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ON THE REPLICA OF LINEAR PRODUCTION GAMES WITH NON ADDITIVE RESOURCES

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Abstract. The replica of the Owen-Granot market linear production game is studied. A suitable way of replicating is devised. The standard convergence of the replica's core to a competitive solution is obtained. The approach presented here could be of wide economic applicability, since it depends on a proper vector in the core of resource games.

1. INTRODUCTION. The linear production game introduced by Owen [3] is a type of market game, which is generated by linear programming optimization problems. In this game it is important the relation between the core and the competitive set. The competitive set is always contained in the core of an LP-game, but both sets are not identical. However, if the set of players is replicated many times, the core of replicated games converges to the competitive set, i.e., the vectors belonging to the core of all replicas of linear production games are the competitive vectors. This result can be considered a consequence of a well-known theorem of Debreu and Scarf [1] which states this convergence in the area of economics. Owen has also proved a special result for linear production games: the convergence after a finite number of replications when the dual optimal set is a single. Samet and Zemel [5] provide a necessary and sufficient condition for finite convergence in LP-games. Granot [2] generalizes this market model dealing with non additive resource vectors.

In this paper we study the replicas of Granot's model. As it is usual in this subject we obtain the convergence of the replica's core. This is done in a very general way, which depends upon an election of Ψ in the cores of the resource games. In a large collection of cases Ψ may be taken either as Shapley value or as the nucleolus. A convergence to that competitive payoff is obtained.

2. NOTATIONS AND PRELIMINARY. For any finite sets N and M, $N \subset M$ means non strict inclusion, $M \setminus N$ means Boolean subtraction, $|N|$ is the cardinal number and 2^N is the power set of N. We shall often use the natural numbers to name the elements of these sets. The symbol R denotes the real numbers and R^N denotes the N-dimensional Cartesian space whose coordinates are indexed by the elements of N. If $x, y \in R^N$ then $x \leq y$ means $x_i \leq y_i$ for all $i \in N$ and 1 denotes the vector $(1, 1, ..., 1)$.

A *cooperative game* with *transferable utility* is a pair (N, v), where N is a nonempty finite set and $v : 2^N \to R$ is a set function that satisfies $v(\emptyset) = 0$. The elements of N are the *players*; a *coalition* is a subset of 2^N and v is the *characteristic function* of the game. A game is *super additive* if one has $v(S) + v(T) \leq v(S \cup T)$ whenever $S \cap T = \emptyset$; if the inequalities hold with the equal signs, the game is *additive*. $x.1 \leq v(N)$. An *imputation* is a vector $x \in R^N$ that satisfy $x.1 = v(N)$, and $x_i \geq v(\{i\})$ for all $i \in N$. The *core* of (N, v) is the set of imputations so that $\sum_{i \in S} x_i \geq \nu(S)$, for all $S \subset N$. A game is *balanced* if its core is non empty.

In the *linear production game* [3] to produce a unit of j-th *good* ($j = 1, 2, ..., m$), requires a_{kj} units of the k-th *resource* ($k = 1, 2, ..., p$) and it can be sold at a *price* c_j. The set of players is denoted by N. As in [2] we will assume that there is a function $b : 2^N \to R^p$ that assigns a resource vector to each coalition, $b(\emptyset) = 0$. A coalition S possesses a total of $b_k(S)$ units of the k-th resource as a whole. A function b_k can be thought of as the characteristic function of the k-th *resource game*. The *generalized linear production game (GLPG)* is defined by the characteristic function v given by the program

(1) $$v(S) = \max\{c.x\}, \quad s.t. \quad a.x \leq b(S), \quad x \geq 0.$$

We assume that the program (1) is feasible and bounded for all $S \subset N$. Therefore, the following dual program defines the same characteristic function.

(2) $$\min\{b(S).y\}, \quad s.t. \quad y.a \geq c, \quad y \geq 0.$$

A GLPG with additive resource function b is a linear production game, taking the vector $b(\{i\})$ as initial resources of the i-th player, $i \in N$.

Theorem A (Granot (1986)). *In a linear production situation as above, if all resource games are balanced, then the corresponding GLPG is also balanced.*

Moreover, a GLPG is super additive if all resource games are super additives, although, neither balance nor super additivity conditions of the resource games are necessary to obtain this property in a GLPG [4].

If the resource games are balanced, let $\Psi[b_k]$ be a vector in the core of the resource game (N, b_k), (k=1,2,...,p). Let y^* be an optimal solution to (2), with $S = N$ (the grand coalition). An imputation u is named *competitive* if it is defined by

(3) $$u_i = \Psi_i[b].y^*, \quad i \in N.$$

Theorem B (Granot (1986)). *In a linear production situation as already described, a competitive imputation is in the core of the GLPG.*

3. RESOURCE FUNCTION EXTENSION. REPLICATED GAME. For each player in the original set N, there are r players of the same type in the r-replica of a game, where r is a positive integer. The symbol rN denotes the set of all players. A suitable extension of the resource function is required.

Definition 1. The *profile* of a coalition $S \subset rN$ is the vector $\chi(S) = (z_i), i \in N$, where z_i is the number of i-type players in S.

Definition 2. The *representation* of a coalition $S \subset rN$ with respect to tN is the set: $S/tN = \{i \in N : z_i \geq t\}$.

Definition 3. Let Ψ be as in (3). The Ψ-*extension* of a resource function b, is the function b^* defined by

$$b_k^*(S) = b_k(S/N) + \Psi[b_k].[\chi(S) - 1]^+, \qquad [a]^+ = \left\{ \begin{array}{ll} a & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{array} \right\}.$$

Definition 4. Let Ψ be as in (3). A Ψ-*convergent extension* b^e is a set function that satisfies:
1) $b^e(S) = b(S/N)$, if $\chi(S) \leq 1$,
2) $b^e(S) = rb(N)$, if $\chi(S) = r1$,
3) $b^e(S) \leq b^*(S)$, for all S, and
4) $\lim_{\chi(S)\to\infty} b^e(S) = b^*(S)$, $\quad [\chi(S) \to \infty$ if $\min\{z_i \mid \chi(S) = (z_i), i \in N\} \to \infty]$.

Definition 5. Given a GLPG, let Ψ be as in (3), and let b^e be a Ψ-convergent extension. The game (rN, v_e) is defined by the program (1), where the resource vector is b^e.

We assume in the sequel that for each coalition $S \subset rN$, the linear program (1) is feasible and bounded so that the optimal objective function values for the various coalitions are finite. It is immediate that if $\chi(S) \leq 1$ then $v_e(S) = v(S)$, and that $v_e(rN) = rv(N)$. So, the characteristic function v_e of a replicated game, will be simply denoted by v.

Any imputation in the core of a r-b^e-replica, assigns an equal payoff to the same type of players (analogous to Theorem 2 in [3]). Therefore, core imputations can be represented by n-dimensional vectors. The following results should be interpreted in this way.

Theorem 1. *A Ψ-competitive imputation belongs to the core of a Ψ-convergent r-b^e-replica, for all r.*

Proof. We will prove that $\Psi[b_k]$ belongs to the core of b_k^e $(k = 1, 2, ..., p)$. Because $\Psi[b_k]$ belongs to the core of each b_k, we have

$$\Psi[b_k].1 = b_k(N), \tag{4}$$

$$\Psi[b_k].\chi(S) \geq b_k(S), \quad S \subset N. \tag{5}$$

Using (4) and Definition 4.2, we obtain $\Psi[b_k].r1 = b_k^e(rN)$.
Now consider $S \subset rN$. Using (5), Definitions 3 and 4.3, we have

$$\Psi[b_k].\chi(S) = \Psi[b_k].[\chi(S) - 1]^+ + \Psi[b_k].\chi(S/N) \geq b_k^e(S).$$

Theorem B completes the proof.

Theorem 2. *Given a GLPG, let Ψ be as in (3), and let b^e be a Ψ-convergent extension. An imputation u is Ψ-competitive if it belongs to the core of the r-b^e-replica, for all $r \geq 1$.*

Proof. Let u be a vector in the core of all replicas. Consider the system:

(6) $$\Psi_i[b].y \leq u_i \ , \ i \in N \ ,$$

(7) $$y.a \geq c \ ,$$

(8) $$y \geq 0.$$

Suppose that this system has a solution. Adding (6) for $i \in N$, results that y is an optimal solution to (2). Hence, (6) holds with the equal signs. We conclude that u is a Ψ-competitive imputation.

If the system (6)-(8) has no solution, consider the dual program of a trivial objective function subject to these constraints:

$$\max\{c.x - uz\} \ , \ \text{s.t.} \quad a.x - \Psi[b].z \leq 0 \ , \ x, z \geq 0.$$

This program is feasible and unbounded. Thus, there will exist vectors x, z such that

(9) $$c.x > uz \ ;$$

(10) $$a.x < \Psi[b].z \ ,$$

(11) $$x \geq 0 \ ,$$

$$(12) \qquad z > 0.$$

The vector z can be chosen with rational numbers so that the strict inequalities (10) and (12) hold. Multiplication of x and z by the common denominator makes z_i positive integer, $i \in N$. Now, multiplication by a positive integer t produces another solution which slackens inequalities in (10). In brief, for any positive integer t, there exists a solution x, z of (9)-(12) such that z_i is positive integer and $z_i > t$, for $i \in N$.

Then, consider a coalition S with $\chi(S) = z$, so that $S/tN = N$, where t is a positive integer such that $\delta = b_k^*(S) - b_k^e(S)$ is small, $\delta \geq 0$ (See Definition 5). Then

$$(13) \qquad b_k^e(S) + \delta = \Psi[b_k].z \ .$$

For a sufficiently small δ, from (10) and (13) we obtain

$$(14) \qquad a.x \leq b^e(S).$$

Now, (11) and (14) are the program (1) constraints. Therefore, x is a feasible solution of (1). So,

$$(15) \qquad v(S) \geq c.x.$$

However, since u belongs to the core of all r-replica,

$$(16) \qquad u.z \geq v(S).$$

But (9), (15) and (16) are contradictory. Therefore, the system (6)-(8) has a solution and the proof is complete.

Theorem 3. *Let b^e be a Ψ-convergent extension of a non negative GLPG, which converges at finite steps at the Ψ-extension b^*. If y^* is the only optimal solution of (2) for the coalition N, then, for a sufficiently large r, the core of the r-b^e-replicated game contains only the Ψ-competitive solution.*

Proof. Let r_1 be an integer large enough so that $b^e(S) = b^*(S)$ if $S/(r_1 - 1)N = N$. Consider $\delta = \min\{b(N).y - v(N)\} > 0$, where the minimal value is taken into the finite set of extreme solutions of the program (2) constraints, such that $y \neq y^*$. Let r_2 be an integer large enough so that $(r_2 - 1)\delta > v(N)$. Let $r = max\{r_1, r_2\}$, and, for any $i_0 \in rN$, let S be the coalition $S = rN - \{i_0\}$.

We will prove that the optimal solution y^* for the coalition rN, is also optimal for S. Let y^S be optimal for the program (2). If $y^S \neq y*$ then

$$(17) \qquad \begin{aligned} b^e(S).y^S = b^*(S).y^S &= \{(r-1)b(N) + \textstyle\sum_{i\neq i_0} \Psi_i[b]\}.y^S \\ &\geq (r-1)b(N).y^S > r\, v(N)\,. \end{aligned}$$

On the other hand, $v(S) < v(rN)$, since the core of (rN, v) is nonempty and v is nonnegative. This fact contradicts (17). So, y^* is optimal for S.

Let u be an imputation in the core of the r-b^e-replica. Then

$$\begin{aligned} ru.1 &= v(rN) = r\; b(N).y^*\ , \text{ and} \\ ru.1 - u_{i_0} \geq v(S) &= b^e(S).y^* = \left[(r-1)b(N) + \textstyle\sum_{i\neq i_0} \Psi_i[b]\right].y^*, \end{aligned}$$

from where

$$(18) \qquad u_{i_0} \leq \left[b(N) - \sum_{i\neq i_0} \Psi_i[b]\right].y^* \leq \Psi_{i_0}[b].y^*.$$

Adding (18) for $i_0 \in N$ we obtain $v(N) = u.1 \leq \sum_{i\in N} \Psi_i[b].y^* = b(N).y^* = v(N)$. Therefore, (18) holds with the equal sign. So that u is the Ψ-competitive solution.

4. EXAMPLES. The three following examples show extensions of a same resource function.

Example 1. Consider a GLPG with two players, two resources and two goods, $b(\{1\}) = (3,1)$, $b(\{2\}) = (2,4)$, $b(\{1,2\}) = (5,7)$ and programs:

$$\begin{array}{rclc} \max\{x_1 + x_2\} & = & v(S) & = \ \min\{b_1(S)y_1 + b_2(S)y_2\} \\ s.t.\ \ x_1 + 2x_2 \leq b_1(S) & & & y_1 + 2y_2 \geq 1 \\ 2x_1 + x_2 \leq b_2(S) & & & 2y_1 + y_2 \geq 1 \\ x_1, x_2 \geq 0 & & & y_1, y_2 \geq 0\,. \end{array}$$

The dual program has three extreme points: $(1,0)$, $(0,1)$ and $(\frac{1}{3},\frac{1}{3})$. However it has one optimal only for the grand coalition: $y^* = (\frac{1}{3},\frac{1}{3})$.

The characteristic function is $v(\{1\}) = 1$, $v(\{2\}) = 2$, $v(\{1,2\}) = 4$ and the core is $C(v) = \{(u_1, u_2) \mid u_1 + u_2 = 4\ ,\ 1 \leq u_1 \leq 2\}$.

The core of the resource games are $C(b_1) = \{(3,2)\}$, $C(b_2) = \{(1+t, 6-t)\ \mid\ 0 \leq t \leq 2\}$. The competitive set is $D(v) = \{(u_1, u_2) \mid u_1 + u_2 = 4\ ,\ \frac{4}{3} \leq u_1 \leq 2\}$.

For the 1[st] resource, $\Psi[b_1] = (3,2)$ and for b_2 the Shapley value is chosen: $\Psi[b_2] = (2,5)$. Hence, the associated competitive solution is: $(\frac{5}{3},\frac{7}{3})$.

Consider the trivial Ψ-extension $b^e = b^*$ that depends only on the profile:

$$\begin{aligned} b_1^*(z_1,z_2) &= 3z_1 + 2z_2, \\ b_2^*(z_1,z_2) &= sgn(z_1) + 4sgn(z_2) + 2sgn(z_1z_2) + 2[z_1-1]^+ + 5[z_2-1]^+. \end{aligned}$$

The following table summarizes the important core data.

r	z_1	z_2	$b_1^*(S)$	$b_2^*(S)$	$v(S)$	core condition
1	1	0	3	1	1	$1 \leq u_1$
1	0	1	2	4	2	$u_1 \leq 2$
1	1	1	5	7	4	$u_1 + u_2 = 4$
2	2	1	8	9	$\frac{17}{3}$	$\frac{5}{3} \leq u_1$
2	1	2	7	12	$\frac{19}{3}$	$u_1 \leq \frac{5}{3}$

The 2-replica core is $C^2(v) = \{(\frac{5}{3}, \frac{7}{3})\}$. Theorem 3 guarantees equality for $r \geq 6$.

In the next example, the extension of the resource function is not Ψ-convergent, as in the strong sense of Definition 4.4. However, examining the Theorem 3 proof, it is concluded that the condition to the convergence may be weakened. For instance, to each $S \subset rN$ and $\epsilon > 0$ if $\chi(S) > 1$ then $b_k^*(tS) - b_k^e(tS) < t\epsilon$ for all sufficiently large integer t.

Example 2. Consider another Ψ-extension b^e of Example 1 function:

$$b_1^e(z_1, z_2) = 3z_1 + 2z_2,$$

$$b_2^e(z_1, z_2) = \left\{ \begin{array}{ccc} 7z_1 + \sum_{i=z_1+1}^{z_2}\{5 - \frac{1}{i}\} & , & \text{if} \quad z_1 < z_2 \\ 7z_2 + \sum_{i=z_2+1}^{z_1}\{2 - \frac{1}{i}\} & , & \text{if} \quad z_1 > z_2 \\ 7z_1 & , & \text{if} \quad z_1 = z_2 \end{array} \right\}.$$

The following table summarizes the important core data.

r	z_1	z_2	$b_1^e(S)$	$b_2^e(S)$	$v(S)$	core condition
1	1	0	3	1	1	$1 \leq u_1$
1	0	1	2	4	2	$u_1 \leq 2$
1	1	1	5	7	4	$u_1 + u_2 = 4$
2	2	1	8	$9 - \frac{1}{2}$	$\frac{17}{3} - \frac{1}{6}$	$\frac{5}{3} - \frac{1}{6} \leq u_1$
2	1	2	7	$12 - \frac{1}{2}$	$\frac{19}{3} - \frac{1}{6}$	$u_1 \leq \frac{5}{3} + \frac{1}{6}$
3	3	2	13	$16 - \frac{1}{3}$	$\frac{29}{3} - \frac{1}{9}$	$\frac{5}{3} - \frac{1}{9} \leq u_1$
3	2	3	12	$19 - \frac{1}{3}$	$\frac{31}{3} - \frac{1}{9}$	$u_1 \leq \frac{5}{3} + \frac{1}{9}$
..	..	..	...	....	....	....
r	r	$r-1$	$5r-2$	$7r - 5 - \frac{1}{r}$	$4r - \frac{7}{3} - \frac{1}{3r}$	$\frac{5}{3} - \frac{1}{3r} \leq u_1$
r	$r-1$	r	$5r-3$	$7r - 2 - \frac{1}{r}$	$4r - \frac{5}{3} - \frac{1}{3r}$	$u_1 \leq \frac{5}{3} + \frac{1}{3r}$
..	..	..	...	....	....	....

The r-replica core is $C^r(v) = \{(u_1, u_2) \ / \ u_1 + u_2 = 4, \ \frac{5}{3} - \frac{1}{3r} < u_1 < \frac{5}{3} + \frac{1}{3r}\}$.

The next example shows that the replica's core can be empty, when the Ψ-convergence condition $b^e(S) \leq b^*(S)$ is not satisfied for all S (See Definition 4.3 and Theorem 1).

Example 3. Consider the Ψ-extension of the example 1 function given by:

$$b_1^e(z_1,z_2) = 3z_1 + 2z_2,$$

$$b_2^e(z_1,z_2) = \left\{ \begin{array}{ccc} 7z_1 + \sum_{i=z_1}^{z_2-1}\{5+\frac{1}{i}\} & , & \text{if } \ 1 \leq z_1 < z_2 \\ 7z_2 + \sum_{i=z_2}^{z_1-1}\{2-\frac{1}{i}\} & , & \text{if } \ z_1 > z_2 \geq 1 \\ b_2^*(z_1,z_2) & & \text{otherwise} \end{array} \right\}.$$

The following table summarizes the important core data. A column $b_2^*(S)$ is added.

r	z_1	z_2	$b_1^e(S)$	$b_2^e(S)$	$b_2^*(S)$	$v(S)$	core condition
1	1	0	3	1	1	1	$1 \leq u_1$
1	0	1	2	4	4	2	$u_1 \leq 2$
1	1	1	5	7	7	4	$u_1 + u_2 = 4$
2	2	1	8	8	9	$\frac{16}{3}$	$\frac{4}{3} \leq u_1$
2	1	2	7	**13**	**12**	$\frac{20}{3}$	$u_1 \leq \frac{4}{3}$
3	3	2	13	$15+\frac{1}{2}$	16	$\frac{28}{3}+\frac{1}{6}$	$\frac{3}{2} \leq u_1$
3	2	3	12	$\mathbf{19+\frac{1}{2}}$	**19**	$\frac{31}{3}+\frac{1}{6}$	$u_1 \leq \frac{3}{2}$

The 3-replica core is empty, because the last two inequalities contradict the others, but $b_2^e(1,2) > b_2^*(1,2)$, for instance.

FINAL REMARK. We would like to emphasize that our results of convergence present certain 'duality'. By this, we mean that the theorems depend upon the choice of Ψ. But, on the other hand, the applicability of the main theorems is powerful since the new agents introduce their resources in a coordinated way measured by Ψ, thus making our results potentially applicable to economic models.

REFERENCES

[1] G. Debreu and H. Scarf, A Limit Theorem on the Core of an Economy, *Internat. Econom. Rev.* **4** (1963) 235-246.

[2] D. Granot, A Generalized Linear Production Model: a Unifying Model, *Math. Programming* **34** (1986) 212-222.

[3] G. Owen, On the Core of Linear Production Games, *Math. Programming* **9** (1975) 358-370.

[4] R.O. Puente, *Estudio del Core en Distintos Juegos de Asignación Ponderados*, Tesis Doctoral, Universidad Nacional de San Luis, San Luis, Argentina, 1991.

[5] D. Samet and Z. Zemel, On the Core and Dual Set of Linear Programming Games, *Math. Oper. Res.* **9** (1984) 309-316.

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