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MEASURES WITH STATIONARY MEAN

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Let (Ω,Γ,μ) be a probability space and T a measurable transformation on (Ω,Γ,μ). In this paper we prove some necessary and sufficient conditions for the existence of

$$\lim_{n\to\infty}\frac{\mu(E)+\mu\left(T^{-1}E\right)+\ldots+\mu\left(T^{-n}E\right)}{n+1},E\in\Gamma.$$

1.INTRODUCTION. Let (Ω,Γ,μ) be a probability space and T a measurable transformation on (Ω,Γ,μ). The basic coding theorems of information theory are applications of the ergodic and the Shannon-McMillan theorems, and it is customary to assume that μ is stationary in order to invoke these results, but several important models arising in information theory are not stationary. In what follows we give some characterizations of measures for which the ergodic theorem holds for T.

We say that μ *is asymptotically mean stationary with respect to* T if for any $E\in\Gamma$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\left(T^{-i}E\right) \text{ exists,}$$

in which case it follows from the Vitali-Hahn-Saks theorem that

$$\overline{\mu}(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\left(T^{-i}E\right)$$

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is a probability measure. $\bar{\mu}$ is obviously stationary with respect to T, that is, $\bar{\mu}\left(T^{-1}E\right)=\bar{\mu}(E)$, $E\in\Gamma$, and we call $\bar{\mu}$ *the stationary mean* of μ. In what follows we merely say that μ is a. m. s.

A measure η on Γ is said to asymptotically dominate μ (with respect to T), if $E\in\Gamma$ and $\eta(E)=0$ implies that $\lim_{n\to\infty}\mu\left(T^{-n}E\right)=0$.

Fix (Ω,Γ,μ) and T for the rest of this paper. We say that a measurable set E is invariant if $T^{-1}E=E$.

R. Gray and J. Kieffer (1980) proved the following results:

THEOREM 1. μ is a. m. s. if, and only if, every bounded measurable $f:\Omega\to R$ satisfies the individual ergodic theorem, that is, $\frac{1}{n}\sum_{i=0}^{n-1} f\circ T^{i}$ converges μ- a. e. as $n\to\infty$

THEOREM 2. Let η be a finite measure on Γ. If η is stationary and asymptotically dominates μ, then μ is a. m. s.

2. THE RESULTS.

THEOREM 3. μ is a. m. s. if and only if there exist a finite measure ν on Γ and a function $\varphi:[0,\infty]\to[0,\infty]$, with $\lim_{t\to0}\varphi(t)=0$, such that

$$\text{(a) } \limsup_{n\to\infty}\frac{\mu(E)+\mu\left(T^{-1}E\right)+\ldots\mu\left(T^{-n}E\right)}{n+1}\le\varphi\left(\nu(E)\right),E\in\Gamma$$

PROOF. Let ν and φ be given. We shall utilize Banach limits , which are also called generalized limits, and may be introduced as follows (see Dunford-Schwartz (1958)). Let S be the space of all bounded sequences $s=\{s_n\}$ of real numbers equipped with the sup norm. Let S_0 be the smallest closed subspace of S containing all sequences $t=\{t_n\}$ such that the sequence $s(t)=\left\{\sum_{j=1}^{n}t_j\right\}$ is an element of S. Since $e=\{1\}$ is not in S_0, it follows from the Hahn-Banach theorem , that there exists a continuous linear functional L on S such that $\|L\|=1$, $L(e)=1$ and $L(s)=0$ if $s\in S_0$. Every such L is said to be a Banach limits on S and is easy to see that :

1) $L\left(\left\{s_n\right\}\right)=L\left(\left\{s_{n+1}\right\}\right)$

2) $L(\{s_n\}) \geq 0$ if s_n is a non-negative sequence.

3) $\liminf s_n \leq L(\{s_n\}) \leq \limsup s_n$

4) If $\{s_n\}$ is a convergent sequence , then $L(\{s_n\}) = \lim_{n\to\infty} s_n$.

Now , we fix a Banach limit L and define for each set $E \in \Gamma$

$$\bar{\mu}(E) = L\left(\left\{\frac{\mu(E)+\mu(T^{-1}E)+\ldots+\mu(T^{-n}E)}{n+1}\right\}\right).$$

Since L is linear, $\bar{\mu}$ is a finitely additive set function. By 2) we have $0 \leq \bar{\mu}(E) \leq 1$, and if $E \subset F$ then $\bar{\mu}(E) \leq \bar{\mu}(F)$. Now, let $\{E_n\}$ be a decreasing sequence of sets in Γ such that $\lim_{n\to\infty} E_n = \varnothing$. From (a) and 3) we obtain $\lim_{n\to\infty} \bar{\mu}(E_n) = 0$, and therefore $\bar{\mu}$ is countably additive. Moreover , by 1) and 4), $\bar{\mu}$ is stationary and $\bar{\mu}(E) = \mu(E)$ for any invariant set $E \in \Gamma$.

Now, let $E \in \Gamma$ satisfy $\bar{\mu}(E) = 0$. Then $\limsup_{n\to\infty} T^{-n}E$ is invariant and has $\bar{\mu}$- measure zero and hence also μ-measure zero. Thus we have $\lim\sup_{n\to\infty} \mu(T^{-n}E) \leq \mu(\limsup_{n\to\infty} T^{-n}E) = 0$. Then $\bar{\mu}$ asymptotically dominates μ and from theorem 2 we conclude that μ is a.m.s. and $\bar{\mu}$ is the stationary mean of μ.

For the converse take $\nu = \bar{\mu}$ and $\varphi(t) = t$.

In the following, we shall assume that T is nonsingular, that is, $E \in \Gamma$ and $\mu(E) = 0$ implies $\mu(T^{-1}E) = 0$. Hence, the operator T defined by $Tf = f \circ T$ is a positive linear contraction on $L_\infty(\Omega, \Gamma, \mu) = L_\infty(\mu)$, and if S is the adjoint operator of T, then S defines a positive linear contraction on $L_1(\Omega, \Gamma, \mu) = L_1(\mu)$.

For any measurable $f: \Omega \to R$ we denote $M_n(K)f$ by $\dfrac{f + K(f) + \ldots + K^n(f)}{n+1}$, where $K = T$ or $K = S$.

THEOREM 4. The following conditions are equivalent :

1) μ is a.m.s.

2) $M_n(S)g$ converges in the norm topology of $L_1(\mu)$ for all $g \in L_1(\mu)$.

3) $M_n(S)1$ is weakly convergent in $L_1(\mu)$.

4) There exists a nonnegative function $h \in L_1(\mu)$ such that $Sh = h$ and $\lim_{n\to\infty} \mu(T^{-n}H) = 1$, where $H = \operatorname{supp} h$.

PROOF. 1) $\Rightarrow$ 2) follows from a mean ergodic theorem (see e.g. U. Krengel (1985), Theorem 2.1.5), and it is obvious that 2) $\Rightarrow$ 3).

3) $\Rightarrow$ 4) : Suppose h is the weak limit of $M_n(S)1$. Hence, h is a nonnegative function such that $Sh = h$ and $\|h\|_1=1$. Let $H = \operatorname{supp} h$. We have $\mu(H-T^{-1}H)=0$, and therefore $H\subset T^{-1}H\subset\ldots\subset T^{-n}H\subset\ldots$, ($\mu$ – a.e.). Thus

$$\lim_{n\to\infty}\mu\left(T^{-n}H\right)=\lim_{n\to\infty}\frac{\mu(H)+\mu\left(T^{-1}H\right)+\ldots\mu\left(T^{-n}H\right)}{n+1}=\lim_{n\to\infty}\int M_n(T)\chi_H\,d\mu$$

$$=\lim_{n\to\infty}\int_H M_n(S)1\,d\mu=1.$$

4) $\Rightarrow$ 1) : We define $\nu(E)=\int_E h\,d\mu$ for $E\in\Gamma$. Hence ν is stationary with respect to T. We shall prove that ν asymptotically dominates μ. If $\nu(E)=0$, then $E^* = \limsup_{n\to\infty} T^{-n}E$ is invariant and has ν-measure 0. Thus $E^*\subset D=\Omega-H$, (μ – a.e.), and therefore $E^*\subset T^{-n}D$ for all $n\geq 0$. Then $\limsup_{n\to\infty}\mu\left(T^{-n}E\right)\leq\mu\left(E^*\right)=0$, and 1) follows from theorem 2.

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