Revista de la
Unión Matemática Argentina
Volumen 39, 1995.

A Riemannian Characterization of Extrinsic 3-Symmetric Spaces

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Abstract

This paper contains a new characterization of extrinsic 3-symmetric spaces. It involves, besides the induced almost complex stucture associated to the 3-symmetric space, only elements related to the Riemannian metric and the isometric imbedding.

1 Introduction

This paper contains a characterization of extrinsic 3-symmetric submanifolds of R^N in terms of the Riemannian metric and the almost complex structure which can be canonically defined in any 3-symmetric space. In fact our result shows that the knowledge of the almost complex structure and its relationship with the metric and second fundamental form are sufficient to recover the 3-symmetric structure which makes them extrinsic 3-symmetric.

Some 3-symmetric spaces have been identified in [2], [1] and [7] as "twistor spaces" over Riemannian symmetric spaces of inner type. It turns out that these spaces are just the extrinsic 3-symmetric ones.

The interest of the theorem included in this note lies on the fact that it is an strictly Riemannian result i. e. it characterizes extrinsic 3-symmetric spaces in terms geometric invariants arising from the metric and the almost complex structure.

The result is the following.

Theorem 1 *Let M^{2n} be a compact simply connected almost Hermitian manifold with almost complex structure J. Assume that $i : M^{2n} \to R^{2n+q}$ is a full isometric imbedding with second fundamental form α. Let ∇ be the Riemannian connection and $R(X,Y,Z,W)$ the Riemannian curvature tensor. Then M^{2n} is an extrinsic 3-symmetric submanifold of R^{2n+q} with symmetry tensor $S = \left(\frac{\sqrt{3}}{2}\right) J - \left(\frac{1}{2}\right) I$ if and only if the following conditions are satisfied.*

*Research partially supported by CONICET and CONICOR of ARGENTINA.

i) S preserves ∇J and $\nabla^2 J$.
ii) $R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$.
iii) $(\nabla_U R)(X,Y,Z,W) + (\nabla_U R)(JX,JY,JZ,JW) = 0$.
iv) $\alpha(JX,JY) = \alpha(X,Y)$.
v) $\nabla^{\perp}_{JU}(\alpha(X,Y)) = \sqrt{3}\nabla^{\perp}_{U}(\alpha(X,Y))$.□

This result extends a theorem due to D. Ferus [3] which characterizes the canonical imbedding of Hermitian Symmetric spaces which, as it is probably well known, are extrinsic k-symmetric for each $k \geq 2$. The next section contains preliminary definitions; the proof of Theorem 1 is contained in section 3.

2 Section

Let M^{2n} be a connected Riemannian manifold, as in [5] we say that M has an s-structure if for each point $p \in M$ there is an isometry θ_p of the Riemannian manifold M for which p is an isolated fixed point. The s-structure is of order $k \geq 2$ if $\theta_p^k = id_M$ for each point p and k is the minimum natural number with this property. The s-structure is called *regular* if $\theta_p \circ \theta_q = \theta_r \circ \theta_p$, where $r = \theta_p(q)$, for every p and q in M.

If we have an imbedding $i : M^{2n} \to R^{2n+q}$ we say that M is an extrinsic k-symmetric submanifold of R^{2n+q} if each θ_p extends to an isometry $\sigma_p : R^{2n+q} \to R^{2n+q}$ such that $\sigma_p\left(T_p(M)^{\perp}\right) = id_{T_p(M)^{\perp}}$. In this paper we consider only 3-symmetric spaces.

In our Riemannian regular s-manifold we may consider the *canonical connection* ∇^c defined by the formula of Graham-Ledger in terms of the tensor $S_p = (\theta_p)_{*p}$ and the Riemannian connection as follows. Let $D(X,Y)$ be the tensor field on M defined by

$$D(X,Y) = \left[\nabla_{\left((I-S)^{-1}X\right)}S\right]\left(S^{-1}Y\right)$$

then ∇^c is defined now as $\nabla^c_X Y = \nabla_X Y - D(X,Y)$. In this way ∇^c is uniquely determined as soon as we have the s-structure defined on M. It is important to indicate that the tensors D and S, as well as the metric on M, are parallel with respect to ∇^c.

Let us recall here a definition given in [8, (2.3)] and used in [6] to characterize R-spaces. The "canonical" covariant derivative of the second fundamental form of an isometrically imbedded k-symmetric space is defined by

$$(\nabla^c_X \alpha)(Y,Z) = \nabla^{\perp}_X(\alpha(Y,Z)) - \alpha(\nabla^c_X Y, Z) - \alpha(Y, \nabla^c_X Z)$$

where, as usual, $\nabla^{\perp}$ denotes the normal connection of the imbedding.

3 Section

First of all we observe that the conditions (i), (ii), (iii) and the hypothesis that M is almost Hermitian with almost complex structure J are precisely the assumptions of [4, p353, (4.5)]. The conclusion of [4] is that *M is a Riemannian locally 3-symmetric space and J is the canonical almost complex structure determined by the tensor S defined by the local cubic isometries of M as* $J = \left(\frac{2}{\sqrt{3}}\right)\left[S + \left(\frac{1}{2}\right) I\right]$.

Remark: It is important to notice that in his definition on p 352, Gray requires that each θ_p be holomorphic in a neighborhood of p with respect to the canonical almost complex structure of the family. This means $(\theta_p)_{*q} \circ J_q = J_{(\theta_p(q))} \circ (\theta_p)_{*q}$ and by the nature of J this implies $(\theta_p)_{*q} \circ S_q = S_{(\theta_p(q))} \circ (\theta_p)_{*q}$ which is the regularity condition of Graham and Ledger. Then the hypothesis of [4, p353, (4.5)] give a *regular* 3-symmetric space.

As we indicated in the previous section we have on M the uniquely defined canonical connection ∇^c. In [8, (1.2)] it is shown that a compact k-symmetric space M, imbedded in R^N, is extrinsic k-symmetric if and only if the following two conditions are satisfied

i) $\nabla^c \alpha = 0$ on M,

ii) $\alpha(SX, SX) = \alpha(X, X)\ \forall X \in T_p(M), p \in M$.

We have to show that, in our situation, the hypothesis (iv) and (v) of (1) imply (i) and (ii) above.

Let us begin by proving (ii). From (iv) it follows that $\alpha(JX, Y) = -\alpha(X, JY)$ and therefore,

$$\begin{aligned} \alpha(SX, SY) &= \left(\tfrac{1}{4}\right) \alpha\left[\left(\left(\sqrt{3}\right) J - I\right) X, \left(\left(\sqrt{3}\right) J - I\right) Y,\right] = \\ &= \left(\tfrac{1}{4}\right)\left[3\alpha(JX, JY) - \alpha\left(\sqrt{3} JX, Y\right) - \alpha\left(X, \sqrt{3} JY\right) + \alpha(X, Y)\right] = \\ &= \left(\tfrac{1}{4}\right)\left[3\alpha(JX, JY) - \alpha\left(\sqrt{3} JX, Y\right) + \alpha\left(\sqrt{3} JX, Y\right) + \alpha(X, Y)\right] = \\ &= \left(\tfrac{1}{4}\right)\left[3\alpha(JX, JY) + \alpha(X, Y)\right] = \alpha(X, Y). \end{aligned}$$

As we indicated in Section 2, the symmetry tensor S is canonically parallel and since $J = \left(\frac{2}{\sqrt{3}}\right)\left[S + \left(\frac{1}{2}\right) I\right]$ we have $\nabla^c J = 0$. Clearly from (iv) we obtain $\nabla^{\perp}_Z (\alpha(JX, Y)) = -\nabla^{\perp}_Z (\alpha(X, JY))$ and then $\alpha(\nabla^c_Z (JX), Y) = -\alpha(\nabla^c_Z X, JY)$. Finally $\alpha(JX, \nabla^c_Z Y) = -\alpha(X, \nabla^c_Z (JY))$ and these equalities add up to

$$(\nabla^c_Z \alpha)(JX, Y) = -(\nabla^c_Z \alpha)(X, JY).$$

By the expression of J in terms of S this becomes

$$(\nabla^c_Z \alpha)(SX, Y) + \left(\frac{1}{2}\right)(\nabla^c_Z \alpha)(X, Y) = -(\nabla^c_Z \alpha)(X, SY) - \left(\frac{1}{2}\right)(\nabla^c_Z \alpha)(X, Y)$$

and therefore

$$(\nabla^c_Z \alpha)(SX, Y) = -(\nabla^c_Z \alpha)(X, (S + I) Y).$$

But since $S^2+S+I=0$ we obtain

$$(\nabla^c_Z\alpha)(SX,Y)=(\nabla^c_Z\alpha)\left(X,S^2Y\right). \tag{1}$$

Now we have the following

Lemma 2 $(\nabla^c_{SZ}\alpha)(SX,Y)=(\nabla^c_Z\alpha)(X,S^2Y)$.

Proof. By definition we have

$$\begin{aligned}(\nabla^c_{SZ}\alpha)(SX,Y)&=\nabla^\perp_{SZ}(\alpha(SX,Y))-\alpha(\nabla^c_{SZ}SX,Y)-\alpha(SX,\nabla^c_{SZ}Y)=\\&=\nabla^\perp_{SZ}(\alpha(SX,S^3Y))-\alpha(S\nabla^c_ZX,Y)-\alpha(SX,\nabla^c_{SZ}S^3Y).\end{aligned}$$

Now by the condition (iv) of the theorem we have

$$\alpha(S\nabla^c_ZX,Y)=\alpha\left(\nabla^c_ZX,S^2Y\right),$$

$$\alpha\left(SX,\nabla^c_{SZ}S^3Y\right)=\alpha\left(SX,\nabla^c_{SZ}S^3Y\right)$$

and also

$$\nabla^\perp_{SZ}\left(\alpha\left(SX,S^3Y\right)\right)=\nabla^\perp_{SZ}\left(\alpha\left(X,S^2Y\right)\right).$$

On the other hand, condition (v) can be written as follows

$$\left(\frac{\sqrt{3}}{2}\right)\nabla^\perp_{JZ}(\alpha(X,Y))=\left(\frac{3}{2}\right)\nabla^\perp_Z(\alpha(X,Y))$$

and this, by the definition of S, yields

$$\nabla^\perp_{SZ}(\alpha(X,Y))=\nabla^\perp_Z(\alpha(X,Y)).$$

From all these equalities we finally get

$$(\nabla^c_{SZ}\alpha)(SX,Y)=\nabla^\perp_Z\left(\alpha\left(X,S^2Y\right)\right)-\alpha\left(\nabla^c_ZX,S^2Y\right)-\alpha\left(X,\nabla^c_ZS^2Y\right)$$

and, by definition, this is the right hand side of the identity that was to be proved. □

By by equation 1 and the Lemma we obtain the identity

$$(\nabla^c_{SZ}\alpha)(SX,Y)=(\nabla^c_Z\alpha)(SX,Y)$$

and by writing X instead of SX we transform this identity into

$$(\nabla^c_{SZ}\alpha)(X,Y)=(\nabla^c_Z\alpha)(X,Y)$$

which in turn may be written as

$$\left(\nabla^c_{(I-S)Z}\alpha\right)(X,Y)=0.$$

Since $(I - S)$ is non singular on M we obtain $(\nabla^c_Z \alpha) = 0$ and this shows that the conditions of the theorem are sufficient.

Let us prove that the conditions are necessary.

In [4, (3.6), p349] there is a proof of the necessity of condition (i) and that (ii) and (iii) are necessary is proved in [4, (3.8)(i), p349] and [4, (3.10), p350] respectively. Notice that, by definition, each θ_p is a holomorphic isometry and so the conditions $\theta(R) = R$ and $\theta(\nabla R) = R$ are satisfied.

That the condition (iv) is necessary, is proved easily as follows:

$$\alpha(JX, JY) = \left(\tfrac{4}{3}\right) \alpha\left(\left(S + \left(\tfrac{1}{2}\right) I\right) X, \left(S + \left(\tfrac{1}{2}\right) I\right) Y\right) =$$
$$= \left(\tfrac{4}{3}\right) \left[\alpha(SX, SY) + \left(\tfrac{1}{2}\right) \alpha(SX, Y) + \left(\tfrac{1}{2}\right) \alpha(X, SY) + \left(\tfrac{1}{4}\right) \alpha(X, Y)\right] =$$
$$\left(\tfrac{4}{3}\right) \left[\left(\tfrac{5}{4}\right) \alpha(X, Y) + \left(\tfrac{1}{2}\right) \alpha\left(\left(S^2 + S\right) X, Y\right)\right] =$$
$$\left(\tfrac{4}{3}\right) \left[\left(\tfrac{5}{4}\right) \alpha(X, Y) - \left(\tfrac{1}{2}\right) \alpha(X, Y)\right] = \alpha(X, Y).$$

In order to finish the proof we first notice that condition (v) is equivalent to the following identity

$$\nabla^{\perp}_{SU}(\alpha(SX, SY)) = \nabla^{\perp}_{Z}(\alpha(X, Y)).$$

In fact, since the second fundamental form satisfies $\alpha(SX, SY) = \alpha(X, Y)$, this last equality is just

$$\nabla^{\perp}_{SU}(\alpha(X, Y)) = \nabla^{\perp}_{Z}(\alpha(X, Y)). \tag{2}$$

Now replacing $S = \left(\frac{\sqrt{3}}{2}\right) J - \left(\frac{1}{2}\right) I$ in the last equality we get

$$\left(\frac{\sqrt{3}}{2}\right) \nabla^{\perp}_{JU}(\alpha(X, Y)) - \left(\frac{1}{2}\right) \nabla^{\perp}_{U}(\alpha(X, Y)) = \nabla^{\perp}_{U}(\alpha(X, Y))$$

which is clearly equivalent to

$$\nabla^{\perp}_{JU}(\alpha(X, Y)) = \sqrt{3}\nabla^{\perp}_{U}(\alpha(X, Y)).$$

Now to prove the equation 2 we need to use that M is extrinsic 3-symmetric i.e. for each $p \in M$ there exists an isometry $\sigma_p : R^{2n+q} \to R^{2n+q}$ such that $\sigma_p \mid T_p(M)^{\perp} = Id_{\left(T_p(M)^{\perp}\right)}$ and $\sigma_p \mid T_p(M) = \theta_p$. Since this is the case we have $SU = (\theta_p)_{*p} U = (\sigma_p)_{*p} U$ and therefore

$$\alpha(SX, SY) = \alpha\left((\theta_p)_{*p} X, (\theta_p)_{*p} Y\right) = (\sigma_p)_{*p} \alpha(X, Y).$$

Then we may write

$$\nabla^{\perp}_{SU}(\alpha(SX, SY)) = \nabla^{\perp}_{((\sigma_p)_{*p} U)}\left((\sigma_p)_{*p} \alpha(X, Y)\right) = \nabla^{\perp}_{U}(\alpha(X, Y)).$$

This proves the validity of the equation 2 and completes the proof of Theorem 1. □

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Recibido en junio de 1994.