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Singular Integral Characterization of Functions with Conditions on the Mean Oscillation on Spaces of Homogeneous Type

by

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Abstract:
In this note we extend Janson's characterization of BMO_φ: $BMO_\varphi = \Lambda_\varphi + \sum_{j=1}^{m} R_j \Lambda_\varphi$, to more general metrics on $\mathbb{R}^n$, including some non-isotropic parabolic distances. We also obtain boundedness properties of singular integral operators on BMO_φ over general spaces of homogeneous type.

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§1. Introduction

Let X be a set. A symmetric function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a quasi-distance if $d(x,y) = 0$ if and only if $x = y$ and there exists a constant K such that the inequality $d(x,z) \leq K(d(x,y) + d(y,z))$ holds for every x, y and z in X. The d-ball with center $x \in X$ and radious $r > 0$ is the set $B(x,r) = \{y \in X : d(x,y) < r\}$. We shall consider X as a topological space with the topology induced by the balls as a system of neighborhoods of each point. Let μ be a positive measure on the Borel σ-algebra of subsets of X. Even when the balls may not be open sets, it is not difficult to show that they are Borel sets. We shall say, following for example [3] or [12], that (X, d, μ) is a space of homogeneous type if the following doubling condition

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$$

holds for some constant A and every $x \in X$ and $r > 0$. If, moreover, μ is a regular measure then continuous functions are dense in $L^p(X)$ for $1 \leq p < \infty$ and, consequently, Lebesgue differentiation theorem holds true.

It is a well known result on spaces of homogeneous type the following theorem proved by R. Macías and C. Segovia in [12]. Given d a quasi-distance on X there exist a quasi-distance d' on X equivalent to d and two constants $C > 0$ and $0 < \beta \leq 1$ such that

$$|d'(x,y) - d'(x,z)| \leq C r^{1-\beta} d'(y,z)^{\beta} \tag{1.1}$$

holds for every x, y, z and r such that $d'(x,y) < r$ and $d'(x,z) < r$. When the space X is equipped with a quasi-distance satisfying (1.1) we say that the space is of order β.

The generalization of Calderón-Zygmund kernels to spaces of homogeneous type becomes more natural when the space is normalized in the following sense, first introduced in [12]. We shall say that (X, d, μ) is a normal space if there exist four positive constants A_1, A_2, K_1 and K_2 such that

$$\begin{aligned} A_1 r \leq \mu(B(x,r)) &\leq A_2 r && \text{for } K_1\mu(\{x\}) \leq r \leq K_2\mu(X) \\ B(x,r) &= X && \text{if } r > K_2\mu(X) \\ B(x,r) &= \{x\} && \text{if } r < K_1\mu(\{x\}). \end{aligned}$$

It is clear that we may assume without loosing generality that $K_1 < 1 < K_2$.

Let us now introduce the main function spaces which concerns us in this paper. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function satisfying the Δ_2 Orlicz's condition $\varphi(2r) \leq$

$C\varphi(r)$ for some positive constant C and every $r > 0$ (see [11]). Given a real function f defined on a space of homogeneous type (X, d, μ) we shall say that f satisfies a Lipschitz-φ condition and we shall write $f \in \Lambda_\varphi$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C\varphi(d(x, y)) \qquad \text{for every } x, y \in X.$$

The infimun of those constants C is a semi-norm which added to the L^∞ norm gives a Banach space structure on Λ_φ. When $\varphi(t) = t^\beta$ for $0 < \beta \leq 1$, Λ_φ is the class of Lipschitz-β functions, which under the hypothesis of μ regular, is dense in every L^p for $p < \infty$. Sometimes we shall write $\Lambda_\varphi(X, d)$ instead of Λ_φ to emphasize the role of the distance.

Let $f \in L^1_{loc}$, i.e. $\int_B |f| \, d\mu < \infty$ for every ball B, we say that f is of φ-bounded mean oscillation and write $f \in BMO_\varphi$ if there exists a constant C such that the inequality

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leq C\varphi(r(B)),$$

holds for every ball B in X, where $r(B)$ is the radious of B and $f_B = \mu(B)^{-1} \int_B f \, d\mu$. If we identify two functions which differ by a constant, BMO_φ becomes a Banach space with the norm

$$||f||_{BMO_\varphi} = \sup_B \frac{1}{\mu(B)\varphi(r(B))} \int_B |f - f_B| \, d\mu,$$

which is equivalent to $\sup_B \inf_{a \in I\!R} \mu(B)^{-1}\varphi(r(B))^{-1} \int_B |f - a| \, d\mu$. In the last section we shall use the notation $BMO_\varphi(X, d, \mu)$ instead of BMO_φ in order to recall the particular structure of space of homogeneous type.

Let us observe that for $\varphi(t) \equiv 1$ the space BMO_φ is BMO and the space Λ_φ is L^∞. More generally the inclusion $\Lambda_\varphi \subset BMO_\varphi$ is always true. On the other hand, except for trivial spaces (X, d, μ), the oposite inclusion is not true for general φ. Nevertheless, when $\varphi(t) = t^\beta$ for $0 < \beta \leq 1$ the spaces Λ_φ and BMO_φ coincide. This result is a consequence of the following generalization to spaces of homogeneous type and general φ of the classical theorem of John and Nirenberg [2]. Let (X, d, μ) a space of homogeneous type with μ a regular measure, then $f \in BMO_\varphi$ if and only if there exist constants a, b and γ such that the inequality

$$\psi_B(t) = b \int_{cr\left(\frac{t}{\gamma\mu(B)}\right)^a}^{r} \frac{\varphi(\xi)}{\xi} \, d\xi \tag{1.3}$$

holds for every ball $B = B(x, r)$, and $t \in (0, \gamma\mu(B))$, where $\psi_B(t)$ is the non increasing rearrangemet of $|f - f_{\tilde{B}}|$ on B, where from now on $\tilde{B}$ is the ball concentric with B and radious $2K$ times that of B, i.e. $\tilde{B} = 2KB$. Let us remark that given $f \in BMO_\varphi$ then

(1.3) holds with $b = C||f||_{BMO_\varphi}$. We recall that the rearrangement on B of a function g, is essentially the inverse function of the distribution of $|g|$ over B. More precisely if

$$\mu_B(s) = \mu(\{x \in B : |g(x)| > s\}), \qquad \text{for } s \geq 0,$$

is the distrubution of $|g|$ on B, then, its rearrangement on B is given by

$$\psi_B(t) = \sup\{s : \mu_B(s) > t\}.$$

For the basic properties of the rearrangement see [16].

The main results of this note are, loosely, the following theorems. The precise statements and proofs are given in sections 3, 4 and 5.

Theorem I: For $\Phi(r) = \int_1^\infty \frac{\varphi(rt)}{t^{1+\beta}}\, dt$, singular integrals are bounded operators from $BMO_\varphi(X, d, \mu)$ into $BMO_\Phi(X, d, \mu)$.

Theorem II: For invariant quasi-distance d on $\mathbb{R}^n$ and μ Lebesgue measure, provided that $BMO(\mathbb{R}^n, d, \mu) \subset L^\infty + \sum_i \mathcal{K}_i L^\infty$ and that (4.8) we have

$$BMO_\varphi(\mathbb{R}^n, d, \mu) \subset \Lambda_\varphi(\mathbb{R}^n, d) + \sum_i \mathcal{K}_i \Lambda_\varphi(\mathbb{R}^n, d).$$

Theorem III: For d the normalized parabolic quasi-distance on $\mathbb{R}^n$ associated to a diagonalizable matrix A, and φ satisfying (4.8) we have the characterization $BMO_\varphi(\mathbb{R}^n, d, \mu) = \Lambda_\varphi(\mathbb{R}^n, d) + \sum_i \mathcal{K}_i \Lambda_\varphi(\mathbb{R}^n, d)$..

§2. Preliminary lemmas

This section contains some lemmas on the behaviour of BMO_φ functions. Even when most of the following result hold true in more general situations, we shall assume in this section that (X, d, μ) is a normal space, of order β and μ is a regular measure, this conditions will be enough for our later application.

(2.1) **Lemma:** Let $f \in BMO_\varphi$ then $\psi_B \in L^p(\mathbb{R}^+)$ for $1 \leq p < \infty$, where ψ_B is the non-increasing rearrangement fo the function $|f(x) - f_{\tilde{B}}|$ on B. Moreover there is a constant C such that

$$\left(\frac{1}{\mu(B)} \int_0^\infty |\psi_B(t)|^p\, dt\right)^{1/p} \leq C||f||_{BMO_\varphi} \varphi(r)$$

holds for every ball B with $r = r(B)$.

Proof: From (1.3) and Minkowski's inequality the result can be obtained in the following way

$$
\begin{aligned}
(\int_0^\infty |\psi_B(t)|^p\,dt)^{1/p} &\leq C\|f\|_{BMO_\varphi}\left(\int_0^\infty \left|\int_{cr(\frac{t}{\gamma\mu(B)})^a}^{r} \frac{\varphi(s)}{s}\,ds\right|^p dt\right)^{1/p}\\
&\leq C\|f\|_{BMO_\varphi}\int_0^r \frac{\varphi(s)}{s}\left(\int_0^\infty \chi_{(cr(\frac{t}{\gamma\mu(B)})^a,r)}(s)\,dt\right)^{1/p} ds\\
&\leq C\|f\|_{BMO_\varphi}\int_0^r \frac{\varphi(s)}{s}\left(\mu(B)(\frac{s}{r})^{1/a}\right)^{1/p} ds\\
&\leq C\|f\|_{BMO_\varphi}\frac{\varphi(r)\mu(B)^{1/p}}{r^{\frac{1}{ap}}}\int_0^r s^{\frac{1}{pa}-1}\,ds\\
&\leq C\|f\|_{BMO_\varphi}\varphi(r)\mu(B)^{1/p}. \blacksquare
\end{aligned}
$$

(2.2) **Lemma:** Let $f \in BMO_\varphi$ then $f \in L^p_{loc}$ and moreover there exists a constant C such that

$$\left(\frac{1}{\mu(B)}\int_B |f|^p\right)^{1/p} \leq C\|f\|_{BMO_\varphi}\varphi(r) + |f_{\tilde{B}}|$$

holds for every ball B.

Proof: Since $\psi_B(t)$ and $|f - f_{\tilde{B}}|$ have the same distribution function, in order to compute $\int_B |f - f_{\tilde{B}}|^p\,d\mu$ we can start by computing $\int_0^\infty \psi_B(t)^p\,dt$. Let B a ball from Lemma (2.1) we get

$$
\begin{aligned}
\left(\int_B |f|^p\,d\mu\right)^{1/p} &\leq \|(f\chi_B\|_p\\
&= \|f - f_{\tilde{B}})\chi_B\|_p + \|f_{\tilde{B}}\chi_B\|_p\\
&\leq (\int_0^\infty |\psi_B(t)|^p)^{1/p} + |f_{\tilde{B}}|\mu(B)^{1/p}\\
&\leq C\|f\|_{BMO_\varphi}\varphi(r)\mu(B)^{1/p} + |f_{\tilde{B}}|\mu(B)^{1/p}. \blacksquare
\end{aligned}
$$

The following lemma is an extension to spaces of homogeneous type and general functions φ of a well known result related to the growth at infinity of $BMO(\mathbb{R}^n)$ function (see for example [10]).

(2.3) **Lemma:** Let $f \in BMO_\varphi$ then for $\alpha > 0$ there is a constant C such that

$$\int_{x\notin B} \frac{|f(x) - f_B|}{d(x,x_0)^{1+\alpha}}\,d\mu(x) \leq C\|f\|_{BMO_\varphi}\int_r^\infty \frac{\varphi(t)}{t^{1+\alpha}}\,dt,$$

where $B = B(x_0, r)$.

Proof: Let $a > 1$ to be chosen later. Let $B_k = B(x_0, a^k r)$ with $k \geq 0$, we observe that $B_0 = B$ then from the doubling condition we have

$$\begin{aligned}|f_{B_{k+1}} - f_{B_k}| &\leq \frac{1}{\mu(B_k)} \int_{B_k} |f - f_{B_{k+1}}| \, d\mu \\ &\leq \frac{A}{\mu(B_{k+1})} \int_{B_{k+1}} |f - f_{B_{k+1}}| \, d\mu \\ &\leq A ||f||_{BMO_\varphi} \varphi(a^{k+1} r).\end{aligned}$$

By iteration we obtain

$$\begin{aligned}|f_{B_k} - f_{B_0}| &\leq A ||f||_{BMO_\varphi} \sum_{i=0}^{k} \varphi(a^i r) \\ &\leq A ||f||_{BMO_\varphi} \int_r^{a^k r} \frac{\varphi(t)}{t} \, dt.\end{aligned} \tag{2.4}$$

We may use this sequence of balls to compute the disired integral in the following way

$$\int_{x \notin B} \frac{|f(x) - f_B|}{d(x, x_0)^{1+\alpha}} \, d\mu(x) = \sum_{k=1}^{\infty} \int_{B_k - B_{k-1}} \frac{|f(x) - f_B|}{d(x, x_0)^{1+\alpha}} \, d\mu(x).$$

If $\{x_0\}$ is an atom then, there exists $j \in \mathbb{Z}$ such that $a^j r \leq K_1 \mu(\{x_0\}) < a^{j+1} r$. Let us set $k_0 = max\{j, 0\}$ then, by choosing $a = \frac{1}{K_1}$, the summ on the right hand side can be taken from $k_0 + 1$, because $B_k - B_{k-1} = \emptyset$ if $k \leq k_0$. Then from the normality hypothesis and (2.4) we have

$$\begin{aligned}\int_{x \notin B} \frac{|f(x) - f_B|}{d(x, x_0)^{1+\alpha}} \, d\mu(x) &\leq C \sum_{k=k_0+1} \frac{\mu(B_k)}{(a^k r)^{1+\alpha}} \frac{1}{\mu(B_k)} \int_{B_k} |f(x) - f_B| \, d\mu(x) \\ &\leq C \sum_{k=k_0+1} \frac{1}{(a^k r)^{\alpha}} \frac{1}{\mu(B_k)} \int_{B_k} |f - f_{B_k}| + |f_{B_k} - f_B| \, d\mu \\ &\leq C \sum_{k=k_0+1}^{\infty} \frac{||f||_{BMO_\varphi}}{(a^k r)^{\alpha}} \left(\varphi(a^k r) + A \int_r^{a^k r} \frac{\varphi(t)}{t} \, dt \right) \\ &\leq C \sum_{k=k_0+1}^{\infty} \frac{||f||_{BMO_\varphi}}{(a^k r)^{\alpha}} \int_r^{a^k r} \frac{\varphi(t)}{t} \, dt.\end{aligned}$$

By setting $F(s) = \int_r^{rs} \frac{\varphi(t)}{t}\, dt$ we obtain that

$$\begin{aligned}
\int_{x\notin B} \frac{|f(x)-f_B|}{d(x,x_0)^{1+\alpha}}\, d\mu(x) &\leq \frac{C||f||_{BMO_\varphi}}{r^\alpha} \sum_{k=k_0+1}^{\infty} \frac{F(a^k)}{(a^k)^\alpha} \\
&\leq \frac{C||f||_{BMO_\varphi}}{r^\alpha} \int_1^\infty \frac{F(t)}{t^{1+\alpha}}\, dt \\
&= \frac{C||f||_{BMO_\varphi}}{r^\alpha} \int_1^\infty \frac{1}{t^{1+\alpha}} \left(\int_1^t \frac{\varphi(sr)}{s}\, ds \right) dt \\
&\leq \frac{C||f||_{BMO_\varphi}}{r^\alpha} \int_1^\infty \frac{\varphi(sr)}{s^{1+\alpha}}\, ds \\
&= C||f||_{BMO_\varphi} \int_r^\infty \frac{\varphi(s)}{s^{1+\alpha}}\, ds. \blacksquare
\end{aligned}$$

Let $\psi \in C^\infty(I\!R)$ be a cutting off function such that $\psi \equiv 1$, if $|x| \leq 1/2$ and $\psi \equiv 0$ if $|x| \geq 1$. Then the function $g_r(x) = \int \psi(\frac{d(x,y)}{r})\, d\mu(y)$ is equivalent to $\mu(B(x,r))$ in fact $A\mu(B(x,r)) \leq g_r(x) \leq \mu(B(x,r))$. Moreover $g_r(x)$ is of class Lipschitz β with constant r^{-1}.

The following kernel $\psi_r(x,y) = \psi(\frac{d(x,y)}{r})g_r^{-1}(x)$ induces the approximate identity operator

$$\psi_r(f)(x) = \int_X \psi_r(x,y) f(y)\, d\mu(y)$$

on L^1_{loc}. Notice that if $f \in L^p$ for $1 \leq p \leq \infty$ the function $\psi_r(f)(x)$ is Lipschitz β and if f is constant then $\psi_r(f)$ is the same constant.

The next lemma shows that ψ_r is a regularizaton operator and provides an estimate which will be usefull in proving Theorem II.

(2.5) **Lemma:** Let $f \in L^\infty(X)$ then there is a constant $C > 0$ such that for $r, s > 0$ we have

$$\sup_{d(x,y)\leq r} |\psi_s(f)(x) - \psi_s(f)(y)| \leq C||f||_\infty (\frac{r}{s})^\beta.$$

Proof: We observe that if $s \leq r$ the result follows immediately because $||\psi_s(f)||_\infty \leq ||f||_\infty$. Then we suppose $r < s$. Given $x, y \in X$ and $r > 0$ such that $d(x,y) \leq r$ we have

$$\begin{aligned}
|\psi_s(f)(x) - \psi_s(f)(y)| &\leq \int |\psi_s(x,z) - \psi_s(y,z)||f(z)|\, d\mu(z) \\
&\leq ||f||_\infty \left[\int \frac{1}{g_s(x)} \left| \psi(\frac{d(x,z)}{s}) - \psi(\frac{d(y,z)}{s}) \right| d\mu(z) \right. \\
&\quad \left. + \int \psi(\frac{d(y,z)}{s}) \left| \frac{1}{g_s(x)} - \frac{1}{g_s(y)} \right| d\mu(z) \right].
\end{aligned}$$

It is clear from the definition of g_s that the second term on the right hand side is bounded by the first, so that it is enough to get the disired bound for this term. Thus, on account of the regularity of ψ and (1.1) we get

$$\begin{aligned}|\psi_s(f)(x)-\psi_s(f)(y)| &\leq \frac{C}{\mu(B(x,s))}\int_{B(x,s)\Delta B(y,s)} \frac{|d(y,z)-d(x,z)|}{s}\,d\mu(z)\\ &\leq \frac{C}{\mu(B(x,s))}\frac{d(x,y)^\beta}{s^\beta}\mu(B(x,s)\Delta B(y,s)).\end{aligned}$$

Notice that for $r<s$, we have $B(x,s)\Delta B(y,s)\subset B(x,2Ks)$ then the result follows from the doubling condition since $\frac{\mu(B(x,s)\Delta B(y,s))}{\mu(B(x,s))}\leq C$ with C independent of x,y and s . ■

The following lemma provides an upper bound for the BMO distance from f to its aproximate $\psi_r(f)$, which will also be applied to prove the Theorem II.

(2.6) **Lemma:** Let $f\in L^1_{loc}(X)$ and $\rho(f,r)=\sup\limits_{\substack{x\in X\\ r'\leq r}}\frac{1}{\mu(B(x,r'))}\int_{B(x,r')}|f(y)-f_{B(x,r')}|\,d\mu(y)$ then there exists a constant C such that

$$||f-\psi_r(f)||_{BMO}\leq C\rho(f,Cr).$$

Proof: Let $B_0=B(x_0,r_0)$ be a given ball. We shall divide the proof in two cases according $r_0\leq r$ or $r<r_0$. Let us first assume $r_0\leq r$ then

$$\begin{aligned}\frac{1}{\mu(B_0)}\int_{B_0}|f-\psi_r(f)|\,d\mu &\leq \frac{1}{\mu(B_0)}\int_{B_0}|f-f_{B_0}|\,d\mu+\frac{1}{\mu(B_0)}\int_{B_0}|\psi_r(f-f_{B_0})|\,d\mu\\ &\leq \rho(f,r_0)+\frac{1}{\mu(B_0)}\int_{B_0}|\psi_r(f-f_{B_0})|\,d\mu.\end{aligned}$$

It remains to bound the last term

$$\begin{aligned}\frac{1}{\mu(B_0)}\int_{B_0}|\psi_r(f-f_{B_0})|\,d\mu(x) &\leq \frac{1}{\mu(B_0)}\int_{B_0}\psi_r(|f-f_{B_0}|)(x)\,d\mu(x)\\ &= \frac{1}{\mu(B_0)}\int_{B_0}\frac{1}{g_r(x)}\int_{B(x,r)}\psi(\frac{d(x,y)}{r})|f-f_{B_0}|(y)\,d\mu(y)\,d\mu(x)\\ &\leq \frac{C}{\mu(B_0)}\int_{\tilde{B}_0}|f-f_{B_0}|\left(\int_{B(y,r)}\frac{d\mu(x)}{\mu(B(x,r))}\right)d\mu(y)\\ &\leq C\rho(f,2Kr_0).\end{aligned}$$

The last inequality follows because there exists $C>0$ such that $B(y,Cr)\subset B(x,r)$ for every $x\in B(y,r)$. Let us assume now that $r<r_0$. We shall use a covering lemma of

Wiener type for which the homogeneous spaces are a natural setting (see for example [5]). For every $x \in B_0 = B(x_0, r_0)$ we consider the ball with center x and radious $r > 0$ then $\mathcal{B} = \{B(x,r) : x \in B_0\}$ is a covering of B_0 then, there exists a sequence of non-overlaping balls $\{B_i : i \in I\!N\} \subset \mathcal{B}$, such that $B_0 \subset \cup_{i=1}^{\infty} \bar{B}_i$ with $\bar{B}_i = B(x_i, Cr)$ and, clearly, $\cup_{i=1}^{\infty} B_i \subset \tilde{B}_0$. Consequently

$$\begin{aligned}
\frac{1}{\mu(B_0)} \int_{B_0} |f - \psi_r(f)|\, d\mu &\leq \frac{1}{\mu(B_0)} \sum_{i=1}^{\infty} \int_{\bar{B}_i} |f - \psi_r(f)|\, d\mu \\
&\leq \frac{1}{\mu(B_0)} \sum_{i=1}^{\infty} \mu(\bar{B}_i) \left[\frac{1}{\mu(\bar{B}_i)} \int_{\bar{B}_i} |f - f_{\bar{B}_i}| + \frac{1}{\mu(\bar{B}_i)} \int_{\bar{B}_i} |f_{\bar{B}_i} - \psi_r(f)| \right] \\
&\leq \frac{C}{\mu(B_0)} \sum_{i=1}^{\infty} \mu(B_i) \left[\rho(f, Cr) + \frac{1}{\mu(\bar{B}_i)} \int_{\bar{B}_i} |f_{\bar{B}_i} - \psi_r(f)| \right].
\end{aligned}$$

Since $\sum_{i=1}^{\infty} \mu(B_i) \leq C\mu(B_0)$ it is enough to prove that $\frac{1}{\mu(\bar{B}_i)} \int_{\bar{B}_i} |f_{\bar{B}_i} - \psi_r(f)| \leq \rho(f, Cr)$. If $x \in \bar{B}_i$ we have that $B(x,r) \subset B(x_i, (c+1)Kr) = \hat{B}_i \subset B(x, 2K^2(C+1)r)$ and $\bar{B}_i \subset \hat{B}_i$ then

$$\begin{aligned}
|f_{\bar{B}_i} - \psi_r(f)| &\leq |f_{\bar{B}_i} - f_{\hat{B}_i}| + |f_{\hat{B}_i} - \psi_r(f)| \\
&\leq \frac{1}{\mu(\bar{B}_i)} \int_{\hat{B}_i} |f - f_{\hat{B}_i}| + |f_{\hat{B}_i} - \frac{1}{g_r(x)} \int_{B(x,r)} \psi\left(\frac{d(x,y)}{r}\right) f(y)\, d\mu(y)| \\
&\leq C\rho(f, (c+1)Kr) + \frac{1}{g_r(x)} \int_{B(x,r)} \psi\left(\frac{d(x,y)}{r}\right) |f_{\hat{B}_i} - f(y)|\, d\mu(y)| \\
&\leq C\rho(f, (c+1)Kr). \blacksquare
\end{aligned}$$

§3. Theorem I

In this section we shall work with a normal space of homogeneous type (X, d, μ), of order β and a regular measure μ. These hypotheses allows us to make use of all the results in §2. Since the singular integral operators which we shall be concerned are to be defined on spaces of generally unbounded funcions, the conditions on the kernel k must be strong enough in order to give a meaning to $\mathcal{K}f$ for $f \in BMO_\varphi$. It turns out that the following conditions will suffice to this end and moreover they will be enough to extend Peetre's [15] result to a very general setting.

(3.1) Let $k : X \times X \to I\!R$ be a measurable function such that

(3.1.1) $|k(x,y)| \leq Cd(x,y)^{-1}$ for every $x \neq y$, (3.1.2) there exists $\alpha \in (0,1]$ and $C > 0$ such that for $2d(z,y) < d(y,x)$ we have

$$|k(x,y) - k(x,z)| + |k(y,x) - k(z,x)| \leq C\frac{d(z,y)^{\alpha}}{d(x,y)^{\alpha+1}},$$

we may assume, without loosing generality, that $\alpha \leq \beta$ where β is the space's order.
(3.1.3) let $0 < r < R < \infty$ then

$$\int_{r<d(x,y)\leq R} k(x,y)\, d\mu(y) = 0 \qquad \text{for every } x \in X$$

$$\int_{r<d(x,y)\leq R} k(x,y)\, d\mu(x) = 0 \qquad \text{for every } y \in X,$$

(3.1.4) the operator $\mathcal{K}_{R,r}f(x) = \int_{r<d(x,y)\leq R} k(x,y)f(y)\, d\mu(y)$ is bounded in $L^2(X)$ with constant which does not depend on R or r.

Now, the hypotheses (3.1) on the kernel k allows us to define the singular integral on L^p in the following way $\mathcal{K}f = lim_{\substack{R\to\infty \\ r\to 0}} \mathcal{K}_{R,r}f$ where the limit is taken in the L^p sense $(1 < p < \infty)$ and almost everywhere if $1 \leq p < \infty$, see for instance [1]). Our first step is to give an extension of this definition to BMO_φ. See for example the book [7] for the definition of singular integral operators on BMO for the euclidean case.

(3.2) **Lemma:** Let K be a kernel satisfying (3.1). Given $f \in BMO_\varphi$ with φ satisfying $\int_1^\infty \frac{\varphi(rt)}{t^{1+\beta}}\, dt < \infty$, then
(3.3) for any ball B with radious r and any $x,y \in B$ we have that the integral

$$S_{\tilde{B}}f(x,y) = \int_{z\notin\tilde{B}} (k(x,z) - k(y,z))f(z)\, d\mu(z)$$

is absolutely convergent. Moreover

$$|S_{\tilde{B}}f(x,y)| \leq C||f||_{BMO_\varphi} \int_1^\infty \frac{\varphi(rt)}{t^{1+\beta}}\, dt + |f_{\tilde{B}}|,$$

(3.4) there is a set of zero measure $\mathcal{N}$ in $X \times X$ such that the function

$$F_y(x) = \mathcal{K}(f\chi_{\tilde{B}})(x) - \mathcal{K}(f\chi_{\tilde{B}})(y) + S_{\tilde{B}}f(x,y)$$

is defined for every $(x,y) \notin \mathcal{N}$ and it does not depend on the ball B containing x and y,
(3.5) given $y_1 \neq y_2$ we have that $F_{y_1}(x) - F_{y_2}(x)$ does not depend on x.

Before proving Lemma (3.2), let us make the following definition and a few remarks on the conditions (3.1) on k.

(3.6) **Definition:** Given $f \in BMO_\varphi$ we define $\mathcal{K}f(x)$, except for additive constants, to be the function $F_y(x)$.

(3.7) **Remarks:** $\mathcal{K}$ is well defined for functions in L^∞ or in Λ_φ. From (3.1.3) and the very definition of $\mathcal{K}$ on BMO we see that $\mathcal{K}1 = 0$. So that can we easily see that

$$F_y(x) = \mathcal{K}((f - f_{\tilde{\tilde{B}}})\chi_{\tilde{B}})(x) - \mathcal{K}(f\chi_{\tilde{B}})(y) + S_{\tilde{B}}(f - f_{\tilde{\tilde{B}}})(x, y). \tag{3.8}$$

Let us also mention that, in several cases, (3.1.4) can be obtained from (3.1.i), $i = 1, 2, 3$. For example, Cotlar's lemma can be applied to get the L^2-norm inequality when the space satisfies a regularity condition of the kind: there exist $\gamma \in (0, 1]$ and $C > 0$ such that

$$\mu(B(x, r)) - \mu(B(x, s)) \leq C(r - s)^\gamma r^{1-\gamma} \tag{3.9}$$

holds for every x and every r, s such that $0 \leq s < r$. The case $\gamma = 1$ was considered in [1] and the case $\gamma > 0$ in [6]. Moreover in [6] is proved that if the space X satisfies (3.9) then $\mathcal{K}1 \in BMO$ if and only if $\mathcal{K}$ is a bounded operator in $L^2(X)$.

Let us also mention that in the paper [14], the same result is obtained with even less restrictive conditions on the space and the kernel.

Proof (Lemma (3.2)):

Proof of (3.3): Given $x \in B = B(x_0, r)$ and $z \notin \tilde{B}$ then $d(x, z) > r$ and $d(x, z)$ is equivalent to $d(x_0, z)$, so we have

$$\begin{aligned} |S_{\tilde{B}} f(x, y)| &\leq C d(x, y)^\beta \int_{d(x_0,z)>2Kr} \frac{|f(z)|}{d(x, z)^{1+\beta}}\, d\mu(z) \\ &\leq C r^\beta \int_{d(x_0,z)>2Kr} \frac{|f(z) - f_{\tilde{B}}|}{d(x_0, z)^{1+\beta}}\, d\mu(z) + \int_{d(x,z)>r} \frac{|f_{\tilde{B}}|}{d(x, z)^{1+\beta}}\, d\mu(z) \\ &\leq C r^\beta \left(\frac{C\|f\|_{BMO_\varphi}}{r^\beta} \int_1^\infty \frac{\varphi(rt)}{t^{1+\beta}}\, dt + \frac{1}{r^\beta} |f_{\tilde{B}}| \right). \end{aligned}$$

Proof of (3.4): We see from Lemma (2.2) that $f\chi_{\tilde{B}} \in L^2(X)$ then $\mathcal{K}(f\chi_{\tilde{B}})(x)$ exists almost everywhere, in the following sense

$$\mathcal{K}(f\chi_{\tilde{B}})(x) = \lim_{\varepsilon \to 0} \int_{d(x,y) \geq \varepsilon} k(x, y) f(y) \chi_{\tilde{B}}(y)\, d\mu(y)$$

for almost all $x \in X$. We observe that if we take $x \in B$ one of such points, it is easy to prove that given B' another ball containing x then also exists $\mathcal{K}(f\chi_{\bar{B}'})(x)$. Consequently there exists a measurable set N with $\mu(N) = 0$ such that for every $x \notin N$ and every ball B containing x, $\mathcal{K}(f\chi_{\bar{B}})(x)$ is finite. Let us write $\mathcal{N} = (N \times X) \cup (X \times N)$. Let B and B' be two balls containing x and y. By decomposing $\chi_{\bar{B}}$ as $\chi_{\bar{B}'} + \chi_{\bar{B}-\bar{B}'} - \chi_{\bar{B}'-\bar{B}}$ we see that the definition of $F_y(x)$ with B or B' coincide.

Proof of (3.5): If we take a ball B with x, y and y' belonging to B, it is easy to see that $F_y(x) - F_{y'}(x) = F_y(y')$. ∎

Let us now state and proof the main result of this section

Theorem I: Let k be a kernel satisfying (3.1) and φ a function such that $\Phi(r) = \int_1^\infty \frac{\varphi(rt)}{t^{1+\beta}}\, dt < \infty$. Then the singular integral $\mathcal{K}$ is bounded as an operator from BMO_φ into BMO_Φ.

Proof: Let $B = B(x_0, r)$ be a given ball, pick a point $x_1 \in B$ and take $\mathcal{K}f(x) = F_{x_1}(x)$. From (3.8) we have

$$\frac{1}{\mu(B)} \int_B |F_{x_1}(x) - \frac{1}{\mu(B)} \int_B F_{x_1}(z)\, d\mu(z)|\, d\mu(x) \leq$$
$$\leq \frac{2}{\mu(B)} \int_B |\mathcal{K}((f - f_{\bar{B}})\chi_{\bar{B}})(x)| + |S_{\bar{B}}(f - f_{\bar{B}})(x, x_1)|\, d\mu(x)$$

Let us get the desired bound for each term on the right separately. For the first one, apply Hölder inequality, the L^2 boundeness of the singular integral, Lemma (2.2) and finally the monotonicity of φ to obtain

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B |\mathcal{K}((f - f_{\bar{B}})\chi_{\bar{B}})(x)|\, d\mu(x) &\leq \frac{1}{\mu(B)^{1/2}} \left(\int_X |\mathcal{K}((f - f_{\bar{B}})\chi_{\bar{B}})(x)|^2\, d\mu(x) \right)^{1/2} \\
&\leq \frac{C}{\mu(B)^{1/2}} \left(\int_{\bar{B}} |f - f_{\bar{B}}|^2(x)\, d\mu(x) \right)^{1/2} \\
&\leq C\|f\|_{BMO_\varphi} \varphi(r) \\
&\leq C\|f\|_{BMO_\varphi} \Phi(r).
\end{aligned}$$

For the second, from (3.3) of lemma (3.2) we conclude that

$$\frac{1}{\mu(B)} \int_B |S_{\bar{B}}(f - f_{\bar{B}})(x, x_1)|\, d\mu(x) \leq C\|f\|_{BMO_\varphi} \Phi(r). \ \blacksquare$$

(3.10) **Corollary:** Let k be a kernel satisfying (3.1), and φ such that $\Phi(r) \leq C\varphi(r)$ then $\mathcal{K}$ is a continuous operator from BMO_φ into BMO_φ and so is from Λ_φ on BMO_φ.

§4. Theorem II

In this section we restrict ourselves to the space $X = I\!R^n$, d a quasi-distance invariant under traslations, i.e. $d(x,y) = d(x-y)$ and μ the Lebesgue measure. We assume that $(I\!R^n, d, \mu)$ is a normal space of homogeneous type of order β. Let us observe that this space is non-atomic and $\mu(X) = \infty$. Let k be a kernel of a singular integral operator satisfying (3.1) such that $k(x,y) = k(x-y)$.

Let g be a given function of class $C^1(I\!R^n)$, such that $g = 1$ on the d-ball $B(x_0, R/2)$ and $g = 0$ outside $B(x_0, R)$. Let h be a Lipschitz-β function with support contained in $B(x_0, R)$ and $\int h\, d\mu = 1$. Then the function

$$\eta(x) = g(x) - h(x) \int g(y) d\mu(y) \tag{4.1}$$

is of class Lipschitz-β with compact support and mean value zero.

(4.2) **Lemma:** With the same notation used in §2 we have

(4.3) $\psi_r(x,y) = \psi_r(y,x)$.

(4.4) $\psi_r \circ \psi_s = \psi_s \circ \psi_r$.

(4.5) If $\mathcal{K}$ is a singular integral operator with kernel k then $\mathcal{K} \circ \psi_r = \psi_r \circ \mathcal{K}$.

(4.6) For r big enough, $supp\psi_r(\psi_r(\eta)) \subset B(x_0, Cr)$ and $|\psi_r(\psi_r(\eta))| \leq \frac{C}{r^{\beta+1}}$.

(4.7) $\int \psi_r(\psi_r(\eta)) = 0$.

Proof of (4.3): Since $d(x,y) = d(x-y)$ we conclude that $g_r(x) = g_r(0) = g_r(y)$.

Proof of (4.4):

$$\begin{aligned}
\psi_r(\psi_s(f))(x) &= \frac{1}{g_r(0)g_s(0)} \int \psi\left(\frac{d(x,y)}{r}\right) \int \psi\left(\frac{d(y,z)}{s}\right) f(z)\, d\mu(z)\, d\mu(y) \\
&= \frac{1}{g_r(0)g_s(0)} \int \psi\left(\frac{d(z,0)}{s}\right) \int \psi\left(\frac{d(x,y)}{r}\right) f(y-z)\, d\mu(y)\, d\mu(z) \\
&= \frac{1}{g_s(0)} \int \psi\left(\frac{d(z,0)}{s}\right) (\psi_r f)(x-z)\, d\mu(z) \\
&= \psi_s(\psi_r(f))(x).
\end{aligned}$$

Proof of (4.5): It is enough to prove the statment on C_0^∞

$$\begin{aligned}
\mathcal{K}(\psi_r(f))(x) &= \lim_{\varepsilon\to 0}\int_{d(x,z)>\varepsilon} k(x,z)\psi_r(f)(z)\,d\mu(z)\\
&= \lim_{\varepsilon\to 0}\int_{d(x,z)>\varepsilon} \frac{k(x,z)}{g_r(0)}\int \psi\left(\frac{d(y,0)}{r}\right) f(z-y)\,d\mu(y)\,d\mu(z)\\
&= \lim_{\varepsilon\to 0}\int \frac{\psi\left(\frac{d(y,0)}{r}\right)}{g_r(0)}\int_{d(x,z)>\varepsilon} k(x,z)f(z-y)\,d\mu(z)\,d\mu(y)\\
&= \int \frac{\psi\left(\frac{d(y,0)}{r}\right)}{g_r(0)}\mathcal{K}f(x-y)\,d\mu(y) = \psi_r(\mathcal{K}f)(x).
\end{aligned}$$

Proof of (4.6): Let R be such that $supp\eta \subset B(x_0,R)$ and take $R<r$

$$\psi_r(\psi_r(\eta))(x) = \frac{1}{g_r(0)^2}\int_{B(x_0,r)}\eta(z)\int_{B(x,r)\cap B(z,r)}\psi\left(\frac{d(x,y)}{r}\right)\psi\left(\frac{d(y,z)}{r}\right)d\mu(y)\,d\mu(z).$$

We observe that if $x \notin B(x_0, 4K^2r)$ then $B(x,r)\cap B(z,r)=\emptyset$ for every $z\in B(x_0,r)$ then $supp\psi_r(\psi_r(\eta)) \subset B(x_0,4K^2r)$. Let $x\in B(x_0,4K^2r)$ then since $\int\eta = 0$ and from the condition of order β on the quasi-distance, we have

$$\begin{aligned}
|\psi_r(\psi_r(\eta))(x)| &\le \frac{C}{g_r(0)^2}\int_{B(x_0,R)}|\eta(z)|\int\psi\left(\frac{d(x,y)}{r}\right)\frac{|d(x,z)-d(y,x_0)|}{r}\,d\mu(y)\,d\mu(z)\\
&\le \frac{C}{g_r(0)^2}\int_{B(x_0,R)}|\eta(z)||d(z,x_0)^\beta r^{-\beta}\,d\mu(z)\int\psi\left(\frac{d(x,y)}{r}\right)d\mu(y)\\
&\le \frac{C}{g_r(0)}r^{-\beta}R^\beta\int_{B(x_0,R)}|\eta(z)|\,d\mu(z)\\
&\le \frac{C}{r^{\beta+1}}.
\end{aligned}$$

Proof of (4.7): It follows immediately because $g_r(x)=g_r(0)$. ■

Let us now state and proof the main result of this section which is an extension of the result in [9].

Theorem II: Let φ be a nondecreasing function satisfying the following growth condition

$$r^\beta\int_r^\infty \frac{\varphi(t)}{t^{1+\beta}}\,dt \le C\varphi(r). \tag{4.8}$$

If there exist m singular integral operators $\mathcal{K}_j$ with kernels salisfying (3.1) such that $BMO \subset L^\infty + \sum_{j=1}^m \mathcal{K}_jL^\infty$ then we also have

$$BMO_\varphi \subset \Lambda_\varphi + \sum_{j=1}^m \mathcal{K}_j\Lambda_\varphi.$$

More precisely, if $f \in BMO_\varphi$ then there exist $g_j \in \Lambda_\varphi$ such that $f = g_0 + \sum_{j=1}^m \mathcal{K}_j g_j$ and $\sum_{j=0}^m ||g_j||_{\Lambda_\varphi} \leq C||f||_{BMO_\varphi}$.

This result and Theorem I, give us a characterization of BMO_φ provided that $BMO \subset L^\infty + \sum_{j=1}^m \mathcal{K}_j L^\infty$. In fact we have the following result

(4.9) **Corollary:** Let φ be a nondecreasing function satisfying (4.8). Assume that there exist m singular integral operators $\mathcal{K}_j$ such that $BMO \subset L^\infty + \sum_{j=1}^m \mathcal{K}_j L^\infty$. Then

$$BMO_\varphi = \Lambda_\varphi + \sum_{j=1}^m \mathcal{K}_j \Lambda_\varphi.$$

Once we have proved the lemmas of section two, Theorem I and Lemma (4.2), the proof of Theorem II follow very closely that of Janson [9].

Proof of Theorem II: Let r_0 be a fixed positive real number. Since we may assume without loosing generality that φ is a nondecreasing continuous function, it is clear that the set of integers i, such that $2^i\varphi(r_0)$ belongs to the image of φ, is an integer interval which may be finite or not. Following [9] we shall denote this interval by $[-L, M]$. Thus for every $i \in [-L, M]$ there exists a positive r_i such that $\varphi(r_i) = 2^i\varphi(r_0)$. Let us now take a function f in BMO_φ with norm equal to one. From Lemma (2.6), monotonicity and Δ_2 condition on φ we have, $||f - \psi_r(f)||_{BMO} \leq C\rho(f, Cr) \leq \bar{C}\varphi(r)$ so that,

$$\begin{aligned} ||\psi_{r_i}(f) - \psi_{r_i+1}(f)||_{BMO} &\leq ||\psi_{r_i}(f) - f||_{BMO} + ||f - \psi_{r_i+1}(f)||_{BMO} \\ &\leq C\left(\varphi(r_i) + \varphi(r_{i+1})\right) \\ &\leq C\varphi(r_i). \end{aligned}$$

By our hypothesis the function $\psi_{r_i}(f) - \psi_{r_{i+1}}(f) \in BMO$ can be written

$$\psi_{r_i}(f) - \psi_{r_{i+1}}(f) = \sum_{j=0}^m \mathcal{K}_j u_j^i,$$

where the functions $u_j^i, j = 0, \cdots, m$ belong to L^∞ and $||u_j^i||_\infty \leq C\varphi(r_i)$ and, for simplicity $\mathcal{K}_0$ is the identity operator. Let us define $v_j^i = \psi_{r_i}(u_j^i) + \psi_{r_{i+1}}(u_j^i)$ and $w_j^i = v_j^i - v_j^i(0)$. From Lemma (2.5) we get the following estimate for the modulus of continuity of w_i^j,

$$\begin{aligned} \omega(w_j^i, r) = \sup_{d(x,y)\leq r} |w_i^j(x) - w_i^j(y)| &= \omega(v_j^i, r) \\ &\leq C\left[C(\frac{r}{r_i})^\beta + (\frac{r}{r_{i+1}})^\beta\right] ||u_j^i||_\infty \\ &\leq C\left(\frac{r}{r_{i+1}}\right)^\beta \varphi(r_i). \end{aligned}$$

Since, moreover $\omega(w_j^i, r) \leq 4||u_j^i||_\infty \leq C\varphi(r_i)$ and $|w_j^i(x)| \leq \omega(w_j^i, d(x, x_0))$ we have

$$\begin{aligned}\sum_i |w_j^i(x)| &\leq \sum_i \omega(w_j^i, r)\\ &\leq \sum_{r_i \leq r} C\varphi(r_i) + C\sum_{r_i > r} (\frac{r}{r_i})^\beta \varphi(r_i)\\ &\leq C\left[\varphi(r) + r^\beta \int_r^\infty \frac{\varphi(t)}{t^{\beta+1}}\, dt\right]\\ &\leq C\varphi(r).\end{aligned}$$

In the third inequality we have used the following estimates, first noticing that the sum $\sum_{r_i \leq r} \varphi(r_i)$ is taken only over $i \leq i_0$ where i_0 is such that $2^{i_0}\varphi(r_0) \leq \varphi(r) < 2^{i_0+1}\varphi(r_0)$ we conclude that $\sum_{r_i \leq r} \varphi(r_i) \leq \varphi(r_0) \sum_{-\infty}^{i_0} 2^i \leq C\varphi(r)$, and second summing by parts we obtain

$$\begin{aligned}\sum_k^m \frac{\varphi(r_i)}{r_i^\beta} &= 2\sum_k^m \frac{\varphi(r_i) - \varphi(r_{i-1})}{r_i^\beta}\\ &= 2\sum_k^{m-1} \left[\frac{\varphi(r_i)}{r_i^\beta} - \frac{\varphi(r_i}{r_{i+1}^\beta}\right] + 2\frac{\varphi(r_m)}{r_m^\beta} - 2\frac{\varphi(r_{k-1})}{r_{k-1}^\beta}\\ &\leq 2\beta \int_{r_k}^\infty \frac{\varphi(t)}{t^{\beta+1}}\, dt.\end{aligned}$$

Consequently $\sum_i w_j^i$ converges absolutely to functions g_j with $\omega(g_j, r) \leq C\varphi(r)$, thus $||g_j||_{\Lambda_\varphi} \leq C$. Since $\mathcal{K}1 = 0$ and from (4.5) and (4.4) of Lemma (4.2) we obtain

$$\begin{aligned}\sum_{j=0}^m \mathcal{K}_j w_j^i &= \sum_{j=0}^m \mathcal{K}_j v_j^i\\ &= \sum_{j=0}^m \psi_{r_i}(\mathcal{K}_j u_j^i) + \psi_{r_{i+1}}(\mathcal{K}_j u_j^i)\\ &= (\psi_{r_i} + \psi_{r_{i+1}}) \sum_{j=0}^m \mathcal{K}_j u_j^i\\ &= (\psi_{r_i} + \psi_{r_{i+1}}) \circ (\psi_{r_i} - \psi_{r_{i+1}})(f)\\ &= \psi_{r_i}(\psi_{r_i}(f)) - (\psi_{r_{i+1}}(\psi_{r_{i+1}}(f)).\end{aligned}$$

Now if $<\ ,\ >$ denotes the usual inner product on L^2 we have

$$\begin{aligned}<\eta, \sum_{j=0}^m \mathcal{K}_j g_j> &= \sum_i <\eta, \sum_{j=0}^m \mathcal{K}_j w_j^i>\\ &= \sum_i <\eta, \psi_{r_i}(\psi_{r_i}(f)) - \psi_{r_{i+1}}(\psi_{r_{i+1}}(f))>\\ &= \lim_{i \to -L} <\eta, \psi_{r_i}(\psi_{r_i}(f))> - \lim_{i \to M} <\eta, \psi_{r_i}(\psi_{r_i}(f))>.\end{aligned}$$

Let us consider only the case $\varphi(0) = 0$ and $\varphi(\infty) = \infty$, the other three situations can be treated in a similar way by changings the functions g_j (see[9]). In our case we have that $L = M = \infty$. So that for $i \to -\infty$ we have $r_i \to 0$ then from (4.3) of Lemma (4.2), we conclude $< \eta, \psi_{r_i}(\psi_{r_i}(f)) > = < \psi_{r_i}(\psi_{r_i}(\eta)), f >$ and then $lim_{i \to -\infty} < \eta, \psi_{r_i}(\psi_{r_i}(f)) > = < \eta, f >$ since $\psi_{r_i}(\psi_{r_i}(\eta))$ converge to η in the L^2 norm when $r_i \to 0$. On the other hand since $r_i \to \infty$ when $i \to \infty$, we get that $supp\eta \subset B(x_0, r_i)$, for i large enough. From (4.3), (4.6) and (4.7) of Lemma (4.2) we have

$$\begin{aligned} < \eta, \psi_{r_i}(\psi_{r_i}(f)) > &= < \psi_{r_i}(\psi_{r_i}(f)), \eta > \\ &= \int_{B(x_0, Cr_i)} \psi_{r_i}(\psi_{r_i}(f))(x) \left(f(x) - f_{B(x_0, Cr_i)} \right) d\mu(x) \\ &\leq C \sup |\psi_{r_i}(\psi_{r_i}(\eta))| \mu(B(x_0, r_i)) \varphi(Cr_i) \\ &\leq \frac{C}{r_i^{\beta}} \varphi(Cr_i) \end{aligned}$$

then $\lim_{i \to \infty} < \eta, \psi_{r_i}(\psi_{r_i}(f)) > = 0$. Consequently we have

$$< \eta, \sum_{j=0}^{m} \mathcal{K}_j g_j > = < \eta, f >$$

for every η with mean value zero, thus, except for an additive constant, $f = \sum_{j=0}^{m} \mathcal{K}_j g_j$. Finally since $||g_j||_{\Lambda_\varphi} \leq C$ and $||f||_{BMO_\varphi} = 1$ we have the desired norm inequality . ∎

§5. Theorem III

To each $n \times n$ diagonalizable matrix A and each $\lambda > 0$, we associate the non isotropic dilations whose matrix is given by $T_\lambda = e^{A log \lambda}$ where $\lambda > 0$. Let us also assume following [8] that the eigenvalues of A are large enough in order to have a unique solution $\rho = \rho(x)$ of $||T_{\frac{1}{\rho}}(x)|| = 1$ such that $\rho(x-y)$ becomes a translation invariant distance on $\mathbb{R}^n$. Moreover being μ the Lebesgue measure on $\mathbb{R}^n$ we have that $(\mathbb{R}^n, \rho, \mu)$ is a space of homogeneous type. Given $B = B(x_0, r)$ a ρ-ball in $\mathbb{R}^n$ we have that $\mu(B) = Cr^\tau$ where $\tau = \sum_i^n a_{ii}$ is the trace of A. The function $d(x, y) = \rho^\tau(x - y)$ is a quasi-distance of order τ^{-1} on $\mathbb{R}^n$ and $(\mathbb{R}^n, d, \mu)$ becomes a normal space of homogeneous type. Coifman and Dahlberg in [4] proved that there exist $2n$ operators $\mathcal{K}_i$ which provide the following characterization of the maximal Hardy space $H^1(\mathbb{R}^n, d, \mu)$

$$H^1(\mathbb{R}^n, d, \mu) = \{f \in L^1(\mathbb{R}^n) : \mathcal{K}_i f \in L^1(\mathbb{R}^n) \quad i = 1, \ldots, 2n\}.$$

For the maximal function definition of Hardy spaces on the even more general situation of space of homogeneous type see [13]. The operators $\mathcal{K}_i$ are defined as multipliers on the Fourier transform by $\widehat{\mathcal{K}_i f}(\xi) = w_i(\xi)\widehat{f}(\xi)$ where w_i is homogeneous of degree zero with respect to T_λ i.e. $w_i(T_\lambda \xi) = w_i(\xi)$, and when restricted to S^{n-1}, $w_i \in C^\infty(S^{n-1})$. We must observe that the w_i can be chosen in such a way that the operators $\mathcal{K}_i$ are singular integrals with kernels satisfying (3.1) on the space of homogeneous type $(I\!R^n, d, \mu)$.
From the atomic decomposition of $H^1(I\!R^n, d, \mu)$ in [13], Macías and Segovia obtain the extension of Fefferman-Stein duality $(H^1)^* = BMO$. This result together with the above mentioned result by Coifman and Dahlberg readily gives

$$BMO(I\!R^n, d, \mu) = L^\infty + \sum_{i=1}^{2n} \mathcal{K}_i L^\infty.$$

Finally, on account of Theorem II, we have the following result

Theorem III: If φ satisfies the growth condition

$$r^{\frac{1}{\tau}} \int_r^\infty \frac{\varphi(t)}{t^{1+\frac{1}{\tau}}}\, dt \leq C\varphi(r)$$

then the space BMO_φ related to the quasi-metric d can be written as

$$BMO_\varphi(I\!R^n, d, \mu) = \Lambda_\varphi(I\!R^n, d) + \sum_{i=1}^{2n} \mathcal{K}_i \Lambda_\varphi(I\!R^n, d).$$

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