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ACCELERATING SIMPLIFIED MONOTONE NEWTON ITERATIONS

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ABSTRACT. Simplified Newton iterations are analyzed in the context given by the hypotheses of the monotone Newton theorem; it is known that they produce upper and lower sequences with monotone convergence to a root of the nonlinear system $Fy = 0$. It is proved here that accurate partial elimination accelerates convergence in both sequences, while retaining monotonicity.

1. INTRODUCTION.

The monotone Newton theorem (See [6]) and some of its generalizations give conditions that generate two monotone sequences, one decreasing and one increasing, both converging at least quadratically to the same root of the nonlinear system $Fy = 0$. This result has three main advantages, when compared with the usual Newton convergence theorem. Firstly, no a priori knowledge of the existence of a root is assumed; secondly, a reliable bound for the error in the determination of the exact root is provided by the vector difference of any two terms, one from the upper sequence and another from the lower one; finally, if these two terms correspond to the same iterate, the bounds on the error thus obtained converge to 0 at least quadratically. On the other hand, actual application of the result implies increased work per iteration, because two linear systems per step, instead of one, must be solved with the same Jacobian matrix. This objection may loose part of its value, if, for instance, a two-processor parallel computer is available.

The hypotheses of the monotone Newton theorem will be assumed throughout this note. In this context, a gain in convergence speed can be obtained by partial reduction of the initial system, if applied accurately; we say that an unknown has been accurately eliminated, if the equation used for the elimination has the same index (See [4]); whether the acceleration thus obtained can be complemented by the reduction of computational work depends on the problem dealt with and the choice of unknowns to be eliminated. Another useful possibility is given by the simplified monotone Newton iterations; with them, updating of the Jacobian matrix is done every p iterations (we consider here only fixed updating steplength p, for the sake of simplicity); in this case, an extension of the monotone Newton theorem says that pairwise monotone convergence is obtained with convergence order greater or equal than $p+1$ (See theorem 4 in [7]).

The main objective in this paper is to prove that accurate partial reduction in the original system, via partial elimination of unknowns, yields improvement in convergence of the simplified Newton iterates, as in the (non simplified) monotone Newton context.

2. THE SIMPLIFIED NEWTON-FOURIER ITERATIONS.

Consider a continuously differentiable function $F : D \subset \Re^n \longrightarrow \Re^n$, for which we seek a solution to the equation

$$Fy = 0 \quad . \tag{2.1}$$

It will be assumed that we have $x^0 \leq y^0$, i.e. $x_i^0 \leq y_i^0$, for $1 \leq i \leq n$, are such that

$$\langle x^0, y^0 \rangle := \{x \in \Re^n / x^0 \leq x \leq y^0\} \subset D,$$

and

$$Fx^0 \leq 0 \leq Fy^0 \quad .$$

We also suppose that F is order convex on $\langle x^0, y^0 \rangle$, i.e.

$$F(\lambda x + (1-\lambda)y) \leq \lambda Fx + (1-\lambda)Fy \quad ,$$

$$\textit{whenever } x \leq y \textit{ or } y \leq x \textit{ and } \lambda \in (0,1) \; .$$

The following result is known as the monotone Newton theorem .

THEOREM 2.1. *Suppose that for each $x \in \langle x^0, y^0 \rangle$, $F'(x)$ is a nonsingular M-matrix (i.e. $(F'(x))_{i,j} \leq 0$, for $i \neq j$, and $F'(x)^{-1}$ is nonnegative). Then the Newton iterates*

$$y^{k+1} := y^k - F'(y^k)^{-1} F y^k, \qquad k = 0, 1, \ldots \tag{2.2}$$

satisfy $y^k \downarrow y^* \in \langle x^0, y^0 \rangle$ as $k \to \infty$ and y^* is the unique solution of (2.1) in $\langle x^0, y^0 \rangle$. Moreover, if F' is isotone on $\langle x^0, y^0 \rangle$(i.e. $x \leq y$ implies $F'(x) \leq F'(y)$), then the Newton-Fourier (N-F) iterates

$$x^{k+1} := x^k - F'(y^k)^{-1} F x^k, \qquad k = 0, 1, \ldots \tag{2.3}$$

satisfy $x^k \uparrow y^*$ as $k \to \infty$. We also have

$$Fx^k \leq 0 \leq Fy^k \quad , \qquad k = 0, 1, \ldots \quad . \tag{2.4}$$

Finally, if for some norm,

$$\|F'(x) - F'(y)\| \leq \gamma \|x - y\| \qquad \forall \quad x, y \in \langle x^0, y^0 \rangle \quad , \tag{2.5}$$

then there exists a constant c such that

$$\left\|y^{k+1} - x^{k+1}\right\| \leq c \left\|y^k - x^k\right\|^2 \qquad k = 0, 1, \ldots \tag{2.6}$$

PROOF: See [6].

REMARK 2.2. Apparently, it has been Baluev (See [2]), who first realized that the iterates in (2.3) generate a complementary sequence to the one given by (2.2); we shall call the couple (x^k, y^k) the Newton-Fourier (N-F) iterates. Other possible formulations of the theorem above are obtained by interchanging the roles of x^0 and y^0, and also by supposing that F is order concave (See Table 13.1 in [6]). Note that isotonocity of F' implies order convexity of F; however, some partial results do not need the isotonicity hypothesis.

In the context given by the theorem above, it is possible to introduce simplified monotone Newton iterations with a fixed steplength $p \geq 1$ as follows:

$$
\begin{aligned}
&For\ k = 0, 1, \ldots \\
&\quad y^{k,0} := y^k \\
&\quad x^{k,0} := x^k \\
&\qquad For\ i = 1, \ldots, p \\
&\qquad y^{k,i} := y^{k,i-1} - F'(y^k)^{-1} F y^{k,i-1} \\
&\qquad x^{k,i} := x^{k,i-1} - F'(y^k)^{-1} F x^{k,i-1} \\
&\quad y^{k+1} := y^{k,p} \\
&\quad x^{k+1} := x^{k,p}
\end{aligned}
\tag{2.7}
$$

These simplified iterations can be useful in two basic situations, i.e. when the Jacobian matrix $F'(y)$ is difficult to calculate, and when $F'(y)$ becomes ill conditioned for y near the root; thus making it impractical to further apply the Newton method. The following theorem extends the monotone Newton theorem to the simplified N-F iterations.

THEOREM 2.3. *The sequences (x^k) and (y^k) satisfy:*
(i) $Fx^{k,i} \leq 0 \leq Fy^{k,i} \quad \forall \quad 1 \leq i \leq p$.
(ii) $x^k \leq x^{k,i-1} \leq x^{k,i} \leq x^{k+1} \leq y^{k+1} \leq y^{k,i} \leq y^{k,i-1} \leq y^k \quad \forall \quad 1 \leq i \leq p$.
(iii) (x^k) and (y^k) both converge to y^*.
(iv) There exists b_p such that

$$\left\|y^{k+1} - x^{k+1}\right\| \leq b_p \left\|y^k - x^k\right\|^{p+1} \qquad k = 0, 1, \ldots \tag{2.8}$$

PROOF: (i) and (ii) can be proved inductively, as the corresponding statements in Theorem 2.1; (iii) then also follows as in Theorem 2.1.
(iv)Note first that, since $\langle x^0, y^0 \rangle$is compact and F' is continuous, there exists b such that

$$\left\|(F'(y))^{-1}\right\| \leq b \quad \forall \quad y \in \langle x^0, y^0 \rangle.$$

It is not difficult to see that the following inequalities hold

$$\begin{aligned}\|y^{k+1}-x^{k+1}\| &= \|y^{k,p}-x^{k,p}\| = \|y^{k,p-1}-x^{k,p-1}-F'(y^k)^{-1}(Fy^{k,p-1}-Fx^{k,p-1})\| \\ &\leq \|F'(y^k)^{-1}\|\,\|F'(y^k)(y^{k,p-1}-x^{k,p-1})-Fy^{k,p-1}+Fx^{k,p-1}\| \\ &\leq b\left\|\int_0^1 \langle F'(y^k)-F'(x^{k,p-1}+t(y^{k,p-1}-x^{k,p-1})), y^{k,p-1}-x^{k,p-1}\rangle\, dt\right\| \\ &\leq b\left(\int_0^1 \|F'(y^k)-F'(x^{k,p-1}+t(y^{k,p-1}-x^{k,p-1}))\|\, dt\right)\|y^{k,p-1}-x^{k,p-1}\| \\ &\leq b\,\gamma\left(\int_0^1 \|y^k-x^{k,p-1}-t(y^{k,p-1}-x^{k,p-1})\|\, dt\right)\|y^{k,p-1}-x^{k,p-1}\| \\ &\leq b\,\gamma\left(\|y^k-x^{k,p-1}\|+\frac{1}{2}\|(y^{k,p-1}-x^{k,p-1})\|\right)\|y^{k,p-1}-x^{k,p-1}\| \\ &\leq b\,\gamma\left(\frac{3}{2}rs\right)\|y^k-x^k\|\,\|y^{k,p-1}-x^{k,p-1}\| \quad ,\end{aligned}$$

where

$$\|\ \ \| \leq r\|\ \ \|_\infty \quad and \quad \|\ \ \|_\infty \leq s\|\ \ \| \quad .$$

Thus, if we set $b_1 := b\gamma(\frac{3}{2}rs)$, we have

$$\|y^{k+1}-x^{k+1}\| \leq b_1\ \|y^k-x^k\|\ \|y^{k,p-1}-x^{k,p-1}\| \quad .$$

If $p=1$, we essentially have (2.6); if $p \neq 1$, we finally get

$$\|y^{k+1}-x^{k+1}\| \leq b_1^{p-1}\ \|y^k-x^k\|^{p+1} \quad .$$

REMARK 2.4. Theorem 2.3 slightly improves a result stated by Wolfe (See Theorem 4 in [7]). Wolfe's proof of (2.8) is based on Satz 4 in [1], which assumes F'' to be continuous: the proof above assumes the somewhat weaker hypothesis (2.5). Also, the inequalities stated here for the inner loops iterates will be needed in the following section.

THEOREM 2.5. *Consider two steplengths p and q with $p < q$, and denote the corresponding N-F sequences by (x_p^k, y_p^k) and (x_q^k, y_q^k). If $kp+i = sq+j$, with $0 \leq i < p$ and $0 \leq j < q$, then*

$$x_q^{s,j} \leq x_p^{k,i} \leq y^* \leq y_p^{k,i} \leq y_q^{s,j} \quad .$$

PROOF: Notice that $y_p^1 = y_q^{0,p}$; thus

$$y_p^{1,1} = y_p^{1,0} - F'(y_p^1)^{-1}Fy_p^{1,0} \leq y_p^{1,0} - F'(y^0)^{-1}Fy_p^{1,0} = y_q^{0,p+1} \quad .$$

An induction argument completes the proof.

3. ELIMINATION AND NEWTON-FOURIER ITERATIONS.

The main result will be proved in this section, i.e. that accurate partial elimination improves convergence of the simplified Newton iterations. In this way we extend the corresponding result in the context of Theorem 2.1 (See Theorem 3.9 in [4]). Some additional material is taken almost verbatim from [4].

Since $F'(y^*)$ is a nonsingular M-matrix, it follows that $\partial_1 f_1(y^*) \neq 0$; the implicit function theorem yields the existence of neighborhoods U of y^*, V of $\overline{y^*} := (y_2^*, \ldots, y_n^*)$ and a function $g : V \to \Re$, such that $f_1(g(\overline{y}), \overline{y}) = 0$; if moreover $y \in U$ is such that $f_1(y) = 0$, then $y_1 = g(\overline{y})\,(\overline{y} := (y_2, \ldots, y_n))$. We assume throughout that $\langle x^0, y^0 \rangle \subset U$ and that $\left\langle \overline{x^0}, \overline{y^0} \right\rangle \subset V$; unnecessary distinction between row and column vectors will be avoided.

It is possible now to eliminate y_1 in (1.1), by means of g, and get the reduced system

$$\overline{F}\,\overline{y} = 0 \quad , \tag{3.1}$$

$$\text{where} \quad \overline{F} := (\overline{f}_i) \quad i = 2, \ldots, n \quad , \quad \overline{y} \in V \quad , \text{and} \quad \overline{f}_i(\overline{y}) := f_i(g(\overline{y}), \overline{y}) \quad .$$

THEOREM 3.1. *The following propositions hold:*

(i) g is isotone on $\left\langle \overline{x^0}, \overline{y^0} \right\rangle$ *and* $x_1^k \leq g(\overline{x^k}) \leq y_1^* \leq g(\overline{y^k}) \leq y_1^k \quad , k = 0, 1, \ldots$.

(ii) $\overline{F}'(\overline{y})$ *is an M-matrix for* $\overline{y} \in \left\langle \overline{x^0}, \overline{y^0} \right\rangle$ *and*

$$(\overline{F}')^{-1}_{ij}(\overline{y}) = (F')^{-1}_{ij}(g(\overline{y}), \overline{y}) \quad , for \quad i \neq 1 \neq j \quad . \tag{3.2}$$

(iii) $\overline{F}'$ *is isotone on* $\left\langle \overline{x^0}, \overline{y^0} \right\rangle$.

(iv) $\overline{F}\,\overline{x^0} \leq 0 \leq \overline{F}\,\overline{y^0}$.

PROOF: See Lemmas 3.2, 3.3, 3.5 and 3.7 in [4].

REMARK 3.2. Theorem 3.1 gives the possibility of applying Theorem 2.1 to the reduced system (3.1) with starting interval $\left\langle \overline{x^0}, \overline{y^0} \right\rangle$; it has been proved in [4] that under appropriate conditions the N-F iterates arising from the reduced system (3.1) converge better than those generated by the original system (See Theorem 3.9 in [4]). Our aim is to extend this result to the simplified Newton sequences; the conclusions in Theorem 3.1, especially (3.2), and Theorem 2.3 as well, will be used in the proof without explicit mention; the reasoning is basically similar to that in [4]. We need the notation corresponding to algorithm (2.7), when applied to the reduced system; let us set

$$\overline{y}^{\,0} := \overline{y^0} \quad and \quad \overline{x}^{\,0} := \overline{x^0} \quad ,$$

and define

$$
\begin{aligned}
& For\ k = 0, 1, \ldots \\
& \overline{y}^{k,0} := \overline{y}^{k} \\
& \overline{x}^{k,0} := \overline{x}^{k} \\
& \qquad For\ i = 1, \ldots, p \\
& \qquad \overline{y}^{k,i} := \overline{y}^{k,i-1} - \overline{F}'(\overline{y}^{k})^{-1}\overline{F}\,\overline{y}^{k,i-1} \\
& \qquad \overline{x}^{k,i} := \overline{x}^{k,i-1} - \overline{F}'(\overline{y}^{k})^{-1}\overline{F}\,\overline{x}^{k,i-1} \\
& \overline{y}^{k+1} := \overline{y}^{k,p} \\
& \overline{x}^{k+1} := \overline{x}^{k,p} \quad .
\end{aligned}
$$

THEOREM 3.3. *The following inequalities hold for* $k = 0, 1, \ldots$ *and* $1 \le i \le p$ *:*

$$
\begin{array}{ccccccccc}
\overline{x_j^{k,i}} & \le & \overline{x}_j^{k,i} & \le & \overline{y^*} & \le & \overline{y}_j^{k,i} & \le & \overline{y_j^{k,i}} \quad , \\
x_1^{k,i} & \le & g(\overline{x}^{k,i}) & \le & y_1^* & \le & g(\overline{y}^{k,i}) & \le & y_1^{k,i} \quad .
\end{array}
$$

PROOF: Suppose that the conclusions hold for a certain k and $0 \le i \le p-1$. For $2 \le j \le n$, we have

$$
\begin{aligned}
\overline{y}_j^{k,i+1} &= \overline{y}_j^{k,i} - (\overline{F}'(\overline{y}^{k})^{-1}\overline{F}(\overline{y}^{k,i}))_j \\
&= \overline{y}_j^{k,i} - \sum_{l=2}^{n} (\overline{F}')_{j,l}^{-1}(\overline{y}^{k}) * \overline{f}_l(\overline{y}^{k,i}) \\
&= \overline{y}_j^{k,i} - \sum_{l=2}^{n} (F')_{j,l}^{-1}(g(\overline{y}^{k}), \overline{y}^{k}) * f_l(g(\overline{y}^{k,i}), \overline{y}^{k,i}) \\
&= \overline{y}_j^{k,i} - \sum_{l=1}^{n} (F')_{j,l}^{-1}(g(\overline{y}^{k}), \overline{y}^{k}) * f_l(g(\overline{y}^{k,i}), \overline{y}^{k,i}) \quad .
\end{aligned}
$$

Now $(g(\overline{y}^{k}), \overline{y}^{k}) \le y^k$ yields $F'(g(\overline{y}^{k}), \overline{y}^{k}) \le F'(y^k)$, which implies

$$
0 \le F'(y^k)^{-1} \le (F')^{-1}(g(\overline{y}^{k}), \overline{y}^{k}) \quad ,
$$

and thus

$$
\overline{y}_j^{k,i+1} \le \overline{y}_j^{k,i} - \sum_{l=1}^{n} (F')_{j,l}^{-1}(y^k) * f_l(g(\overline{y}^{k,i}), \overline{y}^{k,i}) \quad . \tag{3.3}
$$

The inductive hypotheses, including the order convexity of F, also imply that

$$\begin{aligned} Fy^{k,i} - F(g(\overline{y}^{k,i}), \overline{y}^{k,i}) &\leq F'(y^{k,i})(y^{k,i} - (g(\overline{y}^{k,i}), \overline{y}^{k,i})) \\ &\leq F'(y^k)(y^{k,i} - (g(\overline{y}^{k,i}), \overline{y}^{k,i})) \quad , \end{aligned}$$

which yields

$$(g(\overline{y}^{k,i}), \overline{y}^{k,i}) - F'(y^k)^{-1} F(g(\overline{y}^{k,i}), \overline{y}^{k,i}) \leq y^{k,i} - F'(y^k)^{-1} F y^{k,i} = y^{k,i+1} \quad . \tag{3.4}$$

Thus, for $2 \leq j \leq n$, (3.4) means that

$$\overline{y}_j^{k,i} - \sum_{l=1}^{n} (F')_{j,l}^{-1}(y^k) * f_l(g(\overline{y}^{k,i}), \overline{y}^{k,i}) \leq \overline{y}_j^{k,i+1} \quad ,$$

which combined with (3.3) yields

$$\overline{y}^{k,i+1} \leq \overline{y^{k,i+1}} \quad .$$

Since we have $\overline{y^*} \leq \overline{y}^{k,i+1}$, it follows that

$$y_1^* = g(\overline{y^*}) \leq g(\overline{y}^{k,i+1}) \leq g(\overline{y^{k,i+1}}) \leq y_1^{k,i+1} \quad .$$

On the other hand, it is now clear that

$$\overline{x}_j^{k,i+1} = \overline{x}_j^{k,i} - \sum_{l=1}^{n} (F')_{j,l}^{-1}(g(\overline{y}^k), \overline{y}^k) * f_l(g(\overline{y}^{k,i}), \overline{x}^{k,i}) \quad ,$$

and thus

$$\overline{x}_j^{k,i+1} \geq \overline{x}_j^{k,i} - \sum_{l=1}^{n} (F')_{j,l}^{-1}(y^k) * f_l(g(\overline{y}^{k,i}), \overline{x}^{k,i}) \quad . \tag{3.5}$$

We also have

$$\begin{aligned} F(g(\overline{x}^{k,i}), \overline{x}^{k,i}) - Fx^{k,i} &\leq F'(g(\overline{x}^{k,i}), \overline{x}^{k,i})((g(\overline{x}^{k,i}), \overline{x}^{k,i}) - x^{k,i}) \\ &\leq F'(y^k)((g(\overline{x}^{k,i}), \overline{x}^{k,i}) - x^{k,i}) \quad , \end{aligned}$$

whence

$$x^{k,i+1} = x^{k,i} - F'(y^k)^{-1} F x^{k,i} \leq (g(\overline{x}^{k,i}), \overline{x}^{k,i}) - F'(y^k)^{-1} F(g(\overline{x}^{k,i}), \overline{x}^{k,i}) \quad .$$

Thus, by taking account of (3.5), it follows that

$$\overline{x^{k,i+1}} \leq \overline{x}^{k,i+1} \quad ,$$

and we finally obtain

$$x_1^{k,i+1} \leq g(\overline{x^{k,i+1}}) \leq g(\overline{x}^{k,i+1}) \leq g(\overline{y^*}) = y_1^* \quad .$$

REMARK 3.4. If instead of applying accurate elimination, an unknown is eliminated by means of a differently numbered equation, Theorem 3.3 is no longer true (see Remark 3.10 in [4] and the counterexample given there); nevertheless an extension of Theorem 3.11 in [4], comparing the generated sequences for such elimination, can be proved for the simplified N-F iterations in case the elimination satisfies the hypotheses in that theorem. Theorem 3.3 provides a useful tool for the reduction of nonlinear systems that satisfy its hypotheses, in either of the two basic situations mentioned following (2.7); such reduction yields a better convergence; however, its application should take account of the computational cost involved, as in the linear case (See [3]). Thus, theorem 3.3 implies that if the computational cost of the elimination is negligible and the computational cost per iteration is not increased, then a net gain is to be expected with the simplified N-F iterations applied to (3.1). These considerations will be illustrated in the following section.

4. A NUMERICAL EXAMPLE

The example treated in this section is taken from [4]. Thus, verification of the hypotheses in Theorem 2.1 is omitted here. The calculations have been carried out with the double precision of Fortran 5.0 on a PC. Let us define $F : \Re^n \longrightarrow \Re^n$ by

$$\begin{aligned} f_1 &:= \frac{2y_1 - y_2}{h^2} + y_1^3 \;, \\ f_i &:= \frac{2y_i - y_{i-1} - y_{i+1}}{h^2} + y_i^3 \,, \quad 2 \le i \le n-1 \,, \\ f_n &:= \frac{2y_n^3 - y_{n-1}}{h^2} \quad , with\ h := \frac{1}{n+1} \quad . \end{aligned} \tag{4.1}$$

If y_n is eliminated by means of f_n, we get the reduced system

$$\begin{aligned} \overline{f}_1 &= \frac{2y_1 - y_2}{h^2} + y_1^3 \\ \overline{f}_i &= \frac{2y_i - y_{i-1} - y_{i+1}}{h^2} + y_i^3 \,, \quad 2 \le i \le n-2 \,, \\ \overline{f}_{n-1} &= \frac{2y_{n-1} - y_{n-2} - g}{h^2} + y_{n-1}^3 \,, \\ \text{where } g &:= \left(\frac{y_{n-1}}{2}\right)^{\frac{1}{3}} \quad . \end{aligned} \tag{4.2}$$

Consider $n = 10$, $x^0 := (0, \ldots, 0, 0.14, 0.41)$ and $y^0 := (1, \ldots, 1)$. The stopping criteria have been

$$\left\|Fy^{k,i}\right\|_2 < \epsilon := 10^{-13} \quad , \quad \text{and} \tag{4.3}$$

$$\left\|Fx^{k,i}\right\|_2 < \epsilon \tag{4.4}$$

These criteria have ensured machine convergence, i.e. the iterates remain constant thereafter. Table 4.1 describes the necessary work to attain convergence for the iterations (2.7).

Only relevant steplengths have been taken into account in the table. The leftmost column indicates the steplength p; the following two columns indicate, respectively, the values $kp+i$ for which (4.3) and (4.4) are satisfied for the N-F iterations applied to (4.1), whereas the last two columns describe the corresponding values when the iterations are applied to (4.2). The monotone behaviour in each column illustrates Theorem 2.5. For the actual iterates with $p = 1$ see Tables 4.1 and 4.2 in [5].

TABLE 4.1

p	$\|\|Fy^{k,i}\|\|_2 < \epsilon$	$\|\|Fx^{k,i}\|\|_2 < \epsilon$	$\|\|\overline{F}\,\overline{y}^{k,i}\|\|_2 < \epsilon$	$\|\|\overline{F}\,\overline{x}^{k,i}\|\|_2 < \epsilon$
	pk+i			
1	6	8	5	6
2	8	9	6	6
3	10	10	7	8
4	10	11	8	8
5	12	12	8	8
6	13	13	9	9
7	15	15	10	10
8	16	17	10	10
9	16	17	11	11
10	17	18	12	12
11	17	18	12	13
12	17	18	13	13
13	18	19	14	14
14	19	19	15	15
15	20	20	16	16
20	23	24	21	21
23	26	26	24	24
24	27	27	24	24
29	31	32		
30	32	32		
44	45	46		
45	46	46		
82	83	83		
83	83	83		

According to this table, the reduced system behaves numerically better than the original one for every steplength. This example also shows that, if simplified iterations are to be computed throughout with the initial Jacobian matrix, i.e. with no updating at all, then the amount of computational work for the reduced system may be much smaller than the corresponding one for the original system.

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