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ON THE INVERSION OF BESSEL POTENTIALS

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ABSTRACT. The inversion of Bessel potentials by using hypersingular integrals with weighted differences is extended to certain kind of distributions called "causal" and "anticausal" distributions. Here we generalize the article due to B.S. Rubin (cf. [**1**]). In this note we define the operator $(T_\ell^\alpha f)$ called the "causal" hypersingular integral with weighted differences (cf. formula (I,2,1)) and we prove that, in virtue of the formula (I,2,7) $T_\ell^\alpha f \equiv 0$ when $\alpha = 1,3,5,\dots$.
We also evaluate the Fourier transform of $D^\alpha f = \frac{1}{d_{n,\ell}(\alpha)}(T_\ell^\alpha f)$, where $d_{n,\ell}(\alpha)$ is a constant given by the formula (I,2,6) and we conclude that $\mathcal{F}[D_f^\alpha](\xi) = \mathcal{F}[f](\xi)\cdot(1+Q-i0)^{\alpha/2}$, where $f \in S$ and $(1+Q-i0)^{\alpha/2}$ is the Fourier transform of the distributional function $G_{-\alpha}(P+i0,n)\cdot G_\alpha$ (cf. formula (I,1,4)) is a causal (anticausal) analogue of the kernel due to A. Calderón, Aronszajn– Smith and share many properties with the Bessel kernel; for example, $G_{2k}(P+i0,n)$ is an elementary (causal, anticausal) solution of the ultrahyperbolic Klein–Gordon operator iterated k times.
Formula (II,1,4) says that $D^\alpha f = G_{-\alpha}(P+i0,n) * f$.
Finally if we define the Bessel potential as $B^\alpha f = G_\alpha * f$, we conclude that $D^\alpha B^\alpha f = f$, and it finishes the proof of the inversion of Bessel potentials by using "causal" hypersingular integrals.

I.1.

Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbf{R}^n$ the n–dimensional Euclidean space.

$$\langle x, \xi\rangle = \sum_{i=1}^{n} x_i \xi_i \,.$$

$\mathcal{F}[f](\xi) = \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} f(x)dx$ designates the Fourier transform of a function f.

$$(\Delta_t^\ell f)(x) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k f(x-kt)\,. \qquad \text{(I,1,1)}$$

The weighted difference of order ℓ of a function f at the point x, with interval t and weight $\rho(k,t)$, is defined as follows:

$$(\Delta_t^\ell f)(x,\rho) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \rho(k,t) f(x-kt)\,, \tag{I,1,2}$$

$$\mathcal{A}_\ell(\alpha) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k k^\alpha\,. \tag{I,1,3}$$

Let $G_\alpha(P+i0,m,n)$ be the causal distribution defined by

$$G_\alpha(P+i0,m,n) = \frac{2^{1-\frac{\alpha+n}{2}}}{\pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)} (P+i0)^{\frac{1}{2}\left(\frac{\alpha-n}{2}\right)} K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2(P+i0)}\right)\,, \tag{I,1,4}$$

where m is a positive real number, $\alpha \in \mathbf{C}$, K_ν designates the modified Bessel function of the third kind

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}$$

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}$$

(cf. [2], formulae (II,1,1), p. 34). The distribution $(P+i0)^\lambda$ is defined by

$$(P+i0)^\lambda = \lim_{\varepsilon\to 0}\{P + i\varepsilon|t|^2\}^\lambda \tag{I,1,5}$$

where $\varepsilon > 0$, $|t|^2 = t_1^2 + \cdots + t_n^2$, $\lambda \in \mathbf{C}$

$$P = P(t) = t_1^2 + \cdots + t_p^2 - t_{p+1}^2 - \cdots - t_{p+q}^2\,,$$

where $p+q=n$.

The following formula is valid (cf. [2], page 35, formula (II,1;8))

$$\mathcal{F}\left[G_\alpha(P+i0,m,n)\right] = \frac{e^{i\frac{\pi}{2}\alpha}}{(2\pi)^{n/2}}\left[(m^2+Q-i0)\right]^{-\alpha/2}\,,$$

where

$$Q = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2\,,$$

where $p+q=n$.

The auxiliary weight function $\lambda_\alpha(P+i0,n)$ is defined by

$$\lambda_\alpha(P+i0,n) = \frac{2^{1-\frac{n+\alpha}{2}}}{\Gamma\left(\frac{n+\alpha}{2}\right)}(P+i0)^{\frac{n+\alpha}{4}} K_{\frac{n+\alpha}{2}}\left(\sqrt{(P+i0)}\right) \tag{I,1,6}$$

and

$$\lambda_\alpha\left(|t|^2\right) = \frac{1}{\Gamma\left(\frac{n+\alpha}{2}\right)} \int_0^\infty \eta^{\frac{n+\alpha}{2}-1} e^{-\eta-\frac{|t|2}{4\eta}}\, d\eta\ . \tag{I,1,7}$$

The Bessel potential of order α, $\alpha \in \mathbf{C}$, of a function $f \in S$, denoted by $B^\alpha f$, is defined by

$$B^\alpha f = G_\alpha * f\ , \tag{I,1,8}$$

where $*$ designates as usual the convolution.

I.2.

Let f be a function of S; S the Schwartz class of infinitely differentiable functions on $\mathbf{R}^n$ decreasing at infinity faster than $|x|^{-1}$, and let the following hypersingular operator on weighted differences defined by

$$\begin{aligned}(T_\ell^\alpha f)(x) &= \int_{\mathbf{R}^n} \frac{(\Delta_t^\ell f)(x,\lambda_\alpha)}{(P+i0)^{\frac{n+\alpha}{2}}}\, dt \\ &= \int_{\mathbf{R}^n} \frac{\sum\limits_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \lambda_\alpha(k(P+i0)^{1/2}) f(x-kt)}{(P+i0)^{\frac{n+\alpha}{2}}} dt\end{aligned} \tag{I,2,1}$$

where $\alpha \in \mathbf{C}$.

The formula (I,2,1) defines a temperate distribution, then its Fourier transform $\mathcal{F}[T_\ell^\alpha f](\xi)$ exists.

A theorem of S.E. Trione, which permits to evaluate the Fourier transform of distributions of the form $T(P+i0,\lambda)$, $\lambda \in \mathbf{C}$, says: Let $T(P+i0,\lambda)$ be a temperate distribution, then the Fourier transform (being understood in the sense of Schwartz) is

$$\mathcal{F}[T(P+i0,\lambda)] = e^{i\frac{\pi}{2}q}\mathcal{F}[T\{|t|^2,\lambda\}]_{|y|^2\to Q-i0}$$

(cf. **[2]**, pp. 25, formula (I,4,11)). Then, we can evaluate $\mathcal{F}[T^\alpha f](\xi)$:

$$\mathcal{F}[T_\ell^\alpha f](\xi) = \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}(T_\ell^\alpha f)(x,\lambda_\alpha)dx$$

$$= \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}\left\{ \int_{\mathbf{R}^n} \frac{\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda_\alpha(k|t|)f(x-kt)dt}{|t|^{n+\alpha}}\right\}dx$$

$$= \sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k \int_{\mathbf{R}^n} \frac{e^{ik\langle t,\xi\rangle}\mathcal{F}[f](\xi)\lambda_\alpha(k|t|)}{|t|^{n+\alpha}}\,dt\,; \qquad \text{(I,2,2)}$$

and by virtue of the integral expression of $\lambda_\alpha(k|t|)$ we obtain

$$\mathcal{F}[T_\ell^\alpha f](\xi) = \frac{\mathcal{F}[f](\xi)}{\Gamma(\frac{n+\alpha}{2})}\int_{\mathbf{R}^n}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\left\{\int_0^\infty \eta^{\frac{n+\alpha}{2}-1}e^{-\eta-\frac{|kt|^2}{4\eta}}d\eta\right\}\cdot$$

$$\cdot\frac{e^{ik\langle t,\xi\rangle}}{|t|^{n+\alpha}}\,dt\,. \qquad \text{(I,2,3)}$$

The change of the order of integration is licit because the "double" integral converges absolutely (cf. **[1]**). Applying the Bochner's theorem (cf. **[3]**) we obtain

$$\mathcal{F}[T_\ell^\alpha f](\xi) = \frac{\mathcal{F}[f](\xi)\cdot(2\pi)^{n/2}}{\Gamma(\frac{n+\alpha}{2})}\int_0^\infty \eta^{\frac{n+\alpha}{2}-1}e^{-\eta}d\eta\,\cdot$$

$$\cdot\left\{\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\int_0^\infty e^{-\frac{|kt|^2}{4\eta}}(kr|\xi|)^{1-\frac{n}{2}}J_{\frac{n-2}{2}}(kr|\xi|)\frac{dr}{r^{\alpha+1}}\right\}\,. \qquad \text{(I,2,4)}$$

Again, changing the order of integration and applying 6.631 (4) from **[4]**, and by virtue of Trione's theorem we obtain

$$\mathcal{F}[T_\ell^\alpha f](\xi) = d_{n,\ell}(\alpha)(1+Q-i0)^{\alpha/2}\mathcal{F}[f](\xi) \qquad \text{(I,2,5)}$$

where

$$d_{n,\ell}(\alpha)\begin{cases} \dfrac{\pi^{\frac{n}{2}+1}\mathcal{A}_\ell(\alpha)}{2^\alpha\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{\alpha}{2}+1)\sin\frac{\pi\alpha}{2}} & \text{for } \alpha\neq 2,4,6,\dots \\[2ex] \dfrac{(-1)^{\alpha/2}\pi^{n/2}2^{1-\alpha}}{\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{n+\alpha}{2})}\dfrac{d}{d\alpha}\mathcal{A}_\ell(\alpha) & \text{for } \alpha = 2,4,6,\dots \end{cases} \qquad \text{(I,2,6)}$$

It is easily verified that if $\alpha = 1,3,5,\dots$, $\alpha<\ell$, $d_{n,\ell}=0$. (I,2,7)

II.

For $\alpha \neq 1,3,5,\ldots$, we define the generalized Bessel derivative

$$\mathcal{D}^{\alpha} f = \frac{1}{d_{n,\ell}(\alpha)} \int_{\mathbf{R}^n} \frac{(\Delta_t^{\ell} f)(x, \lambda_{\alpha}(P+i0))}{(P+i0)^{\frac{n+\alpha}{2}}} dt \, , \qquad \text{(II,1,1)}$$

By virtue of (I,2,5) is

$$\mathcal{F}[\mathcal{D}^{\alpha} f](\xi) = \mathcal{F}[f](\xi)(1+Q-i0)^{\alpha/2} \qquad \text{(II,1,2)}$$

which can be expressed as follows:

$$\mathcal{F}[\mathcal{D}^{\alpha} f](\xi) = \mathcal{F}\big[G_{-\alpha}(P+i0,n) * f\big](\xi) \, , \qquad \text{(II,1,3)}$$

where $*$ designates, as usual, the convolution. Then,

$$\mathcal{D}^{\alpha} f = G_{-\alpha}(P+i0,n) * f \, . \qquad \text{(II,1,4)}$$

We have defined the Bessel potential of order α

$$B^{\alpha} f = G_{\alpha} * f \, .$$

Then, applying the generalized Bessel derivative to a function $B^{\alpha} f$

$$\mathcal{D}^{\alpha} B^{\alpha} f = G_{-\alpha} * B^{\alpha} f = G_{-\alpha} * \{G_{\alpha} * f\} = \\ = \{G_{-\alpha} * G_{\alpha}\} * f = \delta * f = f \qquad \text{(II.1.5)}$$

Note that $G_{\alpha} \in S$, then the product of convolution (II,1,5) is associative.

The last conclusion is equivalent to the formula (III,9) of [5] which has been obtained by a completely different manner.

Formula (II,1,4) finishes the inversion of Bessel potentials by using causal hypersingular integrals.

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