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TOTAL MULTIGRAPHS. UNICITY OF THEIR PRIMITIVE SUBMULTIGRAPH

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ABSTRACT

In case of graphs without loops or parallel edges Behzad-Radjavi [1] show that if H is the total graph of a graph which is not a cycle or a complete graph then H contains one and only one subgraph G, such that H=T(G). In this work we show it is possible to extend this result to multigraphs and a constructive method for determining the primitive submultigraph is given.

1. INTRODUCTION

We consider finite, connected multigraphs with loops permitted. For simplicity we shall call these simply graph. A graph is trivial if it consists of a single vertex.

Given a graph G, we denote by V(G) the set of its vertices and by E(G) the set of its edges.

With every graph G is associated a graph G^{o}, called the *LINE GRAPH* of G, whose vertices are in a one to one correspondence with the edges of G in such a way that two vertices u',v'(u'$\neq$ v') of G^{o} are joined by as many edges as the corresponding edges u,v of G have common vertices, and also, if u is a loop of G, then at the corresponding vertex u' there is only one loop.

In a similar way we can associate with G a graph H = T(G) called the *TOTAL GRAPH* of G, such that V(H) = V(G) $\cup$ V(G^{o})

and $E(H) = E(G) \cup E(G^{o}) \cup E(C(H))$, where $C(H)$, called the *MIDDLE GRAPH* , has vertices $V(H)=V(G) \cup V(G^{o})$ and edges $[x,u']$ where $x \in V(G)$ is an end vertex of $u \in E(G)$.

H is a total graph if there is a graph G such that $H \cong T(G)$. The preceding definitions have been considered in [2].

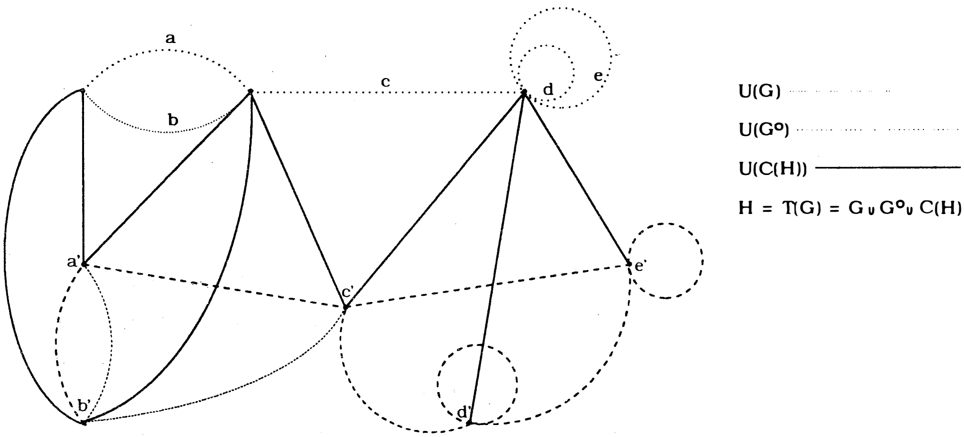

figura 1

REMARKS

If $H = T(G)$ then

1.1. G is connected if and only if H is connected.

1.2. The number of loops of H is even.

1.3. If $u \in E(G)$, in the middle graph of H there are one or two edges with vertex u'.

1.4. If loops or parallel edges are not admitted, G is k-regular of order n if and only if H is 2k-regular of order $n(1+k/2)$.

1.5. If H is the total graph of a cycle (in particular a loop or two parallel edges) or of a complete graph, H cannot be the total graph of another type of graph. (Follows from 1.4 and the definition of total graph applied to a loop or a pair of two parallel edges).

1.6. If H is the total graph of a path of order $n > 2$, H cannot be a total graph of another type of graph .

1.7. If $x \in V(H)$ is the end vertex of at least two loops, then $x \in V(G)$.
1.8. If there are three or more parallel edges $[x,y]$, then $x,y \in V(G)$.
1.9. If $x \in V(H)$ is the end vertex of parallel edges and at least one loop, $x \in V(G)$.

If $x \in V(G)$ we shall denote by d_x and D_x, respectively, the degree of x in G and in $T(G)$. For vertices $u' \in G^o$ we shall denote by $D_{u'}$ their degree in $H = T(G)$.

1.10. If G is not trivial, $D_x - d_x \geq 1$.
1.11. If $x \in V(G)$ and there are h_x loops, $h_x \geq 0$, with end vertex x, then $D_x = 2d_x - h_x$.
1.12. If $u = [x,x] \in E(G)$ and there are h_x loops, $h_x \geq 1$, with vertex x, $D_{u'} = d_x - h_x + 2 = D_x - d_x + 2$. If $d_x = 2$ then $D_{u'} = D_x$, otherwise, $D_{u'} < D_x$.

2. RESULTS

THEOREM 1

Let H be a total graph. If A is a non trivial subgraph of H and A^o is its corresponding line graph in H then there exists a unique graph G such that $A \subseteq G$ and $H = T(G)$.

proof

If $A \subset H$ is not trivial and A^o is its corresponding line graph in H, we can construct $T(A) \subseteq H$. It is obvious that if $T(A) = H$ then $A = G$.

If $T(A) \neq H$, let $\hat{A} = \{ u'_1, u'_2, \ldots, u'_k \}$ be the set of all vertices of $H = T(A)$ adjacent to some vertex of A^o. Now we consider the edges of H with at least one vertex in A, the edges with the other vertex in $V(A^o) \cup \hat{A}$ are in $C(H)$. The remaining ones determine a well defined subgraph $A_1 \subset H$ such that $A \subset A_1$ and its corresponding line graph A_1^o is the graph induced by $V(A^o) \cup \hat{A}$. We can repeat the same argument with A_1 and its corresponding line graph A_1^o and in this way determine a sequence

$$A \subset A_1 \subset A_2 \subset \ldots\ldots \subset A_k$$

such that

$$H = T(A_k) \quad \text{and} \quad A_k = G. \blacksquare$$

Let $\mathbb{A}$ be the family of vertex subsets of a total graph that define cocycles without parallel edges. We say that a subgraph is an *α-subgraph* if it is a graph induced by the minimal elements of $\mathbb{A}$ that have loops and parallel edges.

REMARKS

Let S be an α-subgraph in $H = T(G)$

2.1. S is connected.

2.2. If S has no parallel edges, $|V(S)| = 1$ and its edges are loops.

2.3. If $|V(S)| = 2$, S has at least two parallel edges.

2.4. If $|V(S)| \geq 3$ and H has not three or more parallel edges $[x,y]$ then $S \subseteq G$.

LEMMA 1

If $H = T(G)$ and $A \subseteq G$ is an α-subgraph, then it is possible to determine its corresponding $A^o \subseteq G^o$.

proof

If H has a single subgraph A^o whose vertices are adjacent to the vertices of A just as the total graph definition implies, the lemma is trivial. Otherwise, the corresponding $A^o \subset G^o$ is the one whose vertices are not adjacent to the end vertex of a loop or parallel edges, except those of A. $\blacksquare$

LEMMA 2 (Behzad-Radjavi)

Let G be a connected graph without loops or parallel edges, which is not a path, a cycle or a complete graph and let a_o be a vertex of $H = T(G)$ of maximal degree $2d$.

Then $a_o \in V(G)$ if and only if the graph induced by the neighborhood of a_o has exactly one K_d as subgraph.

From the lemma it follows that the edges $[a_o, x_i]$,

$i \in \{1,2,\ldots,d\}$ such that $x_i \notin K_d$ determine a subgraph $A \subseteq G$ whose corresponding line graph A^o is K_d.

The part a) of the following theorem collects the results obtained in [1].

THEOREM 2

Let H be a total graph, connected, with loops or parallel edges permitted.

a) If H is the total graph of a cycle or a complete graph, H contains more than one subgraph G_i such that $H = T(G_i)$.

b) Otherwise, H has exactly one subgraph G whose total graph is H.

proof

a) If G is a cycle, $G \neq K_3$, (in particular a loop or two parallel edges) H contains two subgraphs G_i such that $H = T(G_i)$.

For $G = K_3$, $T(G)$ is the total graph of each of its eight triangles.

For a complete graph G of order p, $p > 3$, there are exactly p other subgraphs G_i of $T(G)$ with $T(G_i) = T(G)$, $i = 1,\ldots,p$.

b) We consider different cases and for each of them we shall show that there exists only one subgraph G such that $H = T(G)$.

Certainly, if we assume that we know a certain subgraph $A \subset H$ to which we can apply Lemmas 1 or 2, following the constructive method used in the proof of theorem 1 we can determine the desired subgraph G.

That we can choose an adequate subgraph A in all cases is a consequence of the following considerations.

If H has no loops or parallel edges and is not the total graph of a cycle, a complete graph, or a path, to determine A we apply Lemma 2.

If H is the total graph of a path, we take A to be the smallest path which joins the unique vertices of degree 2.(see 1.6).

Otherwise H contains loops or parallel edges. If H contains α-subgraphs such that their vertices satisfy 1.7, 1.8, 1.9 we choose A among them. If in H there are no such α-subgraphs but there is an α-subgraph S such that $|V(S)| \geq 3$ then S can be taken as A.

If all α-subgraphs S consist of only a loop or two parallel edges, we take A to be one of the loops whose vertex has maximum degree in H (see 1.12) or one of the pairs of parallel edges with the minimum number of vertices adjacent to both end vertices.∎

The following theorem is of interest in itself and generalizes the theorem of Behzad and Radjavi [1]. The proof is similar .

THEOREM 3

Let G_1 and G_2 be graphs. Then $T(G_1) \simeq T(G_2)$ if only if $G_1 \simeq G_2$.

proof

It suffices to prove the result for connected graphs. If G_1 and G_2 are isomorphic so are their total graphs. The converse follows from 1.5, 1.6 and theorem 2.∎

REFERENCES

[1] BEHZAD M. and RADJAVI H. "The total group of graph", Proc.Math.Soc.Vol. 19 (1968) n.1, 158-163.

[2] CHIAPPA R., MACCARI A. and ZILIANI A. "A Relation between Total Multigraphs and Total Multidigraphs", Rend. Mat. serie VII,vol. 12 (1992), 1-11.

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