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ON THE COMPACTNESS OF CONNECTED SETS

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ABSTRACT : Let X be a space and U be an open cover of X . Then X is U-chainable if for each $x \in X$, $st^{\infty}(x , U) = X$ and it is U-uniformly chainable if there is an $n \in N$ such that for each $x \in X$, $X = st^{n}(x , U)$. In this paper we characterise connected spaces in terms of U-chainability , connected spaces satisfying finite discrete chain condition in terms of U-uniform chainability and obtain several results which are analogues of the known results for metric spaces .

KEY WORDS AND PHRASES : Chainable , uniformly chainable , strictly chainable, connected , countably compact , compact.

MATHEMATICS SUBJECT CLASSIFICATION (1985)

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1- **INTRODUCTION:** Connetedness and compactness are widely studied in Topology . In 1883 Cantor defined connectedness in metric spaces with the help of ε - chains . At present , however , the Riesz - Hausdorff definition , using the idea of separated sets , is universally accepted . On the other hand , a lot of experimentation has led to several forms of compactness . Compactness and several of its generalizations are defined in terms of open covers ; e.g. compact , countably compact, paracompact , Lindeloff etc . Chainability characterizes connected sets among compact sets in the setting of metric spaces . In the same setting Beer [Be] has characterized compact sets among the connected sets .

In this paper the concept of ε - chainability in metric spaces is generalized by the use of open covers of a topological space. This generalization yields a simple characterization of connectedness in terms of open covers . Several results of Beer [Be] as well as some of the earlier known results are then generalized .

2- **PRELIMINARIES:** In this section we give some basic definition and fundamental facts which are needed in proving the main results in the following sections. Let (X, τ) be a topological space .

Let , μ = a family of open covers of X ; μ_F = a family of open covers of X which induces the fine uniformity on X (when X is Tychonoff) ; μ_0 = the family of all open covers of X .

For each $A \subset X$ and $U \in \mu$

$$st(A , U) = st^1 (A , U) = \cup \{ u \in U : A \cap u \neq \phi \} ;$$

$$st^n(A , U) = st[st^{n-1}(A ,U)] \text{ for } n > 1; \text{ and}$$

$$st^\infty(A , U) = \bigcup_{n=1}^{\infty} st^n(A , U) .$$

We write st(x , U) for st({x} , U) .

(2.1) Definition : Let $U \in \mu$. Then X is *U-chainable* iff for each $x \in X$, $X = st^\infty(x , U)$. X is *μ-chainable* iff for each $U \in \mu$, X is U - chainable .

(2.2) Definition : For $U \in \mu$, X is called *U-locally uniformly chainable* iff for each $x \in X$ there is an $n_x \in \mathbf{N}$ such that $st^{n_x}(x, U) = X$. If n_x is independent of x , we say that X is *U-uniformly chainable* . As in (2.1) we can obviously define μ-locally uniformly chainable and μ-uniformly chainable .

The following fact is well known .

(2.3) Theorem : If X is a Hausdorff paracompact space , then $\mu_F = \mu_0$

(2.4) Lemma : X is U-chainable if and only if it is U-locally uniformly chainable .

Proof : If X is U-locally uniformly chainable for $U \in \mu$, then for each $x \in X$, there is an $n_x \in N$ such that $X = st^{n_x}(x, U)$. Now for each $p \in X$ and $n = 2n_x$, it is easy to see that $X = st^n(p, U)$ i.e. , X is U - uniformly chainable. Converse is obvious .

(2.5) Corollary : X is μ-uniformly chainable if and only if it is μ-locally uniformly chainable .

(2.6) Lemma : Let X be any topological space and $U \in \mu_0$. Then for each $x \in X$, $st^{\infty}(x, U)$ is an open and closed subset of X .

Proof : If $y \notin st^{\infty}(x, U)$, the $st(y, U) \cap st^{\infty}(x, U) = \varnothing$. This shows that $st^{\infty}(x, U)$ is closed and it is obviously open .

The following result characterizes connectedness in terms of chainability which in turn depends upon covers .

(2.7) Theorem : A space X is connected if and only if X is μ_0- chainable .

Proof : The result follows from the facts (a) X is connected if and only if it has no proper open and closed subset and (b) for each $x \in X$ and $U \in \mu_0$, $st^{\infty}(x, U)$ is open and closed .

(2.8) Corollary : A Hausdorff paracompact space X is connected if and only if it is μ_F-chainable .

(2.9) Corollary : A compact Hausdorff space X is connected if and only if it is μ_F- chainable .

(2.10) Remarks : Since a compact Hausdorff space has a unique compatible uniformity , the above results generalize the well known result : a compact metric space is connected if and only if it is ε - chainable for each $\varepsilon > O$ [Be] .

3. COMPACTNESS OF CONNECTED SPACES :

At first we introduce a concept which is analogous to total boundedness in uniform spaces .

(3.1) Definition : X is μ - star compact if and only if for each $U \in \mu$, there is a finite subset F of X such that X = st(F , U) .

Fleischman [Fl] introduced the concept of star compactness , which is μ_0 - star compact and showed that a T_2 space X is star compact if and only if it is countably compact . In case μ is a compatible uniformity on a Tychonoff space X , then μ - star compactness is equivalent to μ- total boundedness .

In case X is a (metric) uniform space with a compatible covering (metric) uniformity μ , μ - chainability plus μ - total boundedness yields μ - uniform chainability (see [Be] page 808) . We analogously have the following result which can be easily shown to hold .

(3.1) Lemma : If X is μ - star compact and μ - chainable , then X is μ - uniformly chainable .

We now characterize μ_0- uniformly chainable spaces .

(3.2) Definition : X is DFCC if and only if every discrete collection of open sets is finite .

(3.3) Theorem : A T_3 space X is μ_0- uniformly chainable if and only if it is connected and is DFCC .

Proof : Suppose X is T_3 and μ_0- uniformly chainable . By Theorem (2.7) , X is connected . If X is not DFCC , then there is a countably infinite discrete collection of open sets $U = \{ u_n : n \in N \}$. Choose $x_n \in U_n$ for each n in N .

For each n in N define

$x_n \in A_1^n \subseteq \overline{A_1^n} \subseteq A_2^n \subseteq \ldots \subseteq A_m^n \subset \ldots \subseteq A_n^n \subseteq \overline{A_n^n} \subseteq u_n$ where A_m is open for each m .

Define :

$$W = X - \cup\{\overline{A_n^n} : n \in N\}.$$

$$V_1^n = A_2^n$$

$$V_2^n = A_3^n - \overline{A_1^n}$$

$$V_{n-1}^n = A_n^n - \overline{A_{n-2}^n}$$

$$V_n^n = u_n - \overline{A_{n-1}^n}.$$

Now let $\mathcal{V} = \{W\} \cup \{V_m^n : m < n , n \in N\}$. It is easy to see that it is a collection of nonempty open sets in X .It is a cover of X . For any x in X , either x is a member of u_n for some n and hence x is in V_m^n for some m and n , or $x \notin \cup_{n \in N} u_n$ in which case x is in W . Consider x in X and n in N . Now there exist infinitely many u_n which do not contain x . Pick p > n such that x is not in u_p . It is easy to see that $st^n(x_p , \mathcal{V}) \subseteq U_p$, so $x_p \notin st^p(x , \mathcal{V})$. x and n are arbitrary , so X cannot be μ_0 - uniformly chainable .

For the converse , assume that X is μ_0 - chainable and has DFCC property but is not μ_0 - uniformly chainable . Then there exists an x in X and U in μ_0 , such that $X \neq st^n(x , U)$ for each n in N .

$$\text{Define } u_i = \begin{cases} st(x,U) \text{ for } i = 1 \\ st^i(x,U) - CL(st^{i-1}(x,U)) \text{ for } i > 1. \end{cases}$$

Since X is μ_0 - chainable , $u_i \neq \emptyset$ for each i in N and $\{u_i : i \in N\}$ is a discrete collection of open sets . So , X is not DFCC , a contradiction .

We now introduce a concept which will provide a characterization of countably compact connected spaces in terms of open covers .

(3.4) Definition : Let $U \in \mu_0$. X is said to be *strictly U - chainable* if and only if there is a finite subset $F = \{x_i : 1 \leq i \leq n\}$ of X such that $X = st(F, U)$ and $x_{i+1} \in st(x_i, U)$ for $1 \leq i \leq n-1$. X is strictly μ - chainable if and only if it is strictly U - chainable for each $U \in \mu$.

From our previous discussion , it is clear that a strictly U - chainable space is U - uniformly chainable .

(3.5) Theorem : A Hausdorff space X is strictly μ - chainable if and only if it is μ - starcompact and μ - chainable .

Proof : Suppose X is μ - starcompact and μ-chainable . Let $U \in \mu$. Then there is a finite set $F = \{x_i : 1 \leq i \leq n\} \subset X$ such that $st(F, U) = X$. Since X is μ-chainable, for each $i \leq n-1$, there is a finite set $F_i = \{x_i^j : 1 \leq j \leq n_i\}$ and $\{u_i^j : 2 \leq j \leq n_i\} \subset U$ such that

(i) $x_i^1 = x_i \in u_i$,

(ii) $x_i^{n_i} = x_{i+1} \in u_i^{n_i}$,

(iii) $x_i^j \in u_i^j \cap u_i^{j+1}$ for $j = 2, 3, 4, \ldots, n_i - 1$.

Then $F^* = \cup\{F_i : 1 \leq i \leq n\}$ provides a strict U - chain as in (3.4) . The converse is obvious .

Combining Theorem 3.5 with Fleischman ' s Theorem [Fl] , we can state the following .

(3.6) Corollary : A Hausdorff space X is countably compact and connected if and only if it is strictly μ_0 - chainable .

We conclude by providing a sequential characterization of μ_0-uniform chainability . For $U \in \mu_0$ we define U-chain distance function $\phi_U : X \times X \rightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$ as follows :

$\phi_U(x, y) =$ n-1 where n is the smallest natural number such that
$y \in st^n(x, U)$.
$= \infty$ if no such n exists .

From Theorem (2.7) it is clear that X is connected if and only if for each $U \in \mu_0$. ϕ_U is finite . If X is connected and $U \in \mu_0$ we say that ;

(a) ϕ_U is constant on a sequence (x_n) if and only if $\{\phi_U(x_n, x_m) : n \neq m\}$ is a singleton ;

(b) ϕ_U is bounded on (x_n) if and only if $\{\phi_U(x_n, x_m) : n \neq m\}$ is finite .

We shall need the following basic theorem of combinatorics .

(3.7) Ramsey's Theorem : Let $r \in \mathbf{N}$ and $\{A_i : 1 \leq i \leq m\}$ a partition of the r-element subsets of **N** . Then there is an infinite subset S of **N** and $i \in \{1, 2, 3, \ldots, m\}$ such that each r-element subset of S belongs to A_i .

For the proof see[3] .

(3.8) Theorem : For a connected space the following are equivalent :

(a) X is μ_0-uniformly chainable .

(b) For each $U \in \mu_0$, each sequence (x_n) in X has a subsequence on which ϕ_U is constant .

(c) For each $U \in \mu_0$, each sequence (x_n) in X has a subsequence on which ϕ_U is bounded .

Proof : (a) $\rightarrow$ (b) . Let $U \in \mu_0$. Since X is μ_0-uniformly chainable , there is an m in **N** such that $\phi_U(x, y) \leq m$ for all x , y in X . If (x_n) is a finite sequence , there is nothing to prove . If (x_n) is an infinite sequence define :

$$A_i = \{ \{x_i, x_k\} : \phi_U(x_i, x_k) = 1\} , \quad 0 \leq i \leq m .$$

Clearly , $\{ A_1, A_2, A_3, \ldots, A_m\}$ is a partition of 2-element subsets of the infinite set $\{ x_n : n \in \mathbf{N} \}$. By Ramsey's Theorem (3.7) , there is an i_0

such that for some infinite subset S of $\{ x_i : i \in \mathbf{N} \}$ all two element subsets of S are in A_i. By arranging the elements S in the natural order we have a subsequence on which ϕ_U is constant.

(b) $\rightarrow$ (c) is obvious.

(c) $\rightarrow$ (a). If X is not μ_0- uniformly chainable, there is a $U \in \mu_0$ and y in X such that $X \neq st^n(y, U)$ for each n in $\mathbf{N}$. Let $x_1 = y$ and $x_n \in st^n(x_1, U) - st^{n-1}(x_1, U)$. Clearly, (x_n) is an infinite sequence such that $\varphi_U(x_n, x_m) \geq |n - m|$. Hence ϕ_U is unbounded on any subsequence of (x_n).

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