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Structural Consequences Stemming from the Existence of a Single Almost Split Sequence

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ABSTRACT. We characterize the structure of the bimodule ${}_{\Delta}Ext_R(C,A)_{\Gamma}$ *with an underlying ring R solely assuming that there exists an almost split sequence of left R-modules* $0 \to A \to B \to C \to 0$. Δ *and* Γ *are quotient rings of End(*${}_RC$*) and End(*${}_RA$*) respectively. The results are dualized under mild assumptions warranting that* ${}_{\Delta}Ext_R(C,A)_{\Gamma}$ *represents a Morita duality. To conclude, a reciprocal result is obtained: Conditions are imposed on* ${}_{\Delta}Ext_R(C,A)_{\Gamma}$ *that warrant the existence of an almost split sequence.*

1- PRELIMINARIES AND NOTATION

This work is motivated by the need to investigate structural properties of ${}_{\Delta}\mathrm{Ext}_R(C,A)_{\Gamma}$ as a Δ - Γ bimodule under the assumption that there exists an almost split sequence $0 \to A \to B \to C \to 0$ (also denoted (a,b) where $a:A \to B$ and $b:B \to C$) of left modules over a ring R. The lifting property must hold on the left and right of the sequence since R need not be an Artin algebra [1]. Δ and Γ are quotient rings of End(${}_RC$) and End(${}_RA$) respectively [2].

Throughout the paper we adopt standard notation. Thus A,B,C,X,Y,Z,... denote left R-modules over the ring R. Moreover, following [1,2], we denote:
$P(X,Y)=\{f \in \mathrm{Hom}_R(X,Y) \mid f$ factors over a projective R-module$\}$
$I(X,Y)=\{f \in \mathrm{Hom}_R(X,Y) \mid f$ factors over an injective R-module$\}$
$\underline{\mathrm{Hom}}_R(X,Y) = \mathrm{Hom}_R(X,Y) / P(X,Y)$; $\overline{\mathrm{Hom}}_R(X,Y) = \mathrm{Hom}_R(X,Y)/I(X,Y)$
$D = \mathrm{End}({}_RC)$; $G = \mathrm{End}({}_RA)$; $\Delta = \underline{\mathrm{End}}({}_RC)$; $\Gamma = \overline{\mathrm{End}}({}_RA)$.

2- RESULTS

The following theorem generalizes a result proven for Artin algebras by Auslander and Reiten ([1], theorem 3.3). The result is a preliminary step to assert the structural properties of the bimodule ${}_{\Delta}\mathrm{Ext}_R(C,A)_{\Gamma}$, as shown below. By contrast with ref. [1], our proof is elementary since it does not make use of functor categories:

Theorem 1:
For every module ${}_R X$, the map $\mathrm{Hom}_R(X,A) \ni g \to \mathrm{Ext}\,(C,g) \in \mathrm{Hom}_\Delta(\mathrm{Ext}_R(C,X), \mathrm{Ext}_R(C,A))$ is a monomorphism of right Γ-modules. (The reader should be reminded that A and C are connected via an almost split sequence.)

Proof: Obviously the map is a homomorphism of right Γ-modules. Let Ext(C,g) be zero. We have to show that g factors over an injective module. Choosing injective extensions, I(X) and I(A) of X and A respectively and extending g to h and j we obtain a commutative diagram (standard notation has been adopted throughout):

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "g"] & I(X) \arrow[r, "\mu"] \arrow[d, "h"] & \mathrm{Coker}(X) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & A \arrow[r, "i''"'] & I(A) \arrow[r, "\mu''"'] & \mathrm{Coker}(A) \arrow[r] & 0
\end{tikzcd} \tag{1}$$

Let the following diagram be a pullback:

$$\begin{tikzcd}
M \arrow[r, "\mu'"] \arrow[d, "g'"] & \mathrm{Coker}(X) \arrow[d, "j"] \\
I(A) \arrow[r, "\mu''"'] & \mathrm{Coker}(A)
\end{tikzcd} \tag{2}$$

Exploiting the properties of a pullback, we get a commutative diagram with the left upper square being a pushout:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "g"] & I(X) \arrow[r, "\mu"] \arrow[d, "g''"] \arrow[dd, bend right=60, "h"'] & \mathrm{Coker}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"] \arrow[d, equal] & M \arrow[r, "\mu'"] \arrow[d, "g'"] & \mathrm{Coker}(X) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & A \arrow[r, "i''"'] & I(A) \arrow[r, "\mu''"'] & \mathrm{Coker}(A) \arrow[r] & 0
\end{tikzcd} \tag{3}$$

To clarify notation, we have denoted Coker(i) by Coker(X) and Coker(i") by Coker(A). We claim that the sequence:

$$\mathrm{Ext}_R(C,X) \xrightarrow{\mathrm{Ext}(C,g)} \mathrm{Ext}_R(C,A) \xrightarrow{\mathrm{Ext}(C,i')} \mathrm{Ext}_R(C,M) \qquad (4)$$

is exact. This follows from the commutativity of the diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_R(C,\mathrm{Coker}(A)) & \xrightarrow{\delta} & \mathrm{Ext}_R(C,X) & \xrightarrow{\mathrm{Ext}(C,\,i)} & \mathrm{Ext}_R(C,I(X)) = 0 \\ \| & & \downarrow \mathrm{Ext}(C,g) & & \downarrow \\ \mathrm{Hom}_R(C,\mathrm{Coker}(A)) & \xrightarrow{\delta'} & \mathrm{Ext}_R(C,A) & \xrightarrow{\mathrm{Ext}(C,i')} & \mathrm{Ext}_R(C,M), \end{array} \qquad (5)$$

where the rows are part of the long exact sequences of Ext, and δ,δ' denote the connecting homomorphisms from Ext° to Ext^1. By assumption Ext(C,g) is zero, therefore Ext(C,i') is a monomorphism and $i'(a,b)\neq 0$. Since (a,b) is almost split, i' turns out to be a splitting monomorphism, and from the three-rows diagram it follows that g factors over the injective module I(X). QED.

The map dealt with in this theorem is actually an isomorphism under the relatively mild additional assumptions R semiperfect and ${}_RC$ finitely presented (cf.[2]). Remarkably, no conditions need to be imposed upon ${}_RX$, in contrast with the results of Auslander and Reiten [1] for Artin algebras.

3- DUALIZATION OF THE RESULTS

Let us fix the setting of reference [2]: R semiperfect; ${}_RC$ finitely presented and $\mathrm{End}({}_RC)$ local ring. Within this context, we intend to explore the consequences of the fact that if ${}_TE_G$ defines a Morita duality, then a Morita duality is induced via the bimodule ${}_\Delta\mathrm{Ext}_R(C,A)_\Gamma$. Let us introduce further notation: $\mathrm{Tr}C_R$= transpose of ${}_RC$; T=End($\mathrm{Tr}C_R$); ${}_TE$=injective hull of T/Ra(T). All the notation is standard (cf.[1]).

There are a number of instances in which ${}_TE_G$ defines a Morita duality [2]:

a) ${}_RA$ is finitely presented and $\mathrm{Tr}C_R$ is purely injective (each pure exact sequence $0 \to \mathrm{Tr}C_R \to M_R$ splits).

b) T is a left Artin ring and ${}_TE$ is finitely generated.

c) $\mathrm{Tr}C_R$ is simple.

d) R is an Artin algebra.

e) R is a ring of finite module type.

We have already shown [2] that if ${}_TE_G$ defines a Morita duality, then ${}_\Delta \mathrm{Ext}_R(C,A)_\Gamma$ is the induced Morita duality. This result is paramount to introduce a dualization of the context presented in the previous section. Accordingly, we shall prove the following results:

Proposition 1:
Let ${}_TE_G$ be a Morita duality. For $n \in N$, let ${}_RY$ be a direct summand of ${}_RC^n$, or let ${}_RC$ be self-projective and ${}_RY$ be an epimorphic image of ${}_RC^n$. Then, ${}_T\mathrm{Hom}_R(C,Y)$ is reflexive with respect to ${}_TE_G$, and ${}_\Delta\underline{\mathrm{Hom}}_R(C,Y)$ is reflexive with respect to ${}_\Delta \mathrm{Ext}_R(C,A)_\Gamma$.

Proof: Under the given assumptions, there is an epimorphism ${}_TT^n \approx {}_T\underline{\mathrm{Hom}}_R(C,C^n) \to {}_T\underline{\mathrm{Hom}}_R(C,Y) \to 0$ which yields the first statement. The second statement follows from well-known properties of the induced Morita duality which one obtains from ${}_TE_G$ by passing over from D and G to Δ and Γ.

At this point, we shall prove the following

Theorem 2:
Let ${}_TE_G$ be a Morita duality. then the following statements are equivalent:
1) ${}_T\underline{\mathrm{Hom}}_R(C,Y)$ is reflexive with respect to ${}_TE_G$.
2) $\underline{\mathrm{Hom}}_R(C,Y) \ni d \to \mathrm{Ext}(d,A) \in \mathrm{Hom}_\Gamma(\mathrm{Ext}_R(Y,A),\mathrm{Ext}_R(C,A))$ is an <u>isomorphism</u> of left T-modules. In this case, ${}_\Delta\underline{\mathrm{Hom}}_R(C,Y)$ and $\mathrm{Ext}_R(Y,A)_\Gamma$ are reflexive with respect to ${}_\Delta\mathrm{Ext}_R(C,A)_\Gamma$.

Proof: Let Ω denote the composition of the G-isomorphisms $\mathrm{Ext}_R(Y,A) = \mathrm{Ext}_R(Y, \mathrm{Hom}_T(\mathrm{Tr}C,E)) \approx \mathrm{Hom}_T(\mathrm{Tor}^R(\mathrm{Tr}C,Y), E) \approx \mathrm{Hom}_T(\underline{\mathrm{Hom}}_R(C,Y), E)$. Then the following diagram commutes, where Σ is the evaluation map:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_R(C,Y) & \xrightarrow{\mathrm{Ext}(-,A)} & \mathrm{Hom}_G(\mathrm{Ext}_R(Y,A), \mathrm{Ext}_R(C,A)) \\ \downarrow \Sigma & & \downarrow \approx \\ \mathrm{Hom}_G(\mathrm{Hom}_T(\underline{\mathrm{Hom}}_R(C,Y),E),E) & \xrightarrow[\mathrm{Hom}(\Omega,\, E)]{} & \mathrm{Hom}_G(\mathrm{Ext}_R(Y,A), E) \end{array} \qquad (6)$$

and thus our assertion follows. QED

Proposition 2:
Let ${}_RA$ be finitely presented and $\mathrm{Tr}C_R$ be a purely injective module. Let ${}_RX$ be any finitely presented module. Then, the following statements hold:

1) $\mathrm{Hom}_R(X,A)_G$ and ${}_T\mathrm{Ext}_R(C,X)$ are reflexive with respect to ${}_TE_G$.
2) $\overline{\mathrm{Hom}}_R(X,A)_G \approx \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$ and
${}_T\mathrm{TrC}\otimes_R X \approx {}_T\mathrm{Hom}_G(\mathrm{Hom}_R(X,A),E)$
3) $\overline{\mathrm{Hom}}_R(X,A)_\Gamma$ and ${}_\Delta\mathrm{Ext}_R(C,X)$ are reflexive with respect to ${}_\Delta\mathrm{Ext}_R(C,A)_\Gamma$.

Proof: Under the above assumptions ${}_TE_G$ is a Morita duality [2], the module TrC_R is reflexive with respect to ${}_TE_G$ and there exists an isomorphism $\mathrm{Hom}_G(A,E)_R \approx \mathrm{TrC}_R$. Let $w: {}_T\mathrm{Hom}_G(A,E)\otimes_R X \to {}_T\mathrm{Hom}_G(\mathrm{Hom}_R(X,A),E)$ denote the natural isomorphism, and let Σ and Σ' be the evaluation maps from ${}_T\mathrm{TrC}$ and $\mathrm{Hom}_R(X,A)_G$ into their biduals with respect to ${}_TE_G$. Then, (1) follows from the commutativity of the diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(X,\mathrm{Hom}_T(\mathrm{TrC},E) = \mathrm{Hom}_R(X,A) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{adj}"] & \mathrm{Hom}_T(\mathrm{Hom}_G(\mathrm{Hom}_R(Y,A),E),E) \arrow[d, "\mathrm{Hom}(\Omega,E)"] \\
\mathrm{Hom}_T(\mathrm{TrC}\otimes_R X,E) & \mathrm{Hom}_T(\mathrm{Hom}_G(A,E)\otimes_R X,E) \arrow[l, "\mathrm{Hom}(\Sigma'\otimes X,E)"]
\end{tikzcd} \qquad (7)$$

As we have the epimorphism $\mathrm{Hom}_R(X,A)_G \to \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$, the module $\mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$ is reflexive and, consequently, ${}_T(\mathrm{Ext}_R(C,X)$ is also reflexive.

(2) We have the isomorphisms:
$\overline{\mathrm{Hom}}_R(X,A)_G \approx \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),\mathrm{Ext}_R(C,A))_G \approx \mathrm{Hom}_T(\mathrm{Ext}_R(C,X),E)_G$
The second statement follows from ${}_T\mathrm{Hom}_G(\mathrm{Hom}_R(X,A),E) \approx$
$\approx {}_T\mathrm{Hom}_G(A,E)\otimes_R X \approx {}_T\mathrm{TrC}\otimes_R X$.
(3) is a consequence of (1). QED

4- UNDER WHAT CONDITIONS DO WE FIND AN ALMOST SPLIT SEQUENCE?

At this point we are in a position to prove a plausible reciprocal of the results expounded previously.

Theorem 3:

Assume the following conditions are satisfied (standard notation is followed):
${}_{\Delta'}\mathrm{Ext}_R(C',A')$ is injective; $\mathrm{Soc}({}_{\Delta'}\mathrm{Ext}_R(C',A'))$ is simple and essential in ${}_{\Delta'}\mathrm{Ext}_R(C',A')$; $\mathrm{Soc}(\mathrm{Ext}_R(C',A')_{\Gamma'}) \supseteq \mathrm{Soc}({}_{\Delta'}\mathrm{Ext}_R(C',A'))$; D' and G' are local rings, and for every ${}_RX$, the map $\mathrm{Ext}(C,-)\colon \mathrm{Hom}_R(X,A') \ni g \to \mathrm{Ext}\,(C',g) \in \mathrm{Hom}_{\Delta'}(\mathrm{Ext}_R(C',X),\mathrm{Ext}_R(C',A'))$ is surjective. Then every nonzero element $(a',b') \in \mathrm{Soc}({}_{\Delta'}\mathrm{Ext}_R(C',A'))$ is almost split.

Proof: Let $g \in Hom_R(A',X)$ be a homomorphism which has no factorization over a'. As $Ext(C', g)$ operates nonzero on a simple essential submodule of $Ext_R(C',A')$, it is a monomorphism. From the injectivity of ${}_{\Delta'}Ext_R(C',A')$, it follows that $Ext(C', g)$ splits, and, from the assumption that $Ext(C',-)$ is an epimorphism, we obtain $g' \in Hom_R(X,A')$ such that $Ext(C', gg')=id$. Thus, the composition gg' is an isomorphism, since, otherwise it would follow that $gg' \in Ra(G')$ and $gg'(a',b')=0$, which is a contradiction. Hence g is a splitting monomorphism and we have shown that (a',b') is almost split on the left side. The lifting property on the right side of the sequence also holds since we have assumed that D' is a local ring. QED

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