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The distributional convolution products of Marcel Riesz' ultra–hyperbolic kernel

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Abstract. *In this note we give a sense to the product of convolution of Marcel Riesz' ultrahyperbolic kernel. First we evaluate the case μ odd (cf. formula (I,2,18)) and then μ even (cf. formula (I,2,25)). It follows from (I,2,26) that the product of convolution $R_\alpha(u) * R_{-2k}(u)$ generalizes the formula $\square^k R_\alpha(u) = R_{\alpha-2k}(u)$ (cf. [8], page 11, formula (V,2)).*

I.1. Introduction.

Let $x = (x_1, \ldots, x_n)$ be a point of R^n. We shall write $x_1^2 + \cdots + x_\mu^2 - x_{\mu-1}^2 - \cdots - x_{\mu+\nu}^2 = u$, $\mu + \nu = n$. By Γ_+ we designate the interior of the forward cone: $\Gamma_+ = \{x \in R^n : x_1 > 0, u > 0\}$, and by $\overline{\Gamma}_+$ we designate its closure. Similarly, Γ_- designates the domain $\Gamma_- = \{x \in R^n, x_1 < 0,\ u > 0\}$ and $\overline{\Gamma}_-$ designate its closure.

Let $F(\lambda)$ be a function of the scalar variable λ, and let $\Phi(x)$ be a function endowed with the following properties:

a) $\Phi(x) = F(u)$,

b) supp $\Phi(x) \subset \overline{\Gamma}_+$,

c) $e^{\langle x,y\rangle}\Phi(x) \in L_1$ if $y \in V_-$,

where,

$$V_- = \{y \in R^n : y_1 > 0,\ y_1^2 + \cdots + y_\mu^2 - y_{\mu+1}^2 - \cdots - y_{\mu+\nu}^2 > 0\}\,.$$

We call recall R the family of functions $\Phi(x)$ which satisfies conditions a), b) and c). Similarly, we call A the family of functions which satisfies conditions:

a') $\Phi(x) = F(x)$,

b') supp $\Phi(x) \subset \overline{\Gamma}_-$,

c') $e^{\langle x,y\rangle}\Phi(x) \in L_1$ if $y \in V_+$, where

$$V_+ = \{y \in R^n : y_1 < 0,\ y_1^2 + \cdots + y_\mu^2 - y_{\mu+1}^2 - \cdots - y_{\mu+\nu}^2 > 0\}\,.$$

We shall consider the following functions of the family R introduced by Nozaki (cf [1], page 72):

$$R_\alpha(u) = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+ \end{cases} \qquad \text{(I,1,1)}$$

Here α is a complex parameter, n the dimension of the space.

The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{\mu-\alpha}{2}\right)}, \tag{I,1,2}$$

μ is the number of positive terms of

$$u = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \ldots - x_{\mu+\nu}^2, \tag{I,1,3}$$

$\mu + \nu = n$.

$R_\alpha(u)$, which is an ordinary function if $Re(\alpha) \geq n$, is a distribution of α.

We shall call $R_\alpha(u)$ the Marcel Riesz' ultra-hyperbolic kernel.

By putting $\mu = 1$ in (I,1,1) and (I,1,2) and remembering the Legendre's duplication formula of $\Gamma(z)$:

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{I,1,4}$$

(cf. [**2**], Vol. I, page 5, formula 15) the formula (I,1,1) reduces to:

$$M_\alpha = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+. \end{cases} \tag{I,1,5}$$

Here $u = x_1^2 - x_2^2 - \cdots - x_n^2$, (I,1,6)

and $H_n(\alpha) = 2^{\alpha-1}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-n+2}{2}\right)$. (I,1,7)

$M_\alpha(u)$ is precisely, the hyperbolic kernel of Marcel Riesz (cf. [**3**], page 31). On the other hand, taking into account (I,1,4), the formula (I,1,2) reduces to

$$K_n(\alpha) = H_n(\alpha)X(\mu,\alpha), \tag{I,1,8}$$

where $H_n(\alpha)$ is defined by (I,1,7) and

$$X(\mu,\alpha) = \frac{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{2+\alpha-\mu}{2}\right)\Gamma\left(\frac{\mu-\alpha}{2}\right)}. \tag{I,1,9}$$

We know (cf. [**2**], page 3, formulae 6 and 7) that the following formulae are valid:

$$\Gamma(z)\Gamma(1-z) = \pi \csc z\pi \tag{I,1,10}$$

and

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi \sec z\pi. \tag{I,1,11}$$

Then, we have

$$X(\mu,\alpha) = \frac{\sin\left(\frac{\mu-\alpha}{2}\right)\pi}{\cos\frac{\alpha}{2}\pi}\,. \tag{I,1,12}$$

From (I,1,2), we have

$$X(\mu,\alpha) = (-1)^{\frac{\mu-1}{2}} \quad \text{if } \mu \text{ is odd} \tag{I,1,13}$$

and

$$X(\mu,\alpha) = \frac{(-1)(-1)^{\mu/2}\sin\frac{\alpha}{2}\pi}{\cos\frac{\alpha}{2}\pi} \quad \text{if } \mu \text{ is even}\,. \tag{I,1,14}$$

Therefore, from (I,1,18) and taking into account (I,1,13) and (I,1,14) we have,

$$K_n(\alpha) = (-1)^{\frac{\mu-1}{2}} H_n(\alpha) \quad \text{if } \mu \text{ is odd} \tag{I,1,15}$$

and

$$K_n(\alpha) = \frac{(-1)(-1)^{\mu/2}\sin\frac{\alpha}{2}\pi}{\cos\frac{\alpha}{2}\pi} H_n(\alpha) \quad \text{if } \mu \text{ is even.} \tag{I,1,16}$$

From [4], pages 276 and 277 and taking into account the formula (I,1,10) we have,

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm\lambda\pi i} P_-^\lambda \tag{I,1,17}$$

$$P_+^\lambda = \Gamma(\lambda)\Gamma(1-\lambda)(2\pi i)^{-1}[e^{\lambda\pi i}(P - i0)^\lambda - e^{-\lambda\pi i}(P \pm i0)^\lambda] \tag{I,1,18}$$

where

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P \geq 0 \\ 0 & \text{if } P < 0\,, \end{cases} \tag{I,1,19}$$

$$P_-^\lambda = \begin{cases} 0 & \text{if } P > 0 \\ (-P)^\lambda & \text{if } P \leq 0\,, \end{cases} \tag{I,1,20}$$

$(P \pm i0)^\lambda = \lim_{\varepsilon\to 0}(P \pm i\varepsilon|x|^2)^\lambda$ ([4], page 275), (I,1,21)

$|x|^2 = x_1^2 + \cdots + x_n^2$ (I,1,22)

$\varepsilon > 0$ and

$P = P(x) = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2\,,$ (I,1,23)

$\mu + \nu = n$ (dimension of the space).

Here λ is a complex parameter. On the other hand, the Fourier transform of the distributions $(P \pm i0)^\lambda$ and P_+^λ are (cf. [4], page 284),

$$\{(P + i0)^\lambda\}^\wedge = a\Big(\lambda + \frac{\mu}{2}\Big)e^{-\frac{\nu\pi i}{2}}(Q - i0)^{-\lambda-\frac{\mu}{2}}\,, \tag{I,1,24}$$

$$\{(P - i0)^\lambda\}^\wedge = a\Big(\lambda + \frac{\mu}{2}\Big)e^{\frac{\nu\pi i}{2}}(Q + i0)^{-\lambda-\frac{\mu}{2}} \tag{I,1,25}$$

and

$$\{P_+^\lambda\}^\Lambda = b\Big(\lambda - \frac{\mu}{2}\Big)\Big[e^{-\pi i(\lambda+\frac{\mu}{2})}(Q-i0)^{-\lambda-\frac{\mu}{2}} - e^{\pi i(\lambda+\frac{\nu}{2})}(Q+i0)^{-\lambda-\frac{\mu}{2}}\Big]\,, \tag{I,1,26}$$

where

$$a\Big(\lambda + \frac{\mu}{2}\Big) = \frac{2^{2\lambda+\mu}\pi^{\mu/2}\Gamma\big(\lambda+\frac{\mu}{2}\big)}{\Gamma(-\lambda)}\,, \tag{I,1,27}$$

$$b\Big(\lambda + \frac{\mu}{2}\Big) = \frac{2^{2\lambda+\mu}\pi^{\frac{\mu-2}{2}}\Gamma(\lambda+1)\Gamma\big(\lambda+\frac{\mu}{2}\big)}{2i}\,, \tag{I,1,28}$$

and

$$Q = Q(y) = y_1^2 + \cdots + y_\mu^2 - y_{\mu+1}^2 - \cdots - y_{\mu+\nu}^2\,. \tag{I,1,29}$$

From [7], page 23, theorem 2, formula (I,3,1), we have

$$(P \pm i0)^\lambda \cdot (P \pm i0)^\mu = (P \pm i0)^{\lambda+\mu} \tag{I,1,30}$$

where λ and μ are complex numbers so that λ, μ and $\lambda+\mu \neq -\frac{\mu}{2} - k$, $k = 0,1,2,\ldots$.

On the other hand, the following formula is valid:

$$P_+^\lambda \cdot P_+^\mu = P_+^{\lambda+\mu}\,, \tag{I,1,31}$$

where λ, μ, $\lambda+\mu \neq -\frac{\mu}{2} - k$, $k = 0,1,2,\ldots$ and λ, μ, $\lambda+\mu \neq -\ell$, $\ell = 1,2,\ldots$.

From (I,1,31) and taking into account (I,1,18) and (I,1,30) we have

$$\begin{aligned}
&c(\lambda+\mu)(2\pi i)^{-1}\big[e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu} - e^{-(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu}\big] = P_+^{\lambda+\mu} = \\
&= P_+^\lambda \cdot P_+^\mu = c(\lambda)(2\pi i)^{-1}c(\mu)(2\pi i)^{-1}\cdot \\
&\quad\cdot \big[e^{\lambda\pi i}(P-i0)^\lambda - e^{-\lambda\pi i}(P+i0)^\lambda\big]\big[e^{\mu\pi i}(P-i0)^\mu - e^{-\mu\pi i}(P+i0)^\mu\big] = \\
&= c(\lambda)c(\mu)(2\pi i)^{-1}\cdot(2\pi i)^{-1}\big[e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu} + \\
&\quad + e^{-(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu}\big] - c(\lambda)c(\mu)(2\pi i)^{-1}\cdot(2\pi i)^{-1}\cdot \\
&\quad\cdot\big[e^{(\lambda-\mu)\pi i}(P-i0)^\lambda(P+i0)^\mu + e^{-(\lambda-\mu)\pi i}(P+i0)^\lambda(P-i0)^\mu\big]\,,
\end{aligned} \tag{I,1,32}$$

where

$$C(\gamma) = \Gamma(\gamma)\Gamma(1-\gamma)\,. \tag{I,1,33}$$

From (I,1,32) and (I,1,33) the following formula is valid:

$$\begin{aligned}
&e^{(\lambda-\mu)\pi i}(P-i0)^\lambda\cdot(P+i0)^\mu + e^{-(\lambda-\mu)\pi i}(P+i0)^\lambda\cdot(P-i0)^\mu = \\
&= [1 - B(\lambda,\mu)]e^{(\lambda+\mu)\pi i}(P-i0)^{\lambda+\mu} + \\
&\quad + [1 + B(\lambda,\mu)]e^{-(\lambda+\mu)\pi i}(P+i0)^{\lambda+\mu}\,,
\end{aligned} \tag{I,1,34}$$

where

$$B(\lambda,\mu) = \frac{c(\lambda+\mu)}{c(\lambda)c(\mu)(2\pi i)^{-1}} \,. \tag{I,1,35}$$

On the other hand, from [9], page 11, formula (I,2,33) we have

$$\begin{aligned}&(P+i0)^{\lambda}\cdot(P-i0)^{\mu}+(P-i0)^{\lambda}\cdot(P+i0)^{\mu} = \\ &= [1-B(\lambda,\mu)](P+i0)^{\lambda+\mu}+[1+B(\lambda,\mu)](P-i0)^{\lambda+\mu}\,,\end{aligned} \tag{I,1,36}$$

where $B(\lambda,\mu)$ is defined by (I,1,35). On the other hand, from (I,1,24), (I,1,25) and (I,1,27), we have

$$\{H_\alpha^{\pm}\}^{\Lambda} = \{H_\alpha(P\pm i0,n)\}^{\Lambda} = e^{\mp\frac{\alpha\pi i}{2}}\left(Q\mp i0\right)^{-\alpha/2}\,, \tag{I,1,37}$$

where

$$H_\alpha^{\pm} = H_\alpha(P\pm i0,n) = e^{\mp\frac{\alpha\pi i}{2}}e^{\pm\frac{\nu\pi i}{2}}a\Big(\frac{\alpha}{2}\Big)(P\pm i0)^{\frac{\alpha-n}{2}} \tag{I,1,38}$$

and

$$a\Big(\frac{\alpha}{2}\Big) = \Gamma\Big(\frac{n-\alpha}{2}\Big)\Big[2^{\alpha}\pi^{n/2}\Gamma\Big(\frac{\alpha}{2}\Big)\Big]^{-1}\,. \tag{I,1,39}$$

The distributional functions $H_\alpha^{\pm}$ have the following properties:

$$H_\alpha^{\pm} * H_\beta^{\pm} = H_{\alpha+\beta}^{\pm} \quad \text{([7], page 41, formula (II,3,11))}, \tag{I,1,40}$$

$$H_\alpha^{\pm} * H_{-2k}^{\pm} = H_{\alpha-2k}^{\pm} \quad \text{([7], page 40, formula (II,3,9))}, \tag{I,1,41}$$

$$H_{-2k}^{\pm} = L^k\{\delta\} \quad \text{([7], page 41, formula (II,3,13))}, \tag{I,1,42}$$

where

$$L^k = \left\{\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_\mu^2}-\frac{\partial^2}{\partial x_{\mu+1}^2}-\cdots-\frac{\partial^2}{\partial x_{\mu+\nu}^2}\right\}^k\,. \tag{I,1,43}$$

I.2. The convolution product of $R_\alpha(u) * R_\beta(u)$.

From (I,1,26) and taking into account (I,1,1), (I,1,15), (I,1,16) and (I,1,28) the Fourier transform of the distribution $R_\alpha(u)$ is

$$\{R_\alpha(u)\}^{\Lambda} = \frac{1}{2}\left[f_\alpha(Q-i0)+f_\alpha(Q+i0)\right] \quad \text{if } \mu \text{ is odd}, \tag{I,2,1}$$

and

$$\{R_\alpha(u)\}^{\Lambda} = \frac{1}{2i}\,\frac{\cos\frac{\alpha}{2}\pi}{\sin\frac{\alpha}{2}\pi}\left[f_\alpha(Q-i0)-f_\alpha(Q+i0)\right] \quad \text{if } \mu \text{ is even}, \tag{I,2,2}$$

where $f_\alpha(Q\pm i0) = e^{\pm\frac{\alpha\pi i}{2}}\left(Q\pm i0\right)^{-\alpha/2}$. (I,2,3)

We observe now that $R_\alpha(u)$ by virtue of (I,1,31), (I,2,1), (I,2,2) and (I,2,3) is a distribution of the class $0'_c$ (the space of rapidly decreasing distributions). Therefore, $R_\alpha(u) \in S'$, where S' is the dual of S and S is the Schwartz of functions ([**5**], page 233).

We conclude by appealing to the classic theorem of Schwartz ([**5**], page 268, formula (II,8,5)) that the following formula is valid:

$$\left\{R_\alpha(u) * R_\beta(u)\right\}^\Lambda = \left\{R_\alpha(u)\right\}^\Lambda \cdot \left\{R_\beta(u)\right\}^\Lambda . \tag{I,2,4}$$

Here $*$ designates, as usual, the convolution.

First, we shall study the case μ odd.

From (I,2,3) and taking into account (I,1,31), (I,2,1) and (I,2,2) we have

$$\begin{aligned}\left\{R_\alpha(u) * R_\beta(u)\right\}^\Lambda &= \frac{1}{4}\left[f_{\alpha+\beta}(Q-i0) + f_{\alpha+\beta}(Q+i0)\right] + \\ &+ \frac{1}{4}\left[f_\alpha(Q-i0)\cdot f_\beta(Q+i0) + f_\alpha(Q+i0)\cdot f_\beta(Q-i0)\right] ,\end{aligned} \tag{I,2,5}$$

if μ is odd.

On the other hand, taking into account the equation (I,1,35) and (I,1,36) for $\lambda = -\frac{\alpha}{2}$, and $\mu = -\frac{\beta}{2}$ we have

$$\begin{aligned}&\frac{1}{4}\left[f_\alpha(Q-i0)\cdot f_\beta(Q+i0) + f_\alpha(Q+i0)\cdot f_\beta(Q-i0)\right] = \\ &= \frac{1}{4}\left[f_{\alpha+\beta}(Q-i0) + f_{\alpha+\beta}(Q+i0)\right] + \\ &+ \frac{C\left(-\frac{\alpha}{2}-\frac{\beta}{2}\right)}{4C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)(2\pi i)^{-1}}\cdot\left[f_{\alpha+\beta}(Q+i0) - f_{\alpha+\beta}(Q-i0)\right] ,\end{aligned} \tag{I,2,6}$$

where $C(\gamma)$ is defined by (I,1,33).

Therefore, from (I,2,5) and taking into account (I,2,1) and (I,2,6) we have,

$$\left\{R_\alpha(u) * R_\beta(u)\right\}^\Lambda = \left\{R_{\alpha+\beta}(u)\right\}^\Lambda + D_{\alpha,\beta}(Q \mp i0) , \tag{I,2,7}$$

where

$$D_{\alpha,\beta}(Q\mp i0) = \frac{C\left(-\frac{\alpha}{2}-\frac{\beta}{2}\right)4^{-1}}{(2\pi i)^{-1}C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)}\left[f_{\alpha+\beta}(Q+i0) - f_{\alpha+\beta}(Q-i0)\right] . \tag{I,2,8}$$

On the other hand, from (I,1,22), (I,1,23) and (I,1,25) we have

$$f_{\alpha+\beta}(Q-i0) = \frac{e^{-(\alpha+\beta)\frac{\pi i}{2}}}{a_n\left(\frac{\alpha+\beta}{2}\right)}\, e^{\frac{\pi\nu i}{2}}\left[(P+i0)^{\frac{\alpha+\beta-n}{2}}\right]^\Lambda \tag{I,2,9}$$

and

$$f_{\alpha+\beta}(Q+i0)=\frac{e^{(\alpha+\beta)\frac{\pi i}{2}}}{a_n\left(\frac{\alpha+\beta}{2}\right)}\,e^{-\frac{\pi\nu i}{2}}\left[(P-i0)^{\frac{\alpha+\beta-n}{2}}\right]^{\Lambda}. \qquad (I,2,10)$$

Therefore, from (I,2,8), (I,2,9) and (I,2,10) we have,

$$D_{\alpha,\beta}(Q\mp i0)=\left\{E_{\alpha,\beta}(P\pm i0)\right\}^{\Lambda}, \qquad (I,2,11)$$

where

$$E_{\alpha,\beta}(P\pm i0)=\frac{C\left(-\frac{\alpha}{2}-\frac{\beta}{2}\right)4^{-1}}{C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)a_n\left(\frac{\alpha+\beta}{2}\right)(2\pi i)^{-1}}\cdot$$
$$\cdot\left[e^{-(\alpha+\beta-\nu)\frac{\pi i}{2}}(P+i0)^{\frac{\alpha+\beta-n}{2}}-e^{(\alpha+\beta-\nu)\frac{\pi i}{2}}(P-i0)^{\frac{\alpha+\beta-n}{2}}\right]. \qquad (I,2,12)$$

On the other hand, from (I,1,38) and (I,1,39), we have

$$\frac{e^{-(\alpha+\beta-\nu)\frac{\pi i}{2}}(P+i0)^{\frac{\alpha+\beta-n}{2}}}{\left[a_n\left(\frac{\alpha+\beta}{2}\right)\right]^{-1}}=H^{+}_{\alpha+\beta} \qquad (I,2,13)$$

and

$$\frac{e^{(\alpha+\beta-\nu)\frac{\pi i}{2}}(P-i0)^{\frac{\alpha+\beta-n}{2}}}{\left[a_n\left(\frac{\alpha+\beta}{2}\right)\right]^{-1}}=H^{-}_{\alpha+\beta}. \qquad (I,2,14)$$

Therefore, from (I,2,12), (I,2,13) and (I,2,14), we have

$$E_{\alpha,\beta}(P\pm i0)=\frac{C\left(-\frac{\alpha}{2}-\frac{\beta}{2}\right)4^{-1}}{C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)(2\pi i)^{-1}}\left[H^{+}_{\alpha+\beta}-H^{-}_{\alpha+\beta}\right]. \qquad (I,2,15)$$

Finally, from (I,2,7), (I,2,11) and (I,2,15), we conclude

$$\left\{R_\alpha(u)*R_\beta(u)\right\}^{\Lambda}=\left\{R_{\alpha+\beta}(u)\right\}^{\Lambda}+\left\{E_{\alpha,\beta}(P\pm i0)\right\}^{\Lambda}=$$
$$=\left\{R_{\alpha+\beta}(u)\right\}^{\Lambda}+\left\{\frac{C\left(-\frac{\alpha-\beta}{2}\right)4^{-1}}{C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)(2\pi i)^{-1}}\left[H^{+}_{\alpha+\beta}-H^{-}_{\alpha+\beta}\right]\right\}=$$
$$=\left\{R_{\alpha+\beta}(u)+\frac{C\left(-\frac{\alpha-\beta}{2}\right)4^{-1}}{C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)(2\pi i)^{-1}}\left[H^{+}_{\alpha+\beta}-H^{-}_{\alpha+\beta}\right]\right\}^{\Lambda}. \qquad (I,2,16)$$

Then, by using the theorem of the unicity for the Fourier transform, we conclude the following interesting formula for $R_\alpha(u)$,

$$R_\alpha(u)*R_\beta(u)=R_{\alpha+\beta}(u)+\frac{C\left(-\frac{\alpha-\beta}{2}\right)4^{-1}}{C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)(2\pi i)^{-1}}\cdot$$
$$\cdot\left[H^{+}_{\alpha+\beta}-H^{-}_{\alpha+\beta}\right],\quad \text{if } \mu \text{ is odd}, \qquad (I,2,17)$$

where

$$\frac{C\left(-\frac{\alpha}{2}-\frac{\beta}{2}\right)4^{-1}}{C\left(-\frac{\alpha}{2}\right)C\left(-\frac{\beta}{2}\right)(2\pi i)^{-1}}=\frac{i}{2}\,\frac{\sin\frac{\alpha}{2}\pi\cdot\sin\frac{\beta}{2}\pi}{\sin\left(\frac{\alpha+\beta}{2}\right)\pi}. \qquad \text{(I,2,18)}$$

Now, we shall study the case μ even.

From (I,2,4) and taking into account (I,2,2) and (I,2,3), we have

$$\begin{aligned} &R_\alpha(u)*R_\beta(u)=\left\{R_\alpha(u)\right\}^\Lambda\cdot\left\{R_\beta(u)\right\}^\Lambda= \\ &=\frac{1}{4}\left(-\frac{1}{i}\right)\left(-\frac{1}{i}\right)\frac{\cos\frac{\alpha}{2}\pi\cos\frac{\beta}{2}\pi}{\sin\frac{\alpha}{2}\pi\sin\frac{\beta}{2}\pi}\left[f_\alpha(Q-i0)-f_\alpha(Q+i0)\right]\cdot \\ &\quad\cdot\left[f_\beta(Q-i0)-f_\beta(Q+i0)\right]= \\ &=\frac{1}{4}\left(-\frac{1}{i}\right)\left(-\frac{1}{i}\right)\frac{\cos\frac{\alpha}{2}\pi\cos\frac{\beta}{2}\pi}{\sin\frac{\alpha}{2}\pi\sin\frac{\beta}{2}\pi}\cdot \\ &\quad\cdot\left\{f_{\alpha+\beta}(Q-i0)-f_{\alpha+\beta}(Q+i0)-G_{\alpha+\beta}(Q\mp i0)\right\}, \end{aligned} \qquad \text{(I,2,19)}$$

where

$$G_{\alpha,\beta}(Q\mp i0)=f_\alpha(Q-i0)\cdot f_\beta(Q+i0)+f_\alpha(Q+i0)\cdot f_\beta(Q-i0). \qquad \text{(I,2,20)}$$

Taking into account the equations (I,1,34) and (I,1,35) for $\lambda=-\frac{\alpha}{2}$ and $\mu=-\frac{\beta}{2}$, we have

$$\begin{aligned} G_{\alpha,\beta}(Q\mp i0)&=f_{\alpha+\beta}(Q-i0)+f_{\alpha+\beta}(Q+i0)+ \\ &+B\left(-\frac{\alpha}{2},-\frac{\beta}{2}\right)f_{\alpha+\beta}(Q+i0)-B\left(-\frac{\alpha}{2},-\frac{\beta}{2}\right)f_{\alpha+\beta}(Q-i0), \end{aligned} \qquad \text{(I,2,21)}$$

where $B\left(-\frac{\alpha}{2},-\frac{\beta}{2}\right)$ is defined by (I,1,35).

From (I,2,19) and (I,2,20), we have

$$\begin{aligned} \left\{R_\alpha(u)*R_\beta(u)\right\}^\Lambda&=\frac{1}{4}\,\frac{\cos\alpha 2\pi\cos\beta 2\pi}{\sin\frac{\alpha}{2}\pi\sin\frac{\beta}{2}\pi}\cdot \\ &\cdot B\left(-\frac{\alpha}{2},-\frac{\beta}{2}\right)\left[f_{\alpha+\beta}(Q-i0)-f_{\alpha+\beta}(Q+i0)\right]. \end{aligned} \qquad \text{(I,2,22)}$$

On the other hand, from (I,2,22) we have

$$\begin{aligned} &\left[f_{\alpha+\beta}(Q-i0)-f_{\alpha+\beta}(Q+i0)\right]= \\ &=(-1)2i\frac{\sin\left(\frac{\alpha+\beta}{2}\right)\pi}{\cos\left(\frac{\alpha+\beta}{2}\right)\pi}\left\{R_{\alpha+\beta}(u)\right\}^\Lambda. \end{aligned} \qquad \text{(I,2,23)}$$

Finally, from (I,2,22), (I,2,23) and taking into account (I,1,35) and (I,1,33), we have

$$\{R_\alpha(u) * R_\beta(u)\}^\Lambda = \frac{1}{4}\Big(-\frac{1}{i}\Big)\Big(-\frac{1}{i}\Big)\frac{\cos\frac{\alpha}{2}\pi\cos\frac{\beta}{2}\pi}{\sin\frac{\alpha}{2}\pi\sin\frac{\beta}{2}\pi}\cdot$$

$$\cdot B\Big(-\frac{\alpha}{2},-\frac{\beta}{2}\Big)\frac{(-1)2i\sin\left(\frac{\alpha+\beta}{2}\right)\pi}{\cos\left(\frac{\alpha+\beta}{2}\right)\pi}\{R_{\alpha+\beta}(u)\}^\Lambda =$$

$$= \frac{\cos\frac{\alpha}{2}\pi\cos\frac{\beta}{2}\pi}{\cos\left(\frac{\alpha+\beta}{2}\right)\pi}\{R_{\alpha+\beta}(u)\}^\Lambda . \tag{I,2,24}$$

Then, by using the theorem of the unicity for the Fourier transform, we conclude the following interesting formula of $R_\alpha(u)$,

$$R_\alpha(u) * R_\beta(u) = \frac{\cos\frac{\alpha}{2}\pi\cos\frac{\beta}{2}\pi}{\cos\left(\frac{\alpha+\beta}{2}\right)\pi}R_{\alpha+\beta}(u)\ , \tag{I,2,25}$$

if μ is even.

Putting in (I,2,17) and (I,2,25) $\beta = -2k$ and taking into account (I,2,18) we have,

$$R_\alpha(u) * R_{-2k}(u) = R_{\alpha-2k}(u)\ . \tag{I,2,26}$$

On the other hand, from [8], page 9, formula (III,9), we have

$$R_{-2k}(u) = \Box^k\delta\ , \tag{I,2,27}$$

where

$$\Box^k = \Big\{\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_\mu^2}-\frac{\partial^2}{\partial_{\mu+1}^2}-\cdots-\frac{\partial^2}{\partial x_{\mu+\nu}^2}\Big\}^k . \tag{I,2,28}$$

From (I,2,27) and (I,2,28) we have

$$\Box^k\{R_\alpha(u)\} = R_{\alpha-2k}(u)\ . \tag{I,2,29}$$

The formula (I,2,29) is given by S.E. Trione in [8], page 11, formula (V,2).

On the other hand, putting $\beta = 2k$ in (I,2,17), (I,2,25) and (I,2,29) and taking into account (I,2,18), we have

$$R_\alpha(u) * R_\beta(u) = R_{\alpha+2k}(u) \tag{I,2,30}$$

and

$$\Box^k R_{2k}(u) = R_0(u) = \delta\ . \tag{I,2,31}$$

From (I,2,31), $R_{2k}(u)$ is the elementary solution of the n–dimensional ultra–hyperbolic operator iterated k–times.

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