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L^p APPROXIMATION OF GENERALIZED BI-AXIALLY SYMMETRIC POTENTIALS WITH FAST GROWTH

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ABSTRACT. The paper deals with growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called generalized bi-axially symmetric potentials (GBSP). We obtain the characterization of q-type and lower q-type of a GBSP, $H \in H_R$, $0 < R < \infty$, in terms of decay of approximation error $E^i_{n,p}(H, R_o), i = 1, 2$.

1 INTRODUCTION

Generalized bi-axially symmetric potentials (GBSP's) are the solutions of elliptic partial differential equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2\alpha+1}{y}\frac{\partial H}{\partial y} + \frac{2\beta+1}{y}\frac{\partial H}{\partial x} = 0, \quad \alpha, \beta > -\frac{1}{2}, \tag{1.1}$$

which are even in x and y *cf.* Gilbert [1]. A polynomial of degree n which is even in x and y is said to be a GBSP polynomial of degree n if it satisfies (1.1). A GBSP H regular about origin can be expanded as

$$H \equiv H(r, \theta) = \sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(cos 2\theta), \tag{1.2}$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $P_n^{(\alpha,\beta)}(t)$ are Jacobi polynomials.

Let $D_R = \{(x, y) : x^2 + y^2 < R,\ 0 < R < \infty\}$ and $\overline{D}_R$ be the closure of D_R. *A GBSP H is said to be regular in D_R if the series (1.2) converges uniformly on every compact subset of D_R.* Let H_R be the class of all GBSP's regular in $D_{R'}$ for every $R' \leq R$ but for no $R' > R$. The functions in the class H_∞ are called entire GBSP's.

McCoy [6] considered the approximation of an entire GBSP H by GBSP polynomials and found the rate of decay of approximation error :

$$E_{n,p}(H, 1) = \inf \|H - g\|_{1,p} = \inf_{g\in\pi_n} \left(\int\int_{\overline{D}_1} \mu(x, y)|H(x, y)|^p dx\ dy \right)^{1/p}$$

in terms of growth parameters associated with the maximum modulus $M(r, H) = \max_\theta |H(r, \theta)|$, where μ is a weight function and $1 \leq p < \infty$.

Also, McCoy [7] considered the approximation of pseudo analytic functions, constructed as complex combination of real valued analytic functions to the Stokes-Beltrami system on the disc. These functions include the GBSP's. He obtained some coefficients and Bernstein type growth theorems on the disc in sup norm.

A GBSP H is said to be regular in $\overline{D}_{R_o}$, the closure of D_{R_o} if it is regular in $D_{R'}$ for some $R' > R_o$. Let $\overline{H}_{R_o}$ be the class of all GBSP's regular on $\overline{D}_{R_o}$. For $H \in \overline{H}_{R_o}$, set

$$||H||_{R_o} = \max_{(x,y)\in \overline{D}_{R_o}} |H(x,y)|,$$

and for $1 \leq p < \infty$,

$$||H||^1_{R_o,p} = \left(\int_0^{2\pi} w(R_o,\theta)|H(R_0,\theta)|^p \, d\theta\right)^{1/p}, \tag{1.3}$$

$$||H||^2_{R_o,p} = \left(\int_{\overline{D}_{R_o}} \overline{w}(x,y)|H(x,y)|^p \, dx \, dy\right)^{1/p}, \tag{1.4}$$

where the functions w and $\overline{w}$ are positive and integrable (in the sense of Lebesgue) such that $\frac{1}{w}$ and $\frac{1}{\overline{w}}$ are bounded and $||\ ||^1_{R_o,p}$ and $||\ ||^2_{R_o,p}$ are L^p-norms on $\overline{H}_{R_o}$. For $H \in \overline{H}_{R_o}$ approximation errors $E^1_{n,p}(H,R_o)$ and $E^2_{n,p}(H,R_o)$ are defined as

$$E^1_{n,p}(H,R_o) = \inf_{g\in\pi_n} ||H-g||^1_{R_o,p} \tag{1.5}$$

$$E^2_{n,p}(H,R_o) = \inf_{g\in\pi_n} ||H-g||^2_{R_o,p}, \tag{1.6}$$

where π_n consists of all GBSP polynomials of degree at most $2n$. The concept of index q, the q-order $\rho(q)$ and lower q-order $\lambda(q)$ were introduced by Sato [8] in order to obtain a measure of growth of the maximum modulus when it is rapidly increasing. Thus, let $M(r,H) \to \infty$ as $r \to R$ and for $q = 2,3,\ldots$, we define

$$\rho_{(q)}(H,R) = \limsup_{r\to R} \frac{\log^{[q]} M(R,H)}{\log(\frac{R}{r-r})},$$

where $\log^{[0]} M(r,H) = M(r,H)$ and $\log^{[q-1]} M(r,H) = \log(\log^{[q-2]} M(r,H))$.

The GBSP $H \in H_R$ is said to have the index q if $\rho_q(H,R) < \infty$ and $\rho_{(q-1)}(H,R) = \infty$. If q is the index of H then $\rho_q(H,R)$ is called the q-order of H. The notions of the index and q-order play a significant role in classifying the rapidly increasing functions analytic in D_R. To compare the growth of two functions analytic in D_R that have same q-orders the distinct growth parameters are used.

We have the following definitions :

Definition 1. A GBSP $H \in H_R$, $0 < R < \infty$ having q-order $\rho_q(H,R)(\rho_q(H,R) > 0)$ is said to have q-type $T_q(H,R)$ and lower q-type $t_q(H,R)$ if

$$\begin{matrix} T_q(H,R) \\ t_q(H,R) \end{matrix} = \lim_{r\to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[q-1]}(r,H)}{\left(\frac{R}{R-r}\right)^{\rho_q(H,R)}}, \quad 0 \leq t_q(H,R) \leq T_q(H,R) \leq \infty.$$

In this paper we study the growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called *generalized bi-axially symmetric potentials* (GBSP's). The GBSP's are taken to be regular in a finite hyperball and influence the growth of their maximum moduli on the rate of decay of their approximation errors in L^p-norm defined by (1.3) and (1.4). The results and methods employed are different from those of McCoy [7]. The text has been divided into three parts. Section 1 consists of introductory exposition of the topic and Section 2 includes some lemmas. Finally, we prove some theorems which characterize the q-type $T_q(H,R)$ and lower q-type $t_q(H,R)$ of a GBSP $H \in H_{R_o}$, $0 < R_o < \infty$, in terms of rate of decay of approximation errors $E^i_{n,p}(H,Ro), 0 < Ro < R < \infty, i = 1,2$.

2 PRELIMINARY RESULTS

In this section we give some lemmas as preliminary results which have been used in the sequel.

Lemma 2.1. Let $H \in H_R$, $R > R_o$. Then there exist GBSP polynomials $g_n \in \pi_n$ such that

$$||H - g_n|| \leq KM(r,H)(n+1)^{\eta+1/2}(R/r)^{2(n+1)}$$

for all r sufficiently near to R and all sufficiently large values of n. Here K is a constant independent of r and n and $\eta = \max(\alpha,\beta)$.

Proof. The proof of this lemma follows from [4].

Lemma 2.2. Let $H \in \overline{H}_R$, $R > R_o$. Then there exist GBSP polynomials $g_n \in \pi_n$ such that

$$E^i_{n,p}(H,R_o) \leq K_i(n+1)^{\eta+1/2}(R_o/r)^{2(n+1)}M(r,H);\ i = 1,2 \tag{2.1}$$

for all r sufficiently near to R and all sufficiently large values of n. Here K_i is a constant depending on R_o, w, and p only and K_2 a constant depending on R_o, $\overline{w}$ and p.

Proof. Using (1.3), (1.4), (1.5), (1.6) and Lemma 2.1 we get the required result.

Lemma 2.3. Let $H \in \overline{H}_R$. Then for $n \geq 1$,

$$|a_n|R_o^{2n} \leq \frac{T^{1/p}(2\pi)^{1/v^*}(2n+\alpha+\beta+1)P(n,\alpha,\beta)\Gamma(n+\eta+1)}{(\eta+1)(n+1)}E^1_{n-1,p}(H,R_o),$$

where $P(n,\alpha,\beta,) = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{(n+\alpha+1)(n+\beta+1)}$.

Proof. By (1.5), for $H \in \overline{H}_{R_o}$there exists a GBSP polynomial $g^*_{n-1} \in \pi_{n-1}$ such that

$$2E^1_{n-1}(H,R_o) \geq ||g^*_{n-1}||^1_{R_o,p} \geq \frac{1}{T^{1/p}}\left(\int_0^{2\pi} |H(R_0,\theta) - g^*_{n-1}(R_o,\theta)|^p\, d\theta\right)^{1/p}, \tag{2.2}$$

since $1/w$ is bounded and we have $w \geq \frac{1}{T}$, $T > 0$. For $p > 1$ choose v^* such that $1/p + 1/v^* = 1$. Using Holder's inequality we get

$$\int_0^{2\pi} |H(R_0,\theta) - g^*_{n-1}(R_o,\theta)|d\theta \leq \left(\int_0^{2\pi} |H(R_0,\theta) - g^*_{n-1}(R_o,\theta)|^p d\theta\right)^{1/p}\left(\int_0^{2\pi} d\theta\right)^{1/v^*}. \tag{2.3}$$

Combining (2.3) and (2.4), we get

$$\begin{aligned} 2E_{n-1,p}(H,R_o) &\geq \frac{1}{2\pi T^{1/p}}\int_0^{2\pi} |H(R_0,\theta) - g^*_{n-1}(R_o,\theta)|d\theta \\ &= \frac{1}{(2\pi)^{1/v^*}T^{1/p}}\int_0^{\pi/2} |H(R_0,\theta) - g^*_{n-1}(R_o,\theta)|d\theta \end{aligned} \tag{2.4}$$

for $p>1$, since GBSP's H and g_{n-1}^* are even in x and y. For $p=1$, (2.4) is obvious with $v^*=0$. From the orthogonality of Jacobi polynomials [9] and uniform convergence of the series (1.2) on $\overline{D}_{R_o}$, we have

$$\frac{a_n R_o^{2n}}{(2n+\alpha+\beta+1)p(n,\alpha\beta)} = 2\int_0^{\pi/2} H(R_0,\theta)p_n^{(\alpha,\beta)}(\cos 2\theta)\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta\, d\theta.$$

Thus, for any $g\in\pi_{n-1}$ we have

$$(2.5)\quad \frac{a_n R_o^{2n} p(n,\alpha,\beta)^{-1}}{(2n+\alpha+\beta+1)} = 2\int_0^{\pi/2}(H(R_0,\theta)-g(R_o,\theta))p_n^{(\alpha,\beta)}(\cos 2\theta)\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta\, d\theta.$$

From [9], we know that

$$(2.6)\qquad \max_{-1\le t\le 1}|p_k^{(\alpha,\beta)}(t)| = \frac{\Gamma(k+n+1)}{\Gamma(n+1)\Gamma(k+1)},\quad \eta=\max(\alpha,\beta).$$

Taking in particular, g_{n-1}^* it follows that

$$(2.7)\quad \frac{a_n R_o^{2n}}{(2n+\alpha+\beta+1)p(n,\alpha\beta)} \le 2\frac{\Gamma(n+\eta+1)}{\Gamma(\eta+1)\Gamma(n+1)}\int_0^{\pi/2}|H(R_0,\theta)-g_{n-1}^*(R_o,\theta)|d\theta.$$

Combining (2.5) and (2.7), the lemma follows.

Lemma 2.4. Let $H\in\overline{H}_{R_o}$. Then for $n\ge 1$, we have

$$|a_n|R_o^{2n+2} \le \frac{\tilde{T}^{1/p}(\pi R_o^2)^{1/v^*}(2n+2)(2n+\alpha+\beta+1)P(n,\alpha,\beta)\Gamma(n+\eta+1)}{\Gamma(\eta+1)\Gamma(n+1)}E_{n-1,p}^2(H,R_o).$$

Proof. By (1.6), for $H\in\overline{H}_{R_o}$, there exists $\tilde{g}_{n-1}\in\pi_{n-1}$ such that

$$\begin{aligned}
2E_{n-1,p}^2(H,R_o) &\ge \|H-\tilde{g}_{n-1}\|_{R_o,p}^2\\
&\ge \frac{1}{\tilde{T}^{1/p}}\Big(\int\int_{\overline{D}_{R_o}}|H(x,y)-\tilde{g}_{n-1}(x,y)|^p\,dx\,dy\Big)^{1/p}\\
(2.8)\qquad &\ge \frac{1}{\tilde{T}^{1/p}(\pi R_o^2)^{1/v^*}}\Big(\int\int_{\overline{D}_{R_o}}|H(x,y)-\tilde{g}_{n-1}(x,y)|dx\,dy\Big)^{1/p},
\end{aligned}$$

where $\tilde{w}=\frac{1}{\tilde{T}}, \tilde{T}>0$ and $1/p+1/v^*=1$. From the orthogonality of Jacobi polynomials and uniform convergence of the series (1.2) on $\overline{D}_{R_o}$, we have for $0\le r\le R$,

$$\frac{a_n r^{2n}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)} = 2\int_0^{\pi/2}(H(r,\theta)-\overline{g}_{n-1}(r,\theta))P_n^{(\alpha,\beta)}(\cos\theta)\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta\, d\theta.$$

Using (2.6), we get

$$\frac{a_n r^{2n}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)} \le \frac{\Gamma(n+\eta+1)}{2\Gamma(n+1)\Gamma)\eta+1)}\int_0^{2\pi}|H(r,\theta_-\tilde{g}_{n-1}(r,\theta)|\, d\theta.$$

Since H and $\tilde{g}_{n-1}$ are even in x and y. Multiplying both sides of the above inequality by $r\,dr$ and integrating from 0 to R_o, we get

$$(2.9)\quad \frac{a_n r^{2n+2}(2n+2)^{-1}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)} \le \frac{\Gamma(n+\eta+1)}{2\Gamma(n+1)\Gamma(\eta+1)} \int\int_{\overline{D}_{R_o}} |H(x,y)-\tilde{g}_{n-1}(x,y)|\,dx\,dy.$$

Combining (2.8) and (2.9) we obtain the required result.

Lemma 2.5. Let $H \in H_R$, $0 < R < \infty (R > R_o)$. Then

$$M(r,H) \le |a_o| + \frac{T^{1/p}(2\pi)^{1/v^*}}{\Gamma(n+1)} M(r,h) .$$

where $h(u) = \sum_{n=1}^{\infty}(2n+\alpha+\beta+1)P(n,\alpha,\beta)\frac{\Gamma(n+\eta+1)}{\Gamma(n+1)}E^1_{n-1,p}(H,R_o)(\frac{u}{R_o})^{2n}$.

Proof. Using (2.6) and Lemma 2.3 we get

$$|\sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(\cos 2\theta)| \le |a_o| + \sum_{n=1}^{\infty} |a_n| r^{2n} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)\Gamma(\eta+1)}$$

or

$$\le |a_o| + \frac{T^{1/p}(2\pi)^{1/v^*}}{\Gamma(n+1)} \sum_{n=0}^{\infty} E^1_{n-1}(H,R_o)(\frac{r}{R_o})^{2n}(2n+\alpha+\beta+1)P(n,\alpha,\beta)\frac{\Gamma(n+\eta+1)}{\Gamma(n+1)}.$$

which corresponds to desired result.

Lemma 2.6. Let $H \in H_R$, $0 < R < \infty$. Then

$$M(r,H) \le |a_o| + \frac{\tilde{T}^{1/p}(\pi)^{1/v^*}}{v^*\Gamma(\eta+1)} R_o^2(1-v^*)M(r,h^*)$$

where $h^*(u) = \sum_{n=1}^{\infty}(2n+2)(2n+\alpha+\beta+1)P(n,\alpha,\beta)\frac{\Gamma(n+\eta+1)}{\Gamma(n+1)}E^2_{n-1,p}(H,R_o)(\frac{u}{R_o})^{2n}$.

Proof. Using Lemma 2.4 the proof has the same analysis as that of Lemma 2.5.

Lemma 2.7. Let $f(z) = \sum_{n=o}^{\infty} a_n z^n$ be analytic in $|z| < R$. Then the function $f(z)$ is of q-order and q-type $T(q)$ if and only if

$$T(q) = B(q)\,V(q),$$

where $B(q) = (\rho+1)^{\rho+1}/\rho^{\rho}$, $A(q) = 1$ for $q = 2$; $B(q) = 1$, $A(q) = 0$ for $q = 3, 4 \ldots$ and $V(q) = \limsup_{n\to\infty}(\log^{[q-2]} n)(\log^+ |a_n|R^n)^{\rho(q)+A(q)}$.

Proof. The lemma can be proved by simple manipulation of the results in [2] and [3].

Lemma 2.8. Let $f(z) = \sum_{k=0}^{\infty} a_{n_k} z^{n_k}$ be analytic in $|z| < R$ and have q-order $\rho(q)$ ($\rho(q) > 0$) and lower q-type $t(q)$. If $\psi(n_k) = |a_{n_k}/a_{n_{k+1}}|^{1/(n_{k+1}-n_k)}$ forms a nondecreasing sequence of k for $k > k_o$, then

$$(2.10)\qquad B(q)t(q) \le \liminf_{k\to\infty}(\log^{[q-2]} n_k)\left(\frac{\log^+ |a_{n_k}|R^{n_k}}{n_k}\right)^{\rho(q)+A(q)}$$

and

$$(2.11)\qquad B(q)t(q) \le L(q)\liminf_{k\to\infty}(\log^{[q-2]} n_{k-1})\left(\frac{\log^+ |a_{n_k}|R^{n_k}}{n_k}\right)^{\rho(q)+A(q)},$$

where $L(q) = \limsup_{k\to\infty}(\log^{[q-2]} n_k/\log^{[q-2]} n_{k-1})$ and $B(q)$ and $A(q)$ have the same meaning as in Lemma 2.7.

Proof. The proof of this lemma is available in [2] and [3].

3 MAIN RESULTS

Theorem 3.1. Let $H \in H_R$ and have q-order $\rho_q(H,R)$ $(0 < \rho_q(H,R) < \infty)$ and q-type $T_q(H,R)$. Then

(3.1) $$G(q) = B(q,H)\, T_q(H,R),$$

where

$$G(q) = \limsup_{n\to\infty}(\log^{[q-2]} n)\left(\frac{\log^+ E^i_{n,p}(H,R_o)(\frac{R}{R_o})^{2n}}{n}\right)^{\rho_q(H,R)+A(q)} \qquad i = 1,2$$

$B(q,H) = \frac{(\rho_2(H,R)+1)^{\rho_2(H,R)+1}}{\rho_2(H,R)^{\rho_2(H,R)}}$, $A(q) = 1$ if $q = 2$ and $B(q,H) = 1$, $A(q) = 0$ if $q = 3,4\ldots.$

Proof. Let $G(q) < \infty$. For given $\epsilon > 0$ and for all $n > n_o(\epsilon)$, we have

$$(\log^{[q-2]} n)\left(\frac{\log^+ E^i_{n,p}(H,R_o)(\frac{R}{R_o})^{2n}}{n}\right)^{\rho_q(H,R)+A(q)} < G(q)+\epsilon$$

or

$$\log^{[q-1]} n + (\rho_q(H,R)+A(q)) + (\log^+\log^+ E^i_{n,p}(H,R_o) + 2n\log\frac{R}{R_o} - \log n < G(q)+\epsilon$$

or

$$\rho_q(H,R)+A(q) > \frac{\log^{[q-1]} n}{\log n - \log^+\log^+ E^i_{n,p}(H,R_o) - 2n\log(R/R_o)} - \frac{log(G(q)+\epsilon)}{\log n - \log^+\log^+ E^i_{n,p}(H,R_o) - 2n\log(R/R_o)}.$$

Let $0 \le T_q(H,R) < \infty$. For given $\epsilon > 0$ and $r > r_o$ (1.7) implies

$$\log M(r,H) < \exp^{[q-2]}\Big\{(T_q(H,R)+\epsilon)(\frac{R}{R-r})^{\rho(H,R)}\Big\}.$$

Using Lemma 2.2, we further have

$$\log^+ E^i_{n,p}(H,R_o)(\frac{R}{R_o})^{2n} \le \log M(r,H) + (\eta+\frac{1}{2})\log(n+1) + 2n\log(\frac{R}{r}) + \log K_i$$

(3.2) $$\le \exp^{[q-2]}\Big\{(T_q(H,R)+\epsilon)(\frac{R}{R-r})^{\rho_q(H,R)}\Big\} + (\eta+\frac{1}{2})\log(n+1) + 2n\log\frac{R}{r} + O(1)$$

Let r be given by the equation

(3.3) $$\prod_{i=0}^{q-2}\exp^{[i]}\Big\{(T_q(H,R)+\epsilon)(\frac{R}{R-r})^{\rho_q(H,R)}\Big\} = \frac{2n(R-r)}{R\rho_q(H,R)}.$$

For $q = 2$, using (3.3) in (3.2) we have for sufficiently large values of n

$$\log^+(E^i_{n,p}(H,R_o)(\frac{R}{R_o})^{2n} \le \frac{(T_\rho(H,R)+\epsilon)^{\frac{1}{\rho_q(H,R)+1}}(2n)^{\frac{\rho_q(H,R)}{\rho_q(H,R)+1}}}{(\rho_q(H,R))^{\frac{\rho_q(H,R)}{\rho_q(H,R)+1}}}(1+\rho_q(H,R))+o(1)$$

On proceeding to limits, the above inequality yields $G(2) \le B(2,H)\ T_2(H,R)$.

Next, for $q = 3,4,\ldots$, (3.3) implies

$$\frac{R}{R-r} = \left(\frac{\log^{[q-2]}}{T_q(H,R)+\epsilon}\right)^{1/\rho_q(H,R)} \quad \text{as} \quad n \to \infty.$$

Using above in (3.2), we have

$$\log^+ E^i_{n.p}(H,R_o)(\frac{R}{R_o})^{2n} < n + (\eta + \frac{1}{2})\log(n+1) + 2\log(\frac{R}{r}) + O(1)$$

or

$$(\log^{[q-2]} n(1+o(1)))\left(\frac{\log^+ E^i_{n.p}(H,R_o)(\frac{R}{R_o})^{2n}}{n}\right)^{\rho_q(H,R)} < (T_q(H,R)+\epsilon)(1+o(1)).$$

Taking limits as $n \to \infty$, we observe that $T_q(H,R) \ge G(q)$ for $q \ge 3$.

To prove the reverse inequality we utilise Lemma 2.5 for the case $i = 1$ and Lemma 2.6 for $i = 2$ and then apply Lemma 2.7 to the functions $h(u)$ and $h^*(u)$.

Theorem 3.2. Let $H \in H_R$ and H have q-order $\rho_q(H,R)$ and lower q-type $t_q(H,R)$. Let n_k be an increasing sequence of natural numbers. Then

$$B(q,H)t_q(H,R) \ge \liminf_{k\to\infty}\left[(\log^{[q-2]} n_{k-1})(\frac{\log^+ E^i_{n_k,p}(H,R_o)(\frac{R}{R_o})^{2n_k}}{n_k})^{\rho_q(H,R)+A(q)}\right].$$

Proof. Let

$$\liminf_{k\to\infty}\left[(\frac{\log^{[q-2]} n_{k-1}}{B(q,H)})(\frac{\log^+ E^i_{n_k,p}(H,R_o)(\frac{R}{R_o})^{2n_k}}{n_k})^{\rho_q(H,R)+A(q)}\right] = \phi(q) \equiv \phi.$$

First suppose that $0 < \phi < \infty$. Then, for $\phi > \epsilon > 0$ and $k > k_o$,

$$\log^+ E^i_{n_k,p}(H,R_o)(\frac{R}{R_o})^{2n_k} > n_k\left\{\frac{(\phi-\epsilon)B(q,H)}{\log^{[q-2]} n_{k-1}}\right\}^{\frac{1}{\rho_q(H,R)+A(q)}}$$

Choose a sequence $\{r_{n_k}\}$ such that

$$2\log\frac{R}{r_{n_k}} = \frac{(\phi-\epsilon)C'(q)}{\log^{[q-2]} n_{k-1}},$$

where $C'(q) = \rho_2(H,R)$ if $q = 2$ and $C'(q) = C'$, $0 < C' < 1$ if $q = 3,4\ldots.$

By Lemma 2.2, if $k > k_o$ and $r_k \le r < r_{k+1}$, then denoting $\rho_q(H,R)+A(q)$ by ρ^*, we get

$$\log M(r,H) \geq \log E^i_{n_{k,p}}(H,R_o)(\frac{R}{R_o})^{2n_k} - (\eta+\frac{1}{2})\log(n_{k+1}) - 2n_k\log(\frac{R}{r_{n_k}}) - \log K_i$$

$$> n_k\left(\frac{(\phi-\epsilon)B(q,H)}{\log^{[q-2]} n_{k-1}}\right)^{\frac{1}{\rho^*}} - (\eta+\frac{1}{2})\log n_{k+1} - n_k\left(\frac{(\phi-\epsilon)C'(q)}{\log^{[q-2]} n_{k-1}}\right)^{\frac{1}{\rho^*}} - \log K_i$$

$$(3.4) \qquad = n_k\left(\frac{(\phi-\epsilon)}{\log^{[q-2]} n_{k-1}}\right)^{\frac{1}{\rho^*}}\left\{B(q,H)^{\frac{1}{\rho^*}} - C'(q)^{\frac{1}{\rho^*}}\right\}+O(1).$$

Using (3.4) we get

$$\log M(r,H) > \frac{\exp^{[q-2]}\left\{(\phi-\epsilon)C'(q)(2\log\frac{R}{r_{n_k}})^{-\rho^*}\right\}}{C'(q)^{\frac{1}{\rho^*}}(2\log\frac{R}{r_{n_k}})^{-1}}\left\{B(q,H)^{\frac{1}{\rho^*}} - C'(q)^{\frac{1}{\rho^*}}\right\}+O(1)$$

$$> \frac{\exp^{[q-1]}\left\{(\phi-\epsilon)C'(q)(2\log\frac{R}{r})^{-\rho^*}\right\}}{(2\log\frac{R}{r})^{-1}}\left[(\frac{B(q,H)}{C'(q)})^{\frac{1}{\rho^*}} - 1\right] + O(1).$$

For $q = 2$, we have

$$\frac{\log M(r,H)}{(\frac{R}{R-r})^{\rho_2(H,R)}} > \frac{(\phi-\epsilon)\rho_2(H,R)(\log R^2/r^2)^{-\rho_2(H,R)}}{(\frac{R}{R-r})^{\rho_2(H,R)}} + o(1).$$

Proceeding to limits as $r \to R$ we get $t_2(H,R) \geq \phi$.

Now, for $q = 3, 4\ldots$,

$$\liminf_{r\to R}\frac{\log^{[q-1]} M(r,H)}{(\frac{R}{R-r})^{\rho_2(H,R)}} \geq \phi C'.$$

Since the above inequality holds for every C', making $C' \to 1$, we get $t_q(H,R) \geq \phi$ for $q = 3,4\ldots$. If $\phi = 0$ the result follows trivially. If $\phi = \infty$, the above inequality with an arbitrary large number in place of $\phi - \epsilon$ gives $t_q(H,R) = \infty$.

Theorem 3.3. Let $H \in H_R$ $0 < r < \infty(R_o < R)$ and H have q-order $\rho_q(H,R)$ and lower q-type $t_q(H,R)$. If $\psi(n_k) = (E^i_{n_{k,p}}(H,R_o)/(E^i_{n_{k+1,p}}(H,R_o))^{1/(n_{k+1}-n_k)}$ forms a nondecreasing function of k for $k > k_o$ and $\log^{[q-2]} n_k \approx \log^{[q-2]} n_{k+1}$ as $k \to \infty$, then

$$B(q,H)t_q(H,R) \leq \liminf_{k\to\infty}\left[(\log^{[q-2]} n_k)(\frac{\log^+ E^i_{n_{k,p}}(H,R_o)(\frac{R}{R_o})^{2n_k}}{n_k})^{\rho_q(H,R)+A(q)}\right]$$

$$(3.5) \qquad \leq L(q)\liminf_{k\to\infty}\left[(\log^{[q-2]} n_{k-1})(\frac{\log^+ E^i_{n_{k,p}}(H,R_o)(\frac{R}{R_o})^{2n_k}}{n_k})^{\rho_q(H,R)+A(q)}\right],$$

where $L(q) = \limsup_{k\to\infty}(\log^{[q-2]} n_k/\log^{[q-2]} n_{k-1})$.

Proof. The proof of the above theorem follows by using Lemma 2.2 and Lemma 2.7 for $i = 1$ to the function $h(u)$ and for $i = 2$ to the function $h^*(u)$.

On combining theorems 2 and 3 we have the following result :

Theorem 3.4. Let $H \in H_R$ $0 < r < \infty (R_o < R)$ and H have q-order $\rho_q(H,R)$ and lower q-type $t_q(H,R)$. If $\psi(n_k) = (E^i_{n_{k,p}}(H,R_o)/(E^i_{n_{k+1,p}}(H,R_o))^{1/(n_{k+1}-n_k)}$ forms a nondecreasing function of k for $k > k_o$ and $\log^{[q-2]} n_k \approx \log^{[q-2]} n_{k+1}$ as $k \to \infty$

$$B(q,H)t_q(H,R) = \liminf_{k\to\infty} \left[(\log^{[q-2]} n_{k-1}) \left(\frac{\log^+ E^i_{n_{k,p}}(H,R_o)(\frac{R}{R_o})^{2n_k}}{n_k} \right)^{\rho_q(H,R)+A(q)} \right].$$

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