t} |k(y)| \Big(\int |g(x-y) - g(x)| v(x) dx \Big) dy < \varepsilon$$

if η and t are small enough.

PROOF OF THEOREM (2.1). Since by Lemma (3.9), $K_R f(x) = K_B f(x) - Lf(x)$ a.e., from the hypothesis that $K_R f$ belongs to $L^1(w)$ we get that $K_B f$ belongs to $L^1(w)$. Besides we observe that Theorem A in [4]holds in the case $p = 1$ and $r = \infty$. Then, it follows that

(a) $\|K_{B_t} f\|_{L^1(v)} \leq c\big(\|f\|_{L^1(w)} + \|K_B f\|_{L^1(w)}\big)$ for every $t > 0$, and

(3.13)

(b) $\lim\limits_{t \to 0+} \|K_{B_t} f - K_B f\|_{L^1(v)} = 0$.

We have

$$K_{R_t} f(x) = -\int_{R_t - B_t} k(y) f(x-y) dy + K_{B_t} f(x) .$$

The integral above is the convolution of f with the kernel $k(y)\chi_{R_t - B_t}(y)$ which is bounded by a constant times $S_t(y)$. Therefore, by Proposition (3.1) and part (a) of (3.13), we have

$$\|K_{R_t} f\|_{L^1(v)} \leq c(\|f\|_{L^1(w)} + \|K_B f\|_{L^1(w)}) .$$

Taking into account Lemma (3.9), part (iii), we obtain part (i) of the theorem. As for part (ii) of the statement of the theorem, let g be a bounded function with bounded support such that $\|f - g\|_{L^1(w)} < \varepsilon$. Then

$$\|K_{R_t} f - K_R f\|_{L^1(v)} = \|K_{B_t} f - (k\,\chi_{R_t - B_t}) * f - K_B f + Lf\|_{L^1(v)}$$
$$\leq \|K_{B_t} f - K_B f\|_{L^1(v)} + \|(k\,\chi_{R_t - B_t}) * (g - f)\|_{L^1(v)}$$
$$+ |L| \, \|f - g\|_{L^1(v)} + \|Lg - (k\,\chi_{R_t - B_t} * g\|_{L^1(v)}$$

Therefore, by part (b) of (3.13), Proposition (3.1) and Lemma (3.10) we obtain part (ii) of the theorem.

REFERENCES

[1] H. Aimar and E.O. Harboure, "On weighted inequalities for non-standard truncations of singular integrals". Revista de la Unión Mat. Argentina 33 (1987),Nro.1-2, 21-30.

[2] O.N. Capri and C. Segovia, "Behavior of L^r-Dini singular integrals in weighted L^1 spaces", Studia Math. 92 (1989), 21-36.

[3] E.O. Harboure, "Non-standard truncations of singular integrals", Indiana Univ. Math. Journal, 28 (1979), 779-790.

[4] L. de Rosa and C. Segovia, "Convergence of truncated singular integrals with two weights", Colloquium Mathematicum, Vol LX/LXI (1990), 579-592.

LILIANA DE ROSA and CARLOS SEGOVIA
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
and
Instituto Argentino de Matemática (CONICET).

Recibido en setiembre de 1993.