Revista de la
Unión Matemática Argentina
Volumen 39, 1994.

ON SOME NOTABLE PLANE SETS, II: DRAGONS

A. Benedek and R. Panzone

ABSTRACT. *We prove in an essentially geometrical way that Heighway's dragons (chinese dragons) are sets of Lebesgue measure 1/2 that tile the plane. We also exhibit the convex hull of the usual dragon and show that the union of two Heighway's dragons is equal to Knuth's twindragon.*

PART I.

1. INTRODUCTION. a) We deal in this paper with the so called chinese dragon, or simply dragon, associated to the (dragon) curve introduced by John E. Heighway, (cf.[D]). It is a self-similar plane set of similarity dimension 2 that can be obtained from the unit interval $u := [0,1]$ and the similarity maps Q and R:

(1) $\quad Q(z) = (1+i)z/2 \qquad\qquad R(z) = 1 + (-1+i)z/2.$

These functions are related as follows:

(2) $\quad R(z) = i[Q(z) - (1+i)/2] + (1+i)/2 .$

In Fig.5, the approximating curve for j=6 :

$$d_{j+1} := \cup\{\ldots\cdot Q^{i}\cdot R^{k}\cdot\ldots(u):\ \ldots+i+k+\ldots=j\}$$

is depicted. This polygonal has 2^{j} sides. Heighway's dragon is a compact set invariant for the iterated function system (Q,R), (cf.[E], Ch.4), that looks like the diagram in Fig. 1.

To prove this assertion (by induction) observe that if one travels along the curve d_{j+1} from 0 to 1 and denotes with D (U) the corners where the curve turns right (left) 90° then the sequence of D's and U's coincides with that given in [D],§1.

b) **Nomenclature:** dim(X) = Hausdorff dimension of X; m(X) = plane Lebesgue measure of X; co(X) = closed convex hull of X;

$\#X$ = cardinality of X; if σ is a similarity of ratio r and $X = \sigma(X')$ then we say that X is **r-similar** to X';
$X_a := \{y;\ |x-y| < a$ for some $x \in X\}$.
T(H) will denote a **tiling** of the plane in the following precise sense: H is a measurable bounded set and $R^2 = \cup\{H(i);\ i=1,2,\dots\}$ where each H(i) is 1-similar to H, verifying

$$i \neq j \Rightarrow m(H(i) \cap H(j)) = 0.$$

Let b=-1+i. F will denote the set of complex numbers whose representation in the binary complex number system (b,{0,1}) are of the form $(0.a_{-1}a_{-2}\dots)_b$. In other words, F is the so called **Knuth's twindragon**, (cf.[K], p.172 and [S], p.330).

c) If instead of the interval u one starts with an isosceles right-angled triangle M in the first quadrant with hypotenuse u, the same limit set is obtained. D(M) will denote this set. In Fig.2 some steps of the process that leads to D(M) can be seen. Q transforms the **mother** M in the right-angled triangle B and A = R(M). A and B are the **daughters** of M; in the second step the triangles C, D, F and E are obtained, see Fig.2,1-3. Here, $F \cup E = = Q(B \cup A)$. **Descendants** of M will be called all the triangles obtained following this procedure; the **order** of descent of a triangle of the process is equal to the number of steps plus 1. That is, the order of descent is counted beginning with 1 for the mother. So, J is a daughter of I but fourth descendant of A. In Fig.2,3-5, appear the offsprings of M of order 4 and 5. In the diagrams of Fig.3 are distiguished the descendants of A from those of B that are drawn with dotted lines. Let P be a triangle r-similar to M. The mappings to obtain the dragon D(P) generated by P, that play the same role as Q and R, will be denoted by $\lambda[P]$ and $\rho[P]$, respectively.

For arbitrary X let $\mathcal{S}(X) = Q(X) \cup R(X)$, $\mathcal{S}^{k+1}(X) = \mathcal{S}\cdot\mathcal{S}^k(X)$. Then, $\mathcal{S}^{k+1}(M)$ tends to D(M) and $\mathcal{S}^{k+1}(M) = \mathcal{S}^k(A \cup B)$. This means that if we replace each triangle P of order k by $\lambda[P](P)$ and $\rho[P](P)$ (its daughters) then we obtain all descendants of order k+1. Thus, we have an *alternative view of the construction of the approximating families of right-angled triangles of equal size with limit* D(M). Then,

(3) $\quad D(M) = D(A) \cup D(B)$; $D(A) = R(D(M))$; $D(B) = Q(D(M))$.

If we pay attention only to the hypotenuse and legs of **M** then the same proof shows that the legs (hypotenuses) of the triangles of order k+1 are the sides of d_{k+1} (d_k).

d) Let G_j be the family of triangles of order of descent equal to j. Then, if Δ is a descendant of order m of $\Delta^\wedge \in G_j$,

(4) $\quad D(\Delta) \subset D(\Delta^\wedge) \quad , \quad D(\Delta)$ is $(1/\sqrt{2})^{m-1}$-similar to $D(\Delta^\wedge)$.

Besides, the hypotenuses of the offsprings of order j are sides of squares in a (**regular**) net of squares with 0 as one vertex, of side $(1/\sqrt{2})^{j-1}$, formed by lines parallel to the coordinate axes if j is odd and at 45° if j is even.

We call **standard a triangulation** of a square if it is the one obtained by means of the diagonals of the square. Thus, of two consecutive regular nets that of smaller width triangulates in a standard way each square of the other.

e) We give two proofs of next Theorem 1. The second is geometrical, resembling that of P. Lévy for his famous curve, ([Y]).

THEOREM 1. Let D=D(M). Then, D=cl(int(D)), m(D)=1/2 and there is a tiling T(D) of the plane.

We also prove that the following result holds, (see [E],p.30).

THEOREM 2. Let M^ be the image of M by the linear map

(5) $\quad t(z) := 1 - z.$

Then, $m(D(M) \cap D(M^\wedge))=0$. Besides, there is an isometric linear map Ω such that $\Omega(F) = D(M) \cup D(M^\wedge)$.

In Fig.4 a diagram of the equality $\Omega(F) = D(M) \cup D(M^\wedge)$ is shown. Since D(M^) is a chinese dragon, F is the union of two dragons. This accounts for the name "twindragon". But F is also the union of two $1/\sqrt{2}$-copies of itself that intersect in a set of measure zero, (see Fig.11). In fact, F is the invariant set for the following similarity maps, (cf.[K], [E] or [B]):

(6) $\quad \Phi_0(z) = z/b \qquad \Phi_1(z) = \Phi_0(z) + 1/b.$

2. **BASIC RESULTS**. Recall that D:=D(M).

LEMMA 1. If descendants of the same order have points in common then these are vertices.

PROOF. The lemma can be proved by induction since if it is true for G_j then it follows from the alternative construction that it also holds for G_{j+1}, QED.

Lemma 1 is also a corollary of [D], Th.4 - that says in particular that a dragon curve traverses each of its line segments exactly once - if one takes into account that the boundary of G_{j+1} is equal to $d_j \cup d_{j+1}$.

LEMMA 2. If U and V are descendants of M but no one is an offspring of the other then $m(D(U) \cap D(V)) = 0$.

PROOF. $m(D(A)\cap D(B)) = 0$ since $m(D) = m(D(A)\cup D(B))$ and $m(D(A)) = $ $= m(D(B)) = m(D/\surd 2) = m(D)/2$. Now, Lemma 2 is a consequence of (4) whenever U is an offspring of A and V of B. From this observation the general case follows, QED.

It applies in particular to descendants of M of the same order.

LEMMA 3. $m(D) \geq 1/2$.

PROOF. Let G_j' be the family of right-angled triangles that have in common with those of G_j the hypotenuse. Because of lemma 1, $m(\cup\{\Delta \in G_j,\ \Delta' \in G_j'\}) = 1/2$. Since $\cup\{\Delta \in G_j\}$ converges in the metric of Hausdorff to D, the lemma follows, QED.

LEMMA 4. Let q be the unit square and $\{M, M', M'', M'''\}$ its standard triangulation where $M'=1+i-M$, $M''=i(1-M)$. Then,

i) $m(D(M) \cap D(M')) = 0$

ii) $D(M) \cup D(M') = D(M'') \cup D(M''')$

iii) $m(D(M) \cap D(M^\wedge)) = 0$.

PROOF. i) follows from Lemma 2 since $m(D(M) \cap D(M'))$ is equal, except for a constant factor, to $m(D(U) \cap D(V))$, U and V as in Fig.3,3.

ii) Observe that $D(R(M)) = R(D(M)) = \lambda[M'''](D(M'''))$. Thus, $D(A) \subset D(M''')$. Repeating this argument one obtains the desired equality.

iii) $\Theta(x+iy) := x-iy$. Thus, $\Theta(M)=M^\wedge$. Because of ii), $D(M^\wedge) \subset D(\Theta(M'')) \cup D(\Theta(M'''))$. Then, to prove iii) it suffices to show that $m(D(M) \cap [D(\Theta(M''))\cup D(\Theta(M'''))]) = 0$. This follows again from Lemma 2 and Fig.3,3 since W, U and V occupy the same relative positions as M, $\Theta(M'')$ and $\Theta(M''')$, respectively, QED.

DEFINITION. $\tau(q) := D(M)\cup D(M')$; $\delta(u) := D(M)\cup D(M^\wedge)$.

In words, $\tau(q)$ is the **mill wheel** associated to q and $\delta(u)$ is the **k-dragon** associated to $[0,1]$. They are depicted in Fig.4.

3. PROOF OF THEOREM 2. Let Ω be the linear map such that

$$0 = \Omega((-1+2i)/5),\ 1 = \Omega((-1-3i)/5),\ (2+i)/5 = \Omega(0),$$

(note that the points to which Ω is applied belong to F). Let (σ_0,σ_1) be the iterated function system of similarities in R^2 defined by $\sigma_0 := \Omega\cdot\Phi_0\cdot\Omega^{-1}$, $\sigma_1 := \Omega\cdot\Phi_1\cdot\Omega^{-1}$. Then,

$$\sigma_0(z) = -\frac{1+i}{2}z + \frac{1+i}{2} \qquad \sigma_1(z) = -\frac{1+i}{2}z + 1$$

and $\Omega(F)$ is the invariant set for this iterated function system. These maps and those defined in (1) can be written as follows,

$$(7)\quad \begin{cases} \sigma_0 = (z-1)/b & \sigma_1 = z/b + 1 \\ Q(z) = -z/b & R(z) = -iz/b + 1 \end{cases}$$

where $b = -1+i$. The following relations hold,

$\sigma_0(1-z) = Q(z)$ $\qquad$ $R\cdot R(z) = -iz/2 + (1+i)/2$

$Q\cdot Q(z) = iz/2$ $\qquad$ $\sigma_1(z) = 1 - \sigma_0(1-z)$

$Q\cdot R(z) = -z/2 + (1+i)/2$ $\qquad$ $R\cdot Q(z) = -z/2 + 1$

$\sigma_0\cdot Q = R\cdot R$ $\qquad$ $\sigma_0\cdot R = 1 - R\cdot Q$

$\sigma_0\cdot(1-R) = Q\cdot R$ $\qquad$ $\sigma_0\cdot(1-Q) = Q\cdot Q.$

Assume A_0 is a nonvoid bounded set. To fix ideas suppose that $A_0 = M$ as in Fig.8. We define A_n recursively:

$$(8)\qquad A_n = Q(A_{n-1}) \cup R(A_{n-1}).$$

We have, for $n \geq 1$, that

$\sigma_0(A_n \cup (1-A_n)) =$

$= \sigma_0\cdot Q(A_{n-1}) \cup \sigma_0\cdot R(A_{n-1}) \cup \sigma_0\cdot(1-Q(A_{n-1})) \cup \sigma_0\cdot(1-R(A_{n-1})) =$

$= R\cdot R(A_{n-1}) \cup (1 - R\cdot Q(A_{n-1})) \cup Q\cdot Q(A_{n-1}) \cup Q\cdot R(A_{n-1})$,

$\sigma_1(A_n \cup (1-A_n)) =$

$= (1 - \sigma_0(1-A_n)) \cup (1 - \sigma_0(A_n)) = (1 - Q\cdot Q(A_{n-1})) \cup$

$\cup (1 - Q\cdot R(A_{n-1})) \cup (1 - R\cdot R(A_{n-1})) \cup R\cdot Q(A_{n-1}).$

In consequence,

$\sigma_0(A_n \cup (1-A_n)) \cup \sigma_1(A_n \cup (1-A_n)) = A_{n+1} \cup (1 - A_{n+1}).$

Since $A_n \cup (1-A_n) \to D(M) \cup D(M^\wedge)$ as $n \to \infty$ in Hausdorff's metric, it follows that $\delta(u) = \Omega(F)$, QED.

FIRST PROOF OF THEOREM 1. There exists a tiling T(F) of the plane according to our definition, (cf.[K],[B]). m(F)=1, Th. 2 and Lemma 4,iii) imply that there is a tiling T(D) with m(D)= =1/2. For the remaining part of the theorem, see b) Th. 5, QED.

Fig. 1

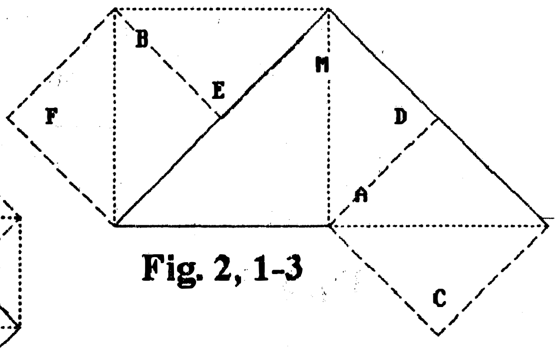

Fig. 2, 1-3

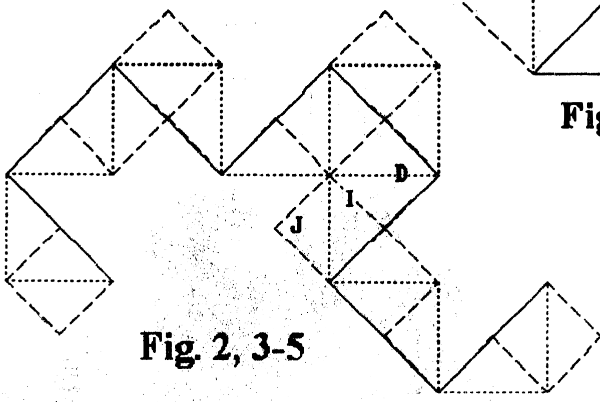

Fig. 2, 3-5

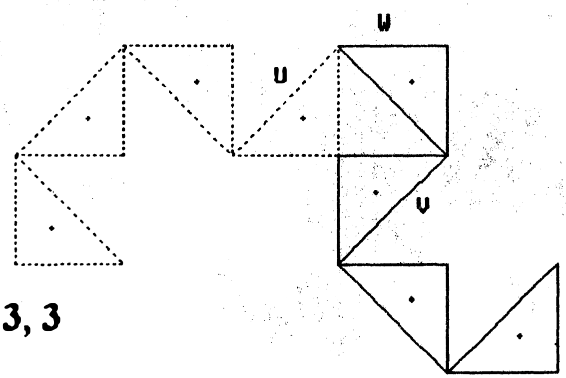

Fig. 3, 3

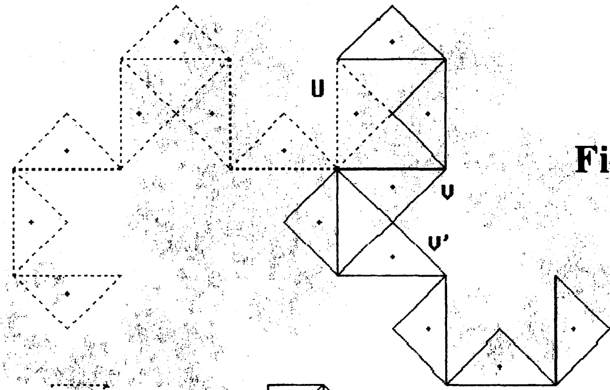

Fig. 3, 4

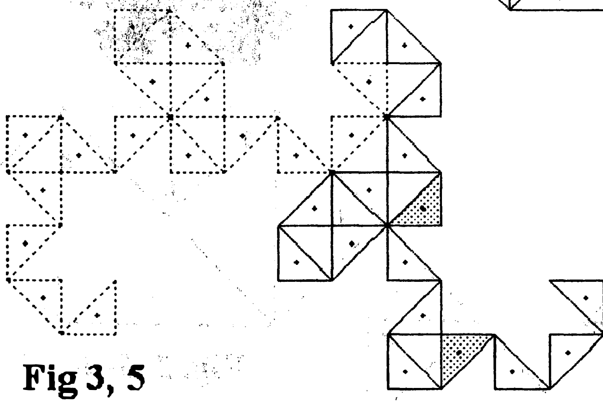

Fig 3, 5

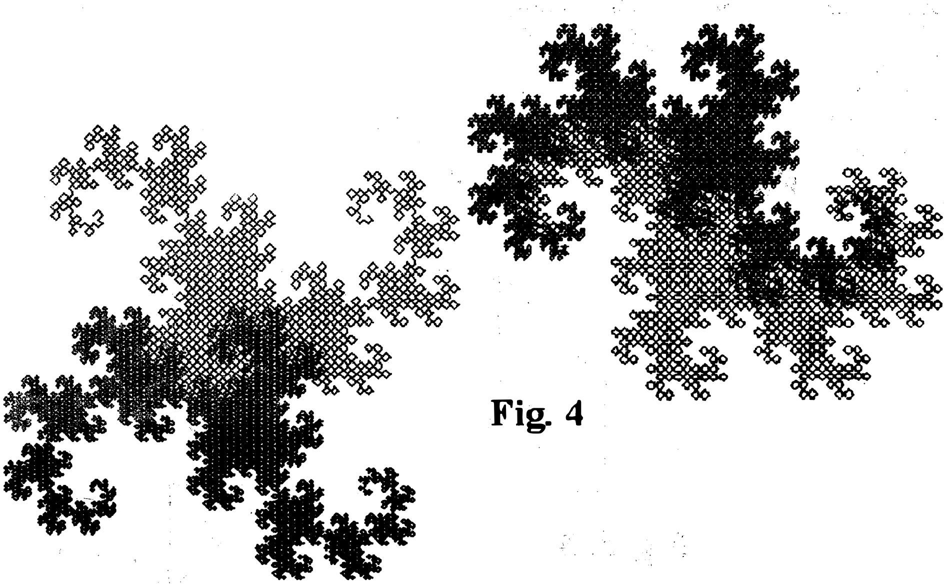

Fig. 4

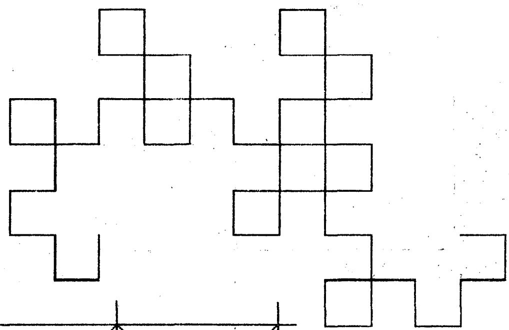

Fig. 5

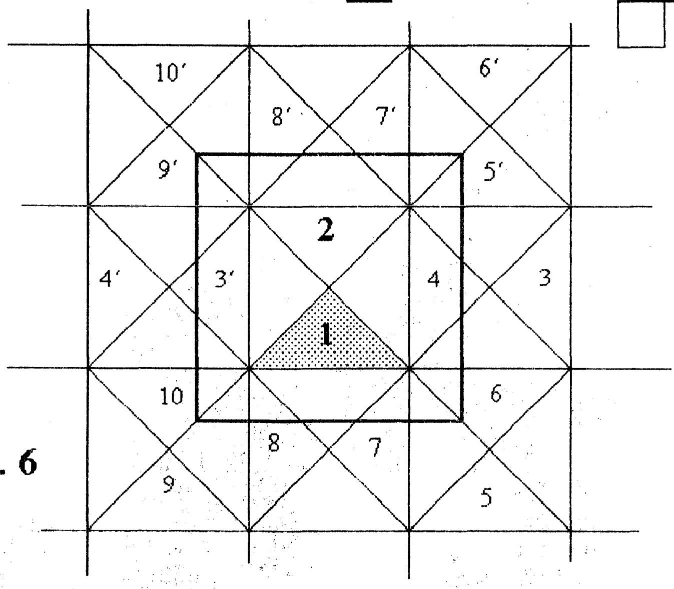

Fig. 6

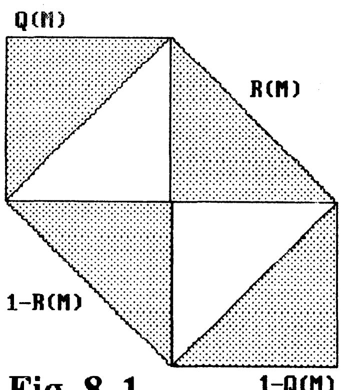

Fig. 8, 1

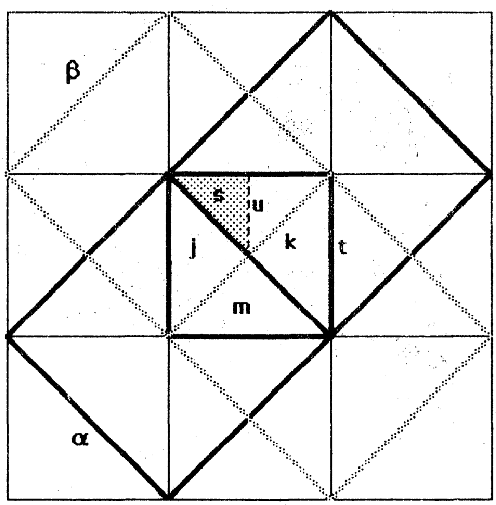

Fig. 7

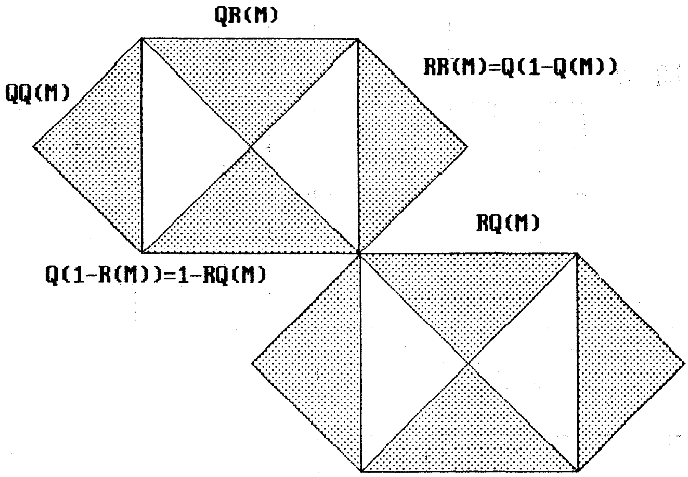

Fig. 8, 2

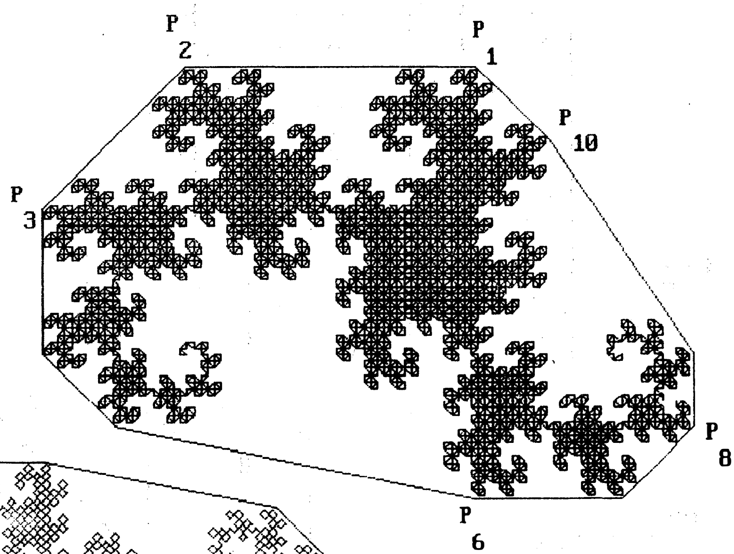

Fig. 9

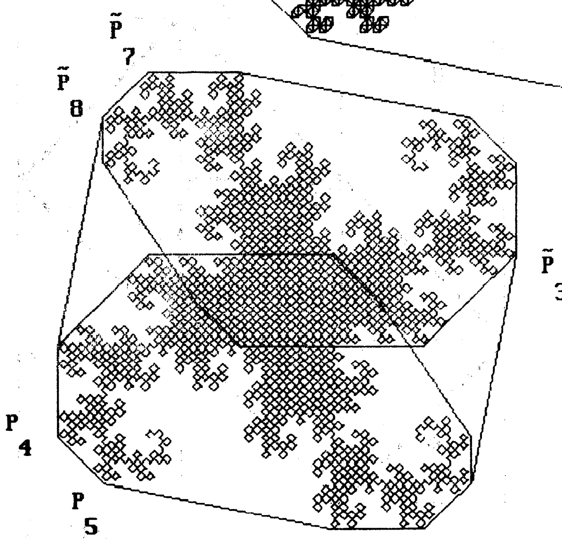

Fig. 10

PART II.

4. THE CONVEX HULL OF HEIGHWAY's DRAGON. In the second part of this paper we show that the convex hull of the dragon D is a decagon and that of a mill wheel, a dodecagon. Theorem 1 becomes an immediate corollary of next Theorem 5 and this provides an alternative proof of that theorem.

THEOREM 3. The convex hull of D is the decagon

$\mathcal{C} := co(P_1,\dots,P_{10})$, where

$P_1 = (2 + 2i)/3 = R(P_7)$	$P_6 = (2 - i)/3 = R(P_2)$
$P_2 = 2i/3 = Q(P_1)$	$P_7 = (3 - i)/3 = R(P_3)$
$P_3 = (-1 + i)/3 = Q(P_2)$	$P_8 = (7 - i)/6 = R(P_4)$
$P_4 = -1/3 = Q(P_3)$	$P_9 = 7/6 = R(P_5)$
$P_5 = -(1 + i)/6 = Q(P_4)$	$P_{10} = (5 + 3i)/6 = R(P_6)$

PROOF. The theorem will follow from i) and ii) :

i) $Q(\mathcal{C}) \subseteq \mathcal{C}$ $\qquad R(\mathcal{C}) \subseteq \mathcal{C}$

ii) $P_j \in D$ for $j = 1,\dots,10$

To prove i) it suffices to show that the images of the vertices of $\mathcal{C}$ by Q and R belong to $\mathcal{C}$. To accomplish this only remains to see that

$R(P_8) = (3 + 4i)/6$	$R(P_9) = (5+7i)/12$
$R(P_{10}) = (2 + i)/6$	$R(P_1) = 1/3$
$Q(P_5) = -i/6$	$Q(P_6) = (3 + i)/6$
$Q(P_7) = (2 + i)/3$	$Q(P_8) = (4 + 3i)/6$
$Q(P_9) = (7 + 7i)/12$	$Q(P_{10}) = (1 + 4i)/6$

are in $\mathcal{C}$ and this is easily checked. In fact, $R(P_8) \in \overline{P_1P_2}$;

$R(P_{10}) \in \overline{P_1P_6}$; $Q(P_7) \in \overline{P_1P_6}$; $Q(P_9) \in \overline{P_5P_1}$; $R(P_1) \in \overline{P_4P_9}$;

$Q(P_{10}) \in \overline{P_1P_2}$; $Q(P_8) \in \overline{P_1P_6}$; $Q(P_5) \in \overline{P_5P_8}$; $Q(P_8) \in \overline{P_1P_6}$

and $R(P_9)$ is in the triangle $P_1P_2Q(P_5)$.

On the other hand, to prove ii) it is enough to show that $P_1 \in D$ since the vertices P_2-P_{10} are obtained from P_1 after repeated applications of Q and R. Observe that no one of these points is in the convex hull of the others, (see Fig.9).

But $R\cdot R\cdot Q\cdot Q(z) = z/4 + (1 + i)/2$ implies that

$$(R^{(2)}Q^{(2)})^{(n)}(z) = z/4^n + ((1 + i)/2)(1 + 4^{-1} + \dots + 4^{1-n}).$$

The right-hand side of the last equality converges to

$(2 + 2i)/3 = P_1$ for $n \to \infty$. Thus, $P_1 \in D$, QED.

THEOREM 4. i) $co(D(M) \cup D(M^\wedge))$ is an octagon,
ii) $co(D(M) \cup D(M'))$ is a dodecagon.

PROOF. i) Because of Th. 2 we have, $co(D(M) \cup D(M')) =$
$= co(\Omega(F)) = \Omega(co(F))$. But, $co(F)$ is an octagon, (cf.[B]).
ii) $t(\mathcal{C}) = co(D(M^\wedge))$ implies that $i + t(\mathcal{C}) = co(D(M'))$.
Since $\tilde{t}(z) := i + t(z) = 1 + i - z$, the vertices of $co(D(M'))$ are

$$\tilde{P}_{10} = (1/6,1/2)\ ;\ \tilde{P}_1 = (1/3,1/3)\ ;\ \tilde{P}_2 = (1,1/3)\ ;\ \tilde{P}_3 = (4/3,2/3)$$
$$\tilde{P}_4 = (4/3,1)\ ;\ \tilde{P}_5 = (7/6,7/6)\ ;\ \tilde{P}_6 = (1/3,4/3)\ ;\ \tilde{P}_7 = (0,4/3)$$
$$\tilde{P}_8 = (-1/6,7/6)\ ;\ \tilde{P}_9 = (-1/6,1).$$

Now it is easy to see (see Fig.10) that

$$co(D(M) \cup D(M')) = co(P_3,P_4,P_5,P_6,P_7,P_8,\tilde{P}_3,\tilde{P}_4,\tilde{P}_5,\tilde{P}_6,\tilde{P}_7,\tilde{P}_8)$$

and that all the points are necessary, QED.

COROLLARY 1. $co(\tau(q)) \subset \tilde{q} := [-1/3,4/3] \times [-1/3,4/3]$.

Let C_0 be the net of squares of sides parallel to the axis that contains q and let $\partial(q)$ be the family of eight squares in $C_0 \setminus \{q\}$ that have points in common with q. Then,

COROLLARY 2. If $q' \in C_0 \setminus \{q\}$ and $co(\tau(q')) \cap co(\tau(q)) \neq \phi$ then $q' \in \partial(q)$.

5. A TESSELATION OF THE PLANE WITH MILL WHEELS. Let us begin with an auxiliary result.

LEMMA 5. Let $T(Y)=\{Y(i)\}$ and $T(Z)=\{Z(i)\}$, $i=1,2,3,...$, be tilings of the plane. Assume that there exists $d < \infty$ such that $dist((Y(i),Z(i)) < d$, $\sup(diam\ Y, diam\ Z) < d$.
Then, $m(Y) = m(Z) > 0$.

PROOF. Let A be a square of sides parallel to the axes of length h and $\Theta = \Theta(h) := \{i;\ A \cap Y(i) \neq \phi\}$. Then,

$$h^2 \le m(Y).\#\Theta(h) \le (h+2d)^2 \tag{9}$$

$$m(Y) = \lim_{h\to\infty} \frac{h^2}{\#\Theta(h)} \tag{10}$$

Let Θ' be the set obtained with Z instead of Y. Then,

$$m(Z) = \lim_{h\to\infty} \frac{h^2}{\#\Theta'(h)} \tag{11}$$

Since $\infty > m(Y) > 0$, $\#\Theta(h)$ is asymptotically equal to $h^2/m(Y)$.
$\#\Theta(h) \simeq \#\Theta'(h)$ follows from next inequalities:

$$\#\Theta(h) \le \#\Theta'(h+4d) \le \#\Theta(h+8d) \tag{12}$$

In consequence, $m(Y)=m(Z)$, QED.

THEOREM 5. Let $\tau:=\tau(q)$.

a) $R^2 = \cup\{\tau(q');\ q'\epsilon\ C_0\}$

b) $cl(int(\tau)) = \tau$

c) $q' \neq q'' \Rightarrow m(\tau(q') \cap \tau(q'')) = 0$

d) $m(\tau) = 1$.

PROOF. a) $\Rightarrow$ b). int(D) $\neq \phi$ because of Baire's Category Theorem. Now b) follows from the self-similarity of D.

a) and c) $\Rightarrow$ d). Apply Lemma 5.

c) The proof consists in showing that for any $q'\epsilon\partial(q)$:

(13) $$m(\tau(q) \cap \tau(q')) = 0.$$

In Fig.6 we see the squares in $\partial(q)$ and the triangles M and M′ renamed as 1 and 2, respectively. It will be enough to prove that for h=3,...,10 and h=3′,...,10′ it holds that

(14) $$m(D(\text{triangle } 1) \cap D(\text{triangle } h)) = 0.$$

In fact, the same equality will follow for 1 replaced by 2 because of the symmetry of the diagram and Lemma 4,ii). We reach the desired result with a previously used argument. In Fig.3,3 there are two triangles of the same order of descent in the same relative position than triangles 1 and 4. The dragons generated by them form a set similar to that engendered by 1 and 4. Then, from Lemma 2 we get (14) for h = 4. Analogously and using the same figure we prove (14) for h = 8, 8′, 10′. Using Fig.3,4, we prove (14) for h = 3, 3′, 4′, 5, 5′, 6, 7, 7′, 9 and 9′. And with Fig.3,5 we get rid of the remaining cases: 6′ and 10.

a) We call C_1 the net of squares defined by the diagonals of the net C_0, C_2 the net obtained with the diagonals of the net C_1 and so on. Then, C_n is called the **derivative** of C_{n-1}. In general, the process that provides a net of squares N′ by means of the diagonals of another, N, is called a **derivation**, (P. Lévy, [Y]). Let $t\ \epsilon\ C_n$ with n so great that diam $t < \varepsilon$. There are two nets of squares: β, α, (see Fig.7), such that C_n is the derivative of α and C_n is also the derivative of β. But these two nets do not have vertices in common, so only one of them has 0 as a vertex, which is then C_{n-1}.

Assume that $\{m, u, j, k\}$ is the standard triangulation of t. To

fix ideas let us consider the triangle u. Define $s:=\rho[u](u)$. If $\beta=C_{n-1}$, let Σ be the triangle of the standard triangulations of the squares belonging to β that contains u. We have in this case $u = \rho[\Sigma](\Sigma)$ and $s = \rho[\rho[\Sigma](\Sigma)](\rho[\Sigma](\Sigma))$, (see Fig.7).
If $\alpha=C_{n-1}$, let Σ be the triangle in the standard triangulations of the squares in α that contains a leg of u but does not contain u. In this case, $s = \lambda[j](\rho[\Sigma](\Sigma))$, that is, $s = \lambda[\rho[\Sigma](\Sigma)](\rho[\Sigma](\Sigma))$. Therefore, in any case, *given* u *from the standard triangulation of* C_n *there is a* Σ *in the standard triangulation of* C_{n-1} *such that* $\rho[u](u)$ *is a daughter of* $\rho[\Sigma](\Sigma)$. Using this result one can prove by induction that there is a triangle $\mathcal{m}$ in the standard triangulation of C_0 such that s is a descendant of order n+2 of $\mathcal{m}$.
The dragon $D(\mathcal{m})$ contains the vertices of its approximating sets. Therefore, any point of t is at distance less than ε of a mill wheel. Because of Lemma 3, $m(\tau(q))\geq 1$. From this and c), we deduce that any bounded region intersects only a finite number of mill wheels, (cf. Lemma 4,ii)). Since $\tau(q)$ is compact and diam $\tau(q) \leq (\sqrt{50})/3$, it follows that any point in R^2 belongs to a mill wheel, QED.

6. **AN ALTERNATIVE PROOF OF TH. 5, a).** $\tau(q)$ is contained in a square of sides 5/3 with the same center as q. The symmetry of the tiling and the compactness of $\tau(q)$ imply that if the mill wheels do not cover the plane it would exist an open set $G \subset q$ not covered by them. Assume that $m(G) = \delta > 0$. With the same notation as in Lemma 5 and $Y = \tau(q)$, $Z = q$, we have :

$$\#\Theta'(h+4).(1-\delta) \geq \#\Theta(h).m(\tau(q)) \geq \#\Theta(h).1.$$

Then, for $h \to \infty$ we obtain : $1-\delta \geq \lim \#\Theta(h)/\#\Theta'(h+4) = 1$, a contradiction, QED.
This proof and that of c) follow the same arguments we already used in [Z].

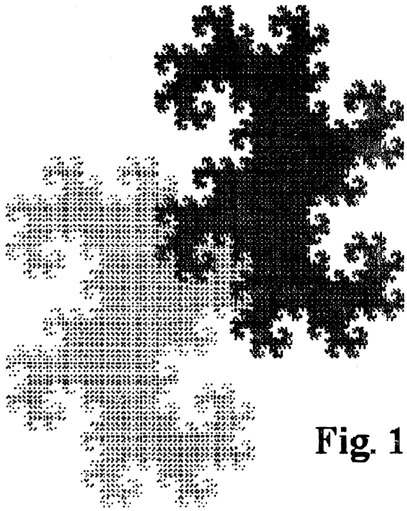

Fig. 11

REFERENCES

[B] Benedek, A. and Panzone, R., The set of gaussian fractions, Actas 2° Congreso "Dr. A. A. R. Monteiro", Univ. Nac. del Sur, (1993)11-40.

[D] Davis, Ch. and Knuth, D.E., Number representations and dragon curves,I, J. of Recreational Mathematics, 3(1970)66-81.

[E] Edgar, G. A., MEASURE, TOPOLOGY and FRACTAL GEOMETRY, Springer, (1990).

[K] Knuth, D. E., THE ART OF COMPUTING PROGRAMMING, Vol. 2, Ch. 4, Addison-Wesley, Reading, (1969).

[L] Lauwerier, H., FRACTALS, Endlessly Repeated Geometrical Figures, Princeton Univ. Press, (1991).

[Y] Lévy, P., Les courbes planes ou gauches et les surfaces composées de parties semblables au tout, J. de l'École Polytechnique, Série III, 7-8(1938)227-291.

[S] Schroeder, M. R., NUMBER THEORY in SCIENCE and COMMUNICATION, Springer-Verlag, (1990).

[Z] Benedek, A. and Panzone, R., Sobre algunos notables conjuntos planos,I. La curva de Lévy, Anales Acad. Cs. Ex. y Nat., Buenos Aires, to appear.

Departamento e Instituto de Matemática
Universidad Nacional del Sur
8000 Bahía Blanca, ARGENTINA.

Recibido en agosto de 1994.