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GEOMETRIC STRUCTURE OF THE FAREY-BROCOT SEQUENCE

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ABSTRACT: The Farey-Brocot (F-B) sequence can be found in a number of problems, both in mathematics and in the empirical sciences. The problem of finding a general formula $a_n(j)$ for the j^{th} element in the n^{th} F-B sequence is partially solved in this paper, by giving information on what are the rational numbers in the n^{th} F-B sequence. First we show that it is sufficient to know the denominators in the n^{th} F-B; then we normalize the set of these denominators so that the first and last ones are the extremes of the unit interval and, as n tends to infinity, we obtain a limit set $\Omega\subset[0,1]$. We study the geometrical structure of Ω, which is endowed with strong self-similar geometry, described here in a precise quantitative way.

INTRODUCTION. Let us review some of the physical problems in which the Farey-Brocot (F-B) sequence shows itself. Per Bak [1] explored the forced-pendulum problem by means of a dynamical system. For certain critical values of the parameters involved we have that, when plotting the winding number w as a function g of the internal frequency ω of the system, then $w = g(\omega)$ is a Cantor-type staircase, i.e. an increasing function in the unit interval, constant on each interval of resonance I_k, $k \in I\!N$. We also have that $\mu^1\{[0,1] - \sum_{k\in I\!N} I_k\} = \mu^1(\Omega) = 0$, where μ^1 is the usual Lebesgue measure on $I\!R$. The set Ω has Hausdorff dimension $d_H(\Omega) = 0.868 \pm 0.0002$. Also the constant value of g over each I_k is a rational number.

Cvitanovic, Jensen, Kadanoff, and Procaccia [2] discovered a property of this staircase: Let $\frac{P}{Q}$ and $\frac{P'}{Q'}$ be the values of w for a pair of intervals of resonance I and I', such that all intervals of resonance in the gap between I and I' are smaller in size than both I and I'. Then there is an interval of resonance I'' in this gap such that the corresponding constant value of w is $\frac{P''}{Q''} = \frac{P+P'}{Q+Q'}$, and we will see below that F-B rationals are constructed exactly in this way. Also, this interval I'' is the widest of all intervals of resonance in the gap between I and I'. This is a purely empirical finding.

Cvitanovic et al. [2] and Halsey et al. [3] have different examples of other physical phenomena exhibiting Cantor-type staircases with the F-B arrangement, and we can find this arrangement in some of the staircases shown in [1] including the chemical reaction of Belusov-Zabotinsky.

Finally, Bruinsma and Bak [4] studied the magnetic structure of ferromagnetic quasicrystals by plotting the ratio of up spins against the strength of magnetic field applied to the quasicrystalline structure, when only 2 values of each spin are allowed, i.e. + and -, or up and down. Again the result is a Cantor staircase with the Farey arrangement.

In our attempt at finding a mathematical model of the multifractal spectra associated with the physical phenomena referred to so far, we found all the necessary tools within Hyperbolic Geometry [5]. But again, the most important tiling of the Hyperbolic plane is precisely the Farey tiling [6]. We wish to stress the empirical connection between the Farey arrangement in the staircases referred to above, and the underlying fractal sets $\Omega = [0,1] - \sum_{k \in I\!N} I_k$, each I_k a resonance interval. Although the sets Ω involved in the corresponding literature are not as completely self-similar as, e.g., the Koch snowflake, they still possess important self-similar characteristics, which become apparent in a multifractal decomposition of Procaccia of each such Ω. Thus, it becomes relevant to look for self-similar properties when studying the F-B geometric structure.

SECTION 1

SECTION 1.1 Let $\frac{a}{b}$ and $\frac{c}{d}$ be two positive rational numbers such that $\frac{a}{b} < \frac{c}{d}$. We have that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. (1)

Also $\begin{vmatrix} a & a+c \\ b & b+d \end{vmatrix} = \begin{vmatrix} a+c & c \\ b+d & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$. Let $\frac{0}{1}$ and $\frac{1}{1}$ be $\frac{a}{b}$ and $\frac{c}{d}$. Then (1) can be written as $\frac{0}{1} < \frac{1}{2} < \frac{1}{1}$ (Step 1). With $\frac{0}{1}$ and $\frac{1}{2}$ and with $\frac{1}{2}$ and $\frac{1}{1}$ taking turns as $\frac{a}{b}$ and $\frac{c}{d}$ we have $\frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1}$ (Step 2). Iterating this procedure once more we obtain Step 3: $\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \ldots$ and so on.

The n^{th} step of this procedure is known as the Farey-Brocot (F-B) sequence of order n. If we started the F-B procedure more generally, from numbers $\frac{a}{b}$ and $\frac{c}{d}$, as in (1), then the 4^{th} F-B sequence would be

$$\frac{a}{b}, \frac{4a+c}{4b+d}, \frac{3a+c}{3b+d}, \frac{5a+2c}{5b+2d}, \frac{2a+c}{2b+d}, \frac{5a+3c}{5b+3d}, \frac{3a+2c}{3b+2d}, \frac{4a+3c}{4b+3d},$$

$$\frac{a+c}{b+d}, \frac{3a+4c}{3b+4d}, \frac{2a+3c}{2b+3d}, \frac{3a+5c}{3b+5d}, \frac{a+2c}{b+2d}, \frac{2a+5c}{2b+5d}, \frac{a+3c}{b+3d}, \frac{a+4c}{b+4d}, \frac{c}{d}.$$

Notice that each one of the $2^n + 1 = 2^4 + 1$ elements in this n^{th} (4^{th}) F-B sequence is of the form $\frac{Na+Mc}{Nb+Md}$. We are, therefore, interested in the (N, M) pairs obtained in the n^{th} F-B sequence. These (N, M) pairs for different values of n form the so called Brocot Tree in Fig.1. Notice that in the 2^{nd} step of the tree we obtain the symmetric (N, M) pairs (2,1) and (1,2), whereas in the next step the pairs (3,1) and (3,2) are the symmetric of (2,3) and (1,3). This symmetry, carried down through all the steps in the tree, permits as to study just one half of the tree, e.g. the half shown in Fig.2. Let us draw the reduced tree in Fig.2 up to the 3^{rd} step. Let us leave (1,1) out (see Fig.3). If we write down the 1^{st} coordinate of each (N, M) pair, going from top to bottom and from left to right –i.e. 1, 2, 3, 3, 4, 5, 5, 4– we obtain the 2^{nd} coordinate of the next (4^{th}) step in the tree of Fig.2. Therefore we are interested in studying the evolution of the 1^{st} coordinate only of each (N, M) pair, i.e. the numerical tree shown in Fig.4. We invite the reader to notice that each number in this tree is the sum of two numbers obtained in previous steps in a unique way. Again, there is a strong symmetry between left and right halves of the tree, and we will work on one half only.

SECTION 1.2

In what follows n will enumerate the step θ_n, and with k we will denote the location of the number in the step θ_n, so a_k^n is in the k^{th} place in step θ_n. In Fig.5 we observe the numerical tree developed from step θ_0 till θ_6. Studying the numerical tree we observe

$$\begin{cases} a_{2k-1}^{n+1} = a_k^n, & k = 1, ..., 2^n + 1 \\ a_{2k}^{n+1} = a_k^n + a_{k+1}^n, & k = 1, ..., 2^n \end{cases} \tag{2}$$

Henceforth, we will concentrate in the study of the integers that appear in step θ_n, as $n \longrightarrow \infty$.

SECTION 2

SECTION 2.1

The Fibonacci sequence is the sequence $\{u_n\}_{n \in I\!N}$, such that $u_n = u_{n-1} + u_{n-2}$ and $u_{-1} = 1$; $u_0 = 0$. Let us locate these Fibonacci numbers in our numerical tree. In Fig.6 (full line) we observe these numbers as indicated by the zig-zag $\mathcal{Z}(1,2)$ that starts from $u_2 = 1$ and $u_3 = 2$ from left to right.

SECTION 2.2

The "Fibonacci-type" sequence is obtained by $u_n = u_{n-1} + u_{n-2}, ...$ but u_{-1} and u_0 are any other pair (x, y). In Fig.6 we see a Fibonacci-type sequence with $(x, y) = (2, 5)$ indicated by the dotted zig-zag line, $\mathcal{Z}(x, y) = \mathcal{Z}(2, 5)$, drawn starting from right to left, as opposed to the ordinary Fibonacci sequence zig-zag $\mathcal{Z}(1, 2)$ drawn from left to right (full line). In the same figure, with $(x, y) = (1, 3)$ we have again a left to right zig-zag. Other left-to-right and right-to-left examples are shown in the same figure. Next, let us compare Fig.6 with Fig.7. In Fig.6 we can observe $\mathcal{Z}(1,2)$, $\mathcal{Z}(1,3)$, $\mathcal{Z}(2,5)$, $\mathcal{Z}(3,8)$ and $\mathcal{Z}(5,13)$ intertwined in a certain way. In Fig.7 $\mathcal{Z}(1,3)$, $\mathcal{Z}(1,4)$, $\mathcal{Z}(3,7)$, $\mathcal{Z}(4,11)$ and $\mathcal{Z}(7,18)$ are intertwined in exactly the same way as the $\mathcal{Z}(x, y)$ in Fig.6,

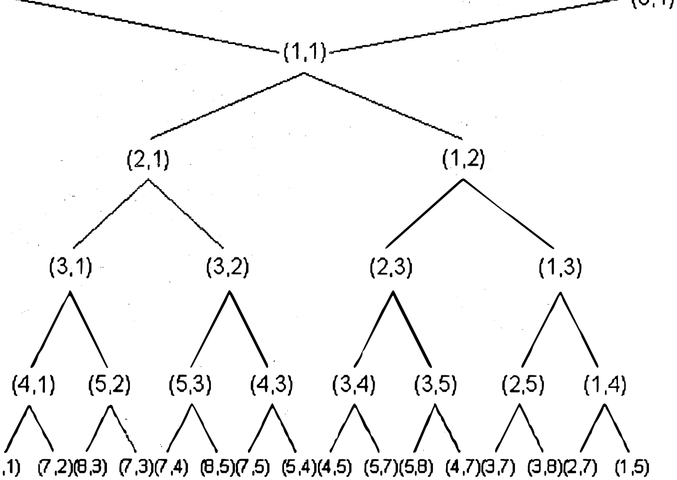

(1,1)
(2,1)
(1,2)
(3,1)
(3,2)
(2,3)
(1,3)
(4,1)
(5,2)
(5,3)
(4,3)
(3,4)
(3,5)
(2,5)
(1,4)
(7,2)(8,3) (7,3)(7,4) (8,5)(7,5) (5,4)(4,5) (5,7)(5,8) (4,7)(3,7) (3,8)(2,7) (1,5)

Fig.1

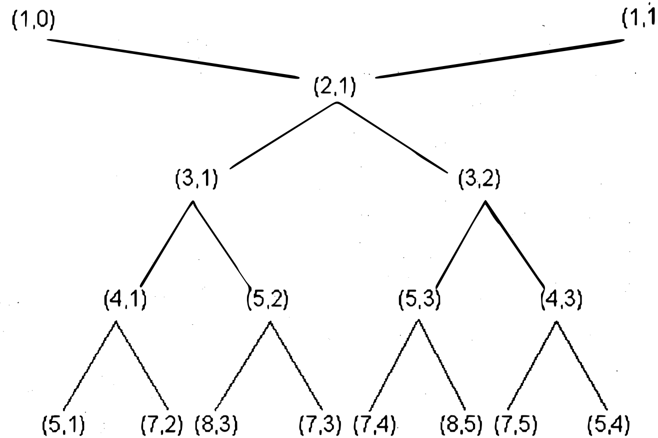
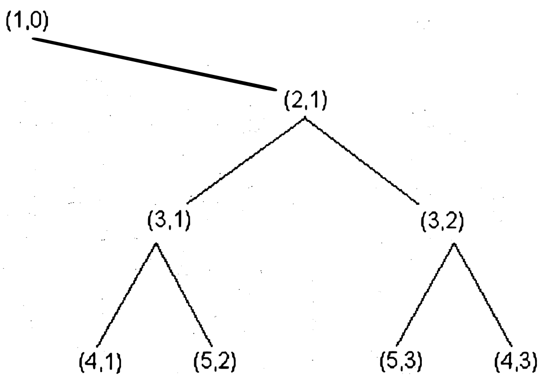
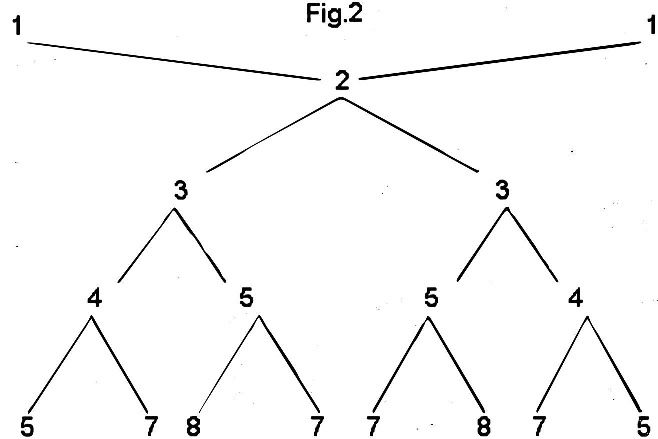

(1,0)
(1,1)
(2,1)
(3,1)
(3,2)
(4,1)
(5,2)
(5,3)
(4,3)
(5,1) (7,2) (8,3) (7,3) (7,4) (8,5) (7,5) (5,4)

Fig.2


(1,0)
(2,1)
(3,1)
(3,2)
(4,1)
(5,2)
(5,3)
(4,3)


1
1
2
3
3
4
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5 7 8 7 7 8 7 5

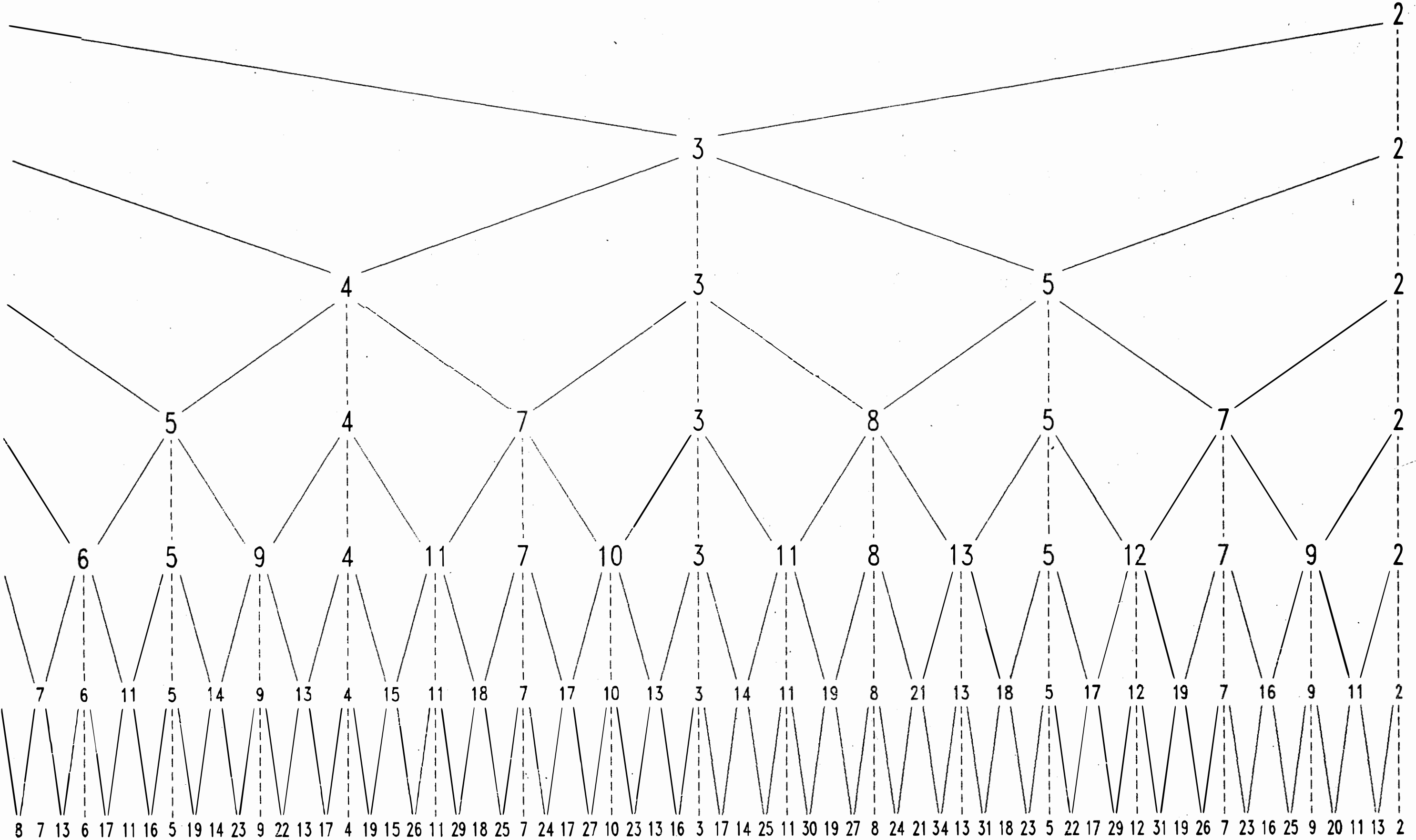
2
3 2
4 3 5 2
5 4 7 3 8 5 7 2
6 5 9 4 11 7 10 3 11 8 13 5 12 7 9 2
7 6 11 5 14 9 13 4 15 11 18 7 17 10 13 3 14 11 19 8 21 13 18 5 17 12 19 7 16 9 11 2
8 7 13 6 17 11 16 5 19 14 23 9 22 13 17 4 19 15 26 11 29 18 25 7 24 17 27 10 23 13 16 3 17 14 25 11 30 19 27 8 24 21 34 13 31 18 23 5 22 17 29 12 31 19 26 7 23 16 25 9 20 11 13 2

Fig. 5

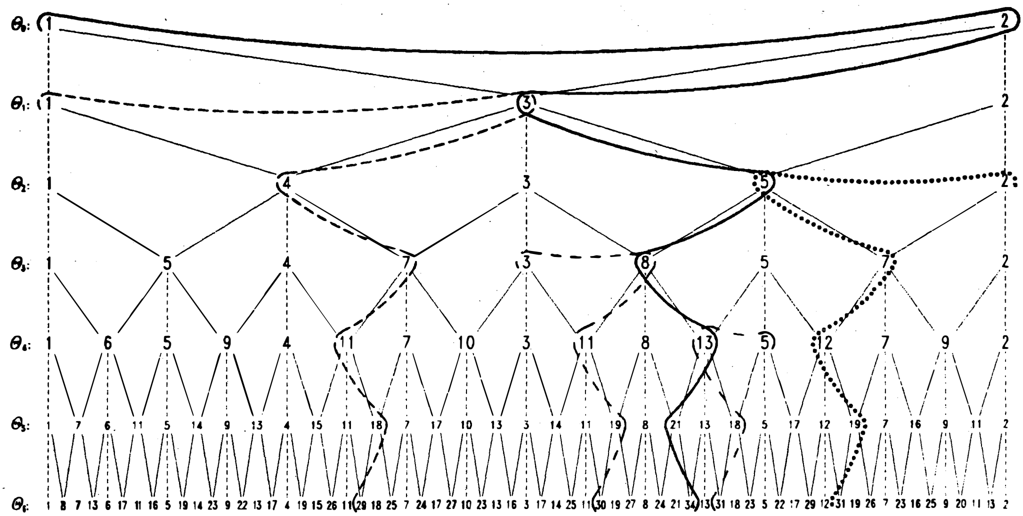

Fig. 6

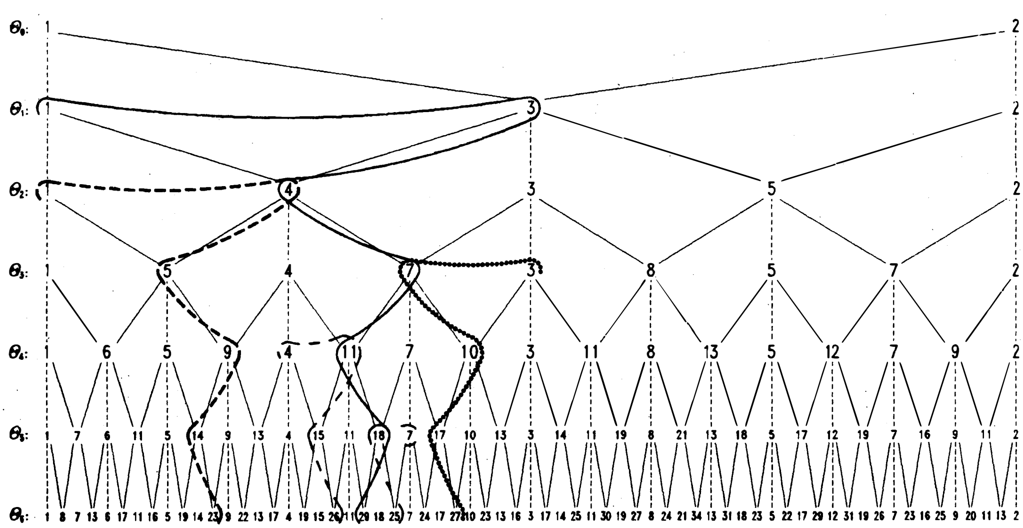

Fig. 7

as can be seen by direct observation. What one sees in Fig.7 is but what is seen in Fig.6 except for a "change of scale": The $\mathcal{Z}(x,y)$ in Fig.7 are simply smaller than those in Fig.6. Were it not for this change of scale the two diagrams are structurally identical. The expression "self-similar" will denote this connection between the diagrams in the two Figs. In Fig.6, we notice that $\mathcal{Z}(1,2)$ generated $\mathcal{Z}(1,3)$ (and also $\mathcal{Z}(2,5)$, etc.), and, in Fig.7 this same $\mathcal{Z}(1,3)$, in turn, generated $\mathcal{Z}(1,4)$ (and many others). This remark, and a moment of reflection, show that all steps θ_n of the numerical tree can be reached by a $\mathcal{Z}(x_k, y_k)$ generated by a $\mathcal{Z}(x_{k-1}, y_{k-1})$ generated by ... by $\mathcal{Z}(1,2)$, the original Fibonacci zig-zag. It should be remarked that the initial (x,y) from which a $\mathcal{Z}(x,y)$ arises has a unique $(k, k+1)$ location in a unique step θ_n of the numerical tree. It should also be remarked that, given the self-similarity observed in the structures drawn from Figs.6 and 7, then the study of our numerical tree can be started in any $\mathcal{Z}(x,y)$—not necessarily $\mathcal{Z}(1,2)$— for any well defined initial pair of values (x,y).

SECTION 3

In what follows, we will take a certain zig-zag $\mathcal{Z}(x,y)$; for a certain initial pair (x,y), then take the associated smaller zig-zags intertwined with it —Figs.6 and 7— and we will put this whole visual structure (generated by (x,y)) in formulas.

SECTION 3.1 - NOTATION

Given an initial pair (x,y), we define the elements of $\mathcal{Z}(x,y)$ by: $\alpha_{-1} = x$, $\alpha_0 = y$, $\alpha_n = \alpha_{n-1} + \alpha_{n-2}$, and with "$\alpha$" we will simply denote the whole sequence $\{\alpha_n\}_{n \geq -1}$. Notice that α is $\mathcal{Z}(x,y)$ —the first zig-zag from the pair (x,y). We will keep denoting with greek letters β, γ... all Fibonacci-type zig-zags. α will be called the "first generation sequence from the pair (x,y)". Let us define the (infinite) second generation sequence from (x,y). A sequence $\beta = \{\beta_n\}_{n \geq -1}$ is a Fibonacci-type sequence of second generation from (x,y) when there exists $m \in \mathbb{N}$ such that the initial pair fullfils $(\beta_{-1}, \beta_0) = (\alpha_{m-2}, \alpha_m)$. In this case, we will say that the initial pair of β was generated in the step m of the sequence α, and we will indicate this fact by the notation β^m instead of just β. In step n of the first generation sequence α (from the pair (x,y)) we would, all in all, have: the n^{th} term of α ($\alpha_n = F_n x + F_{n+1} y$, where F_k is the k^{th} Fibonacci number), the $(n-1)^{th}$ term from β^1 ($\beta^1_{n-1} = F_{n-1}\alpha_{-1} + F_n \alpha_1$), the $(n-2)^{th}$ term from sequence β^2 ($\beta^2_{n-2} = F_{n-2}\alpha_0 + F_{n-1}\alpha_2$),..., the second term of β^{n-2} ($\beta^{n-2}_2 = F_2 \alpha_{n-4} + F_1 \alpha_{n-2}$), the first term of β^{n-1} ($\beta^{n-1}_1 = F_1 \alpha_{n-3} + F_0 \alpha_{n-1}$) and the initial pair of data for β^n —that is $\beta^n_{-1} = \alpha_{n-2}$ and $\beta^n_0 = \alpha_n$. In general, $\beta^m_j = F_j \alpha_{m-2} + F_{j+1} \alpha_m \;\forall\; j \geq -1$.

SECTION 3.2

In what follows we will assume $2x \leq y$, where (x, y) is our initial pair (this entails no loss of generality, as $2x = y$ for the first proper Fibonacci sequence with initial pair $(x,y) = (1,2)$, and $2x < y$ for all other Fibonacci-type sequence pairs).

Proposition. *Let $n \in I\!N$; $n \geq 2$, therefore:*

a) $2\beta^m_{-1} < \beta^m_0 \quad \forall\ m = 1, ..., n-1;$

b) $\beta^m_j \leq \beta^m_{n-m} \quad \forall\ j = 1, ..., n-m; \quad m$ *as in a);*

c) $\beta^m_{n-m} < \alpha_n, \quad m$ *as in a);*

d) $\beta^m_{n-m+2} = \beta^{m-1}_{n-m+2} + \beta^{m-2}_{n-m+2}, \quad m = 3, ..., n+1;$

e) $\beta^m_{n-m+2} = \beta^m_{n-m+1} + \beta^m_{n-m}, \quad m = 1, ..., n-1.$

Proof. Trivial.

Notice that $d)$ and $e)$ imply that, if one knows $\beta^1_1, \beta^1_2, \beta^2_1$, and β^2_2 one can know all the other β^h_k. Thus, we make the different values of different sequences β^m notationally independent of the values in the sequence α, and also independent of the particular values of x and y.

Let $n \in I\!N$, for $k = 1, ..., n$; let us consider all elements β^k_{n-k+1} —in the $(n+1)^{th}$ step of α they are the elements of any second generation β sequence. In terms of $\beta^1_1, \beta^1_2, \beta^2_1$, and β^2_2 these numbers can be easily seen to be

$$\beta^k_{n-k+1} = F_{k-2}\left(F_{n-k-1}\beta^1_1 + F_{n-k}\beta^1_2\right) + F_{k-1}\left(F_{n-k-1}\beta^2_1 + F_{n-k}\beta^2_2\right). \tag{3}$$

Equally easy it is to see that $\beta^1_2 \leq \beta^2_1$ and $2\beta^2_1 + \beta^1_1 \leq \beta^2_2 + \beta^1_2$. (4)

Notice that the finite sequence β^k_{n-k+1}, for $k = 1, ..., n$ (i.e. $\beta^1_{n-1+1}, \beta^2_{n-2+1}, ..., \beta^n_{n-n+1}$) is not in monotonically increasing order. We want to permute the indices such that the permuted finite sequence is re-ordered in an increasing way. The permutation is $\prod_{j=1}^{\left[\frac{n+1}{4}\right]} (2j, n+2-2j)$, denoted by σ_n. This σ_n is a product of $\left[\frac{n+1}{4}\right]$ two-term cycles $2j \quad n+2-2j$ applied in succession, for $j = 1, ..., \left[\frac{n+1}{4}\right]$, in a way illustrated in the following example: Let $n = 14$; then our β^k_{n-k+1}, $k = 1, ..., n$ are

$$(S): \ \beta^1_{14}, \beta^2_{13}, \beta^3_{12}, \beta^4_{11}, \beta^5_{10}, \beta^6_9, \beta^7_8, \beta^8_7, \beta^9_6, \beta^{10}_5, \beta^{11}_4, \beta^{12}_3, \beta^{13}_2 \text{ and } \beta^{14}_1$$

whereas $\left[\frac{n+1}{4}\right] = 3$. Therefore, $\sigma_{14} = \prod_{j=1}^{3} (2j,\ 16-2j) = (2,14).(4,12).(6,10)$ switches positions 2 and 14, then 4 and 12 and after that 6 and 10, thus:

$$(S): \ \beta^1_{14}, \beta^2_{13}, \beta^3_{12}, \beta^4_{11}, \beta^5_{10}, \beta^6_9, \beta^7_8, \beta^8_7, \beta^9_6, \beta^{10}_5, \beta^{11}_4, \beta^{12}_3, \beta^{13}_2, \beta^{14}_1$$

$$(S'): \ \beta^1_{14}, \beta^{14}_1, \beta^3_{12}, \beta^4_{11}, \beta^5_{10}, \beta^6_9, \beta^7_8, \beta^8_7, \beta^9_6, \beta^{10}_5, \beta^{11}_4, \beta^{12}_3, \beta^{13}_2, \beta^2_{13}$$

$$(S''): \ \beta^1_{14}, \beta^{14}_1, \beta^3_{12}, \beta^{12}_3, \beta^5_{10}, \beta^6_9, \beta^7_8, \beta^8_7, \beta^9_6, \beta^{10}_5, \beta^{11}_4, \beta^4_{11}, \beta^{13}_2, \beta^2_{13}$$

$$(S'''): \ \beta^1_{14}, \beta^{14}_1, \beta^3_{12}, \beta^{12}_3, \beta^5_{10}, \beta^{10}_5, \beta^7_8, \beta^8_7, \beta^9_6, \beta^6_9, \beta^{11}_4, \beta^4_{11}, \beta^{13}_2, \beta^2_{13}$$

Sequence S''' is now in increasing order as we can verify directly. Furthermore from sequence S''':

$$\underbrace{\beta_{14}^{1},\ \beta_{1}^{14}},\ \underbrace{\beta_{12}^{3},\ \beta_{3}^{12}},\ \underbrace{\beta_{10}^{5},\ \beta_{5}^{10}},\ \underbrace{\beta_{8}^{7},\ \beta_{7}^{8}},\ \underbrace{\beta_{6}^{9},\ \beta_{9}^{6}},\ \underbrace{\beta_{4}^{11},\ \beta_{11}^{4}},\underbrace{\beta_{2}^{13},\ \beta_{13}^{2}}$$

we can readly verify that

$$\begin{aligned}
\beta_1^{14}-\beta_{14}^{1} &= F_{13}\left(\beta_1^2-\beta_2^1\right)=F_{13}A=F_{2.7-1}A=F_{2j(1)-1}A\\
\beta_3^{12}-\beta_{12}^{3} &= F_{9}\left(\beta_1^2-\beta_2^1\right)=F_{9}A=F_{2.5-1}A=F_{2j(2)-1}A\\
\beta_5^{10}-\beta_{10}^{5} &= F_{5}\left(\beta_1^2-\beta_2^1\right)=F_{5}A=F_{2.3-1}A=F_{2j(3)-1}A\\
\beta_7^{8}-\beta_{8}^{7} &= F_{1}\left(\beta_1^2-\beta_2^1\right)=F_{1}A=F_{2.1-1}A=F_{2j(4)-1}A\\
\beta_9^{6}-\beta_{6}^{9} &= F_{3}\left(\beta_1^2-\beta_2^1\right)=F_{3}A=F_{2.2-1}A=F_{2j(5)-1}A\\
\beta_{11}^{4}-\beta_{4}^{11} &= F_{7}\left(\beta_1^2-\beta_2^1\right)=F_{7}A=F_{2.4-1}A=F_{2j(6)-1}A\\
\beta_{13}^{2}-\beta_{2}^{13} &= F_{11}\left(\beta_1^2-\beta_2^1\right)=F_{11}A=F_{2.6-1}A=F_{2j(7)-1}A
\end{aligned}$$

where the sequence of subindices "$j(k)$", $k=1,\dots,7$ of the Fibonacci numbers $F_{2j(k)-1}$ at the right hand side of each equation follows the law

$$\begin{array}{rccccccc}
k: & 1 & 2 & 3 & 4 & 5 & 6 & 7\\
j(k): & 7 & 5 & 3 & 1 & 2 & 4 & 6
\end{array}$$

From S''' we can calculate other differences:

$$\beta_{14}^{1},\ \underbrace{\beta_{1}^{14},\ \beta_{12}^{3}},\ \underbrace{\beta_{3}^{12},\ \beta_{10}^{5}},\ \underbrace{\beta_{5}^{10},\ \beta_{8}^{7}},\ \underbrace{\beta_{7}^{8},\ \beta_{6}^{9}},\ \underbrace{\beta_{9}^{6},\ \beta_{4}^{11}},\ \underbrace{\beta_{11}^{4},\beta_{2}^{13}},\ \beta_{13}^{2}$$

$$\begin{aligned}
\beta_{12}^{3}-\beta_{1}^{14} &= F_{11}B=F_{2.6-1}B=F_{2j(1)-1}B\\
\beta_{10}^{5}-\beta_{3}^{12} &= F_{7}B=F_{2.4-1}B=F_{2j(2)-1}B\\
\beta_{8}^{7}-\beta_{5}^{10} &= F_{3}B=F_{2.2-1}B=F_{2j(3)-1}B\\
\beta_{6}^{9}-\beta_{7}^{8} &= F_{1}B=F_{2.1-1}B=F_{2j(4)-1}B\\
\beta_{4}^{11}-\beta_{9}^{6} &= F_{5}B=F_{2.3-1}B=F_{2j(5)-1}B\\
\beta_{2}^{13}-\beta_{11}^{4} &= F_{9}B=F_{2.5-1}B=F_{2j(6)-1}B
\end{aligned}$$

where, in this case, $B=\beta_2^2-2\beta_1^2+\beta_2^1-\beta_1^1$. The law $k\to j(k)$, $k=1,\dots,6$, follows

$$\begin{array}{rcccccc}
k: & 1 & 2 & 3 & 4 & 5 & 6\\
j(k): & 6 & 4 & 2 & 1 & 3 & 5
\end{array}$$

In the case where n is odd the situation is slightly more complicated, but analogous to that just depicted. In the general case we have:

Theorem. *Let the β^p_q, p and q in $I\!N$, be defined as above. We have:*

a) Given the finite sequence β^k_{n-k+1}, $k=1,...,n$; the reordered sequence $\beta^{\sigma_n(k)}_{n-\sigma_n(k)+1}$, $k=1,...,n$ is placed in increasing order.

b) Let $n\equiv 0(2)$.

b_1) Then we have $\beta^{n+2-2k}_{2k-1}-\beta^{2k-1}_{n+2-2k}=F_{2j(k)-1}A$, $k=1,...,\frac{n}{2}$, where the function $k\to j(k)$ is

$$\begin{array}{lcccccccc} k: & 1 & 2 & 3 & 4 & \ldots & \frac{n}{2}-3 & \frac{n}{2}-2 & \frac{n}{2}-1 & \frac{n}{2} \\ j(k): & \frac{n}{2} & \frac{n}{2}-2 & \frac{n}{2}-4 & \frac{n}{2}-6 & \ldots & \frac{n}{2}-7 & \frac{n}{2}-5 & \frac{n}{2}-3 & \frac{n}{2}-1 \end{array}$$

and $A=\beta^2_1-\beta^1_2$.

b_2) $\beta^{2k+1}_{n-2k}-\beta^{n+2-2k}_{2k-1}=F_{2j(k)-1}B$, $k=1,...,\dfrac{n}{2}-1$, where B is as in our example, and $j(k)$ follows an analogous spiral-like configuration.

c) Let $n\equiv 1(2)$. The case where n is odd will be done sticking five subcases c_1)...c_5). Notice that in the example above ($n=14$) we had two spiral-like laws for $k\to j(k)$: the first spiral began with $k=1$, $j(1)=7$, whereas in the second spiral —one turn shorter— we had $j(1)=6$. In what follows we need to distinguish these cases with, say, notations $j_7(k)$ and $j_6(k)$ for the two spirals just depicted— and with notation $j_n(k)$ in the general case when $j(1)=n$. With this in mind we have:

c_1) $\beta^{n+2-2k}_{2k-1}-\beta^{2k-1}_{n+2-2k}=F_{2j_{[\frac{n}{2}]}(k)}A$, $k=1,...,\left[\frac{n+1}{4}\right]$, where A is as in b_1).

c_2) $\beta^{2k+1}_{n-2k}-\beta^{n+2-2k}_{2k-1}=F_{2j_{[\frac{n}{2}]-1}(k)}B$, $k=1,...,\left[\frac{n}{4}\right]$, where B is as in b_2).

c_3) $\left|\beta^{[\frac{n}{2}]+2}_{[\frac{n}{2}]}-\beta^{[\frac{n}{2}]+1}_{[\frac{n}{2}]+1}\right|=C$, where $C=\beta^2_2-\beta^2_1-\beta^1_1$.

c_4) $\beta^{n+1-2k}_{2k}-\beta^{2k}_{n+1-2k}=F_{2j_{[\frac{n}{2}]}(k)}A$, $k=\left[\frac{n+1}{4}\right]+1,...,\left[\frac{n}{2}\right]$ (A as before).

c_5) $\beta^{2k+2}_{n-1-2k}-\beta^{n+1-2k}_{2k}=F_{2j_{[\frac{n}{2}]-1}(k)}B$, $k=\left[\frac{n}{4}\right]+1,...,\left[\frac{n}{2}\right]-1$ (B as before).

Proof: The proof of this theorem is both long and very technical. The details are in [7].

SECTION 4: NORMALIZATION - THE HALIOTIS PARVUS

SECTION 4.1. THE SPIRAL DIAGRAM.

According to the proposition in section 3 we have: $\beta^2_{n-1}\le\alpha_{n+1}$ and $\alpha_{n+1}-\beta^2_{n-1}=F_{n-1}x=F_{n-1}B$. We start with an observation: For clarity let us work with the concrete example of section 3, where n is even ($n=14$). The β^p_q involved in increasing order, and the geometrical configuration given by the differences in parts (b) of the theorem can be put together in one diagram, thus:

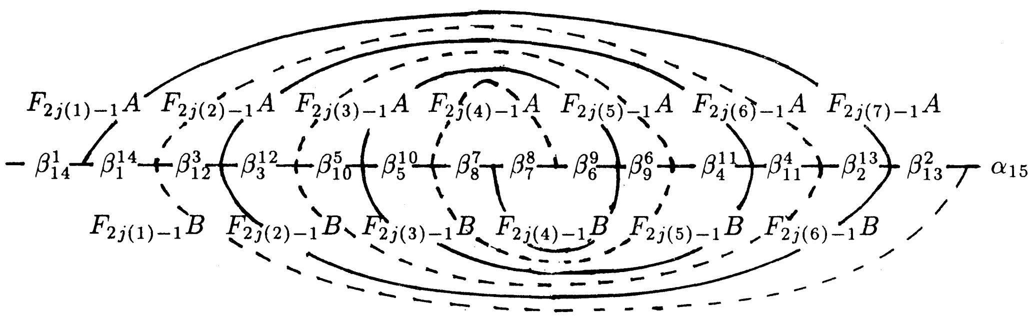

Diagram (D)

Notice that $\frac{F_{2j_7(7)-1}A}{F_{2j_7(1)-1}A} = \frac{F_{11}}{F_{13}} = \frac{\phi^{11}-\psi^{11}}{\phi^{13}-\psi^{13}} = \frac{1}{\phi^2}\cdot\frac{1-\frac{1}{\phi^{22}}}{1-\frac{1}{\phi^{26}}}$, a quantity very much like $\frac{1}{\phi^2}$; here $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. We invite the reader to follow the spiral in order to verify that, e.g., $\frac{F_{2j_7(2)-1}A}{F_{2j_7(7)-1}A}$ (5) and $\frac{F_{2j_7(4)-1}A}{F_{2j_7(5)-1}A}$ (6) are, likewise, very much like $\frac{1}{\phi^2}$. The same can be seen for the other spiral joining the B-differences. For growing values of n, these quotients —analogous to (5) and (6)— are more and more like $\frac{1}{\phi^2}$. For short we will group all the β's and α's in diagram (D) with the abbreviation: $S_n(x,y) = \left\{\beta_{n-\sigma_n(k)+1}^{\sigma_n(k)}, \alpha_{n+1}\right\}$ and we will keep in mind that such an $S_n(x,y)$ is a pair of spirals. Notice that, if $(x,y) = (1,2)$ —and only in this case— we do not have two spirals, but one.

SECTION 4.2. NORMALIZATION

Let us go back to step θ_m in section 1 (see Fig.5). The smallest number in this finite sequence is always 1, and the largest is F_{m+3}. If we normalize step θ_m, by F_{m+3}, for growing values of m we obtain:

a) Each normalized set $\left\{\frac{x}{F_{m+3}}, x \in \theta_m\right\} = \omega_m$ will be maximally contained in $\left[\frac{1}{F_{m+3}}, 1\right]$. In the limit $n \to \infty$, we obtain a set Ω maximally contained in $[0,1]$.

b) Notice that our $S_m(x,y)$, like the one in diagram (D), is in some step θ_ℓ. On the value of this ℓ we can say the following: let us recall that all elements in $S_m(x,y)$ started in an initial (x,y) pair, and this (x,y) pair had a unique location in a certain θ_p... then $\ell = p + m + 1$. Once we normalize this θ_ℓ (by $F_{\ell+3}$ all elements in $S_m(x,y)$ have their own corresponding location inside the normalized ω_ℓ..., and for growing values of ℓ they acquire their precise location in Ω.

c) Diagram (D) —and all such $S_m(x,y)$ diagrams— is mapped by the normalizing process $F_{m+3} \to \omega_m \to \Omega$ into a corresponding diagram D_Ω in $\Omega \subset [0,1]$. On this normalized D_Ω we may observe that:

c$_1$) It is still, like (D), a spiral diagram.

c$_2$) Quotients like (5) and (6) in section 4.1 —and all such quotients in that section— are now exactly $\frac{1}{\phi^2}$, instead of "very much like $\frac{1}{\phi^2}$" or "more like $\frac{1}{\phi^2}$". There is an element in Nature with exactly these characteristics, i.e. a spiral with a $\frac{1}{\phi^2}$ growth per half turn: it is the Haliotis Parvus (H-P). We will use this H-P=H-P(x, y) expression in order to denote a D_Ω diagram of a normalized $S_m(x, y)$.

c$_3$) Going to part (c) of the Theorem we observe that, when n is odd, the corresponding normalized diagram D_Ω will be no different from the case in which n is even (Theorem, part (b)), since the constant C in part (c) will have gone to zero.

c$_4$) Since, as we saw, $\min_{1\leq k\leq n} \beta^k_{n-k+1} = \beta^1_n$, and the corresponding maximum is β^2_{n-1}, then (see part (a) of the Theorem) we have $\beta^2_{n-1} - \beta^1_n = F_n y - F_{n+1} x$ (7)
Notice that (7) implies that in the normalized set Ω, the corresponding H-P(x,y) has a certain length associated with the pair (x, y).

SECTION 5: IRRATIONAL NUMBERS $i(\Omega)$.

In order to facilitate the study of the geometrical structure of Ω we will associate a real number —that will turn out to be irrational— $i(\text{H-P}(x, y)) = i(x, y)$ to any H-P (x, y). Such a real number will simply be the centre of the spiral H-P(x, y) of infinite turns, i.e. the point around which the spiral coils. Let us calculate $i(1, 2)$, i.e., the centre of the H-P spiral located more to the right in Ω—actually H-P$(1,2)$ touches the right boundary point of $[0, 1]$.

Let $n \in I\!N$, and let us consider the corresponding $S_n(1,2)$. We know that $(1,2) \in \theta_0$, $A = y - 2x = 0$, $B = x = 1$ and $S_n(1,2) \subset \theta_{n+1}$. Its left-most element is $\beta^1_n = F_{n+2} + 2F_{n+1}$. Since we know all differences of each two adjacent $\beta^{\sigma_n(k)}_{n-\sigma_n(k)+1}$ (see the Theorem) we can find the centre of the not-normalized $S_n(1,2)$ by adding to β^1_n all the differences between adjacent β's, starting from β^1_n, till we reach the smallest spiral turn. This sum Σ_n is $\beta^1_n + \sum_{j=1}^{[\frac{n}{4}]} F_{n-4j+1}$. Since the biggest element in θ_{n+1} is F_{n+4}, then, when normalizing Σ_n by it, we have

$$i(1,2) = \lim_{n\to\infty} \frac{\Sigma_n}{F_{n+4}} = \lim_{n\to\infty} \frac{F_{n+2} + 2F_{n+1} + \frac{1}{\sqrt{5}}\left[\frac{1}{\phi+2}(\phi^n - 1) - \frac{1}{\psi+2}(\psi^n - 1)\right]}{F_{n+4}} = \frac{2\phi}{\phi+2}$$

Analogously, we have $\quad i(x, y) = \frac{2}{\phi^2(\phi+2)}\phi^{-k}(x + \phi y)$ (8)
for any other suitable pair (x, y), where "k" is the only integer such that $(x, y) \in \theta_k$. The difference between coefficients $\frac{2}{\phi^2(\phi+2)}$ and $\frac{2\phi}{\phi+2}$ hinges on the fact that $A = 0$ only for $(x, y) = (1, 2)$.

SECTION 6: SELF SIMILARITY OF Ω.

SECTION 6.1 IRRATIONAL NUMBERS OF THE FIRST GENERATION.

Let us assume now that all H-P(x, y) have been replaced by, or associated with, a certain irrational $i(x, y)$. Let us refer to these numbers as of the zero generation, and let us denote them by $i_\Omega^{(0)}(x, y)$, for suitable (x, y)-pairs. Let us choose a certain pair (x, y), in, say, θ_n. Let us observe in Fig.6 that the corresponding $\mathcal{Z}(x, y)$ ($\mathcal{Z}(1,2)$ in Fig.6) is directly interwined with other $\mathcal{Z}$'s, e.g. $\mathcal{Z}(1,3), \mathcal{Z}(2,5), \mathcal{Z}(3,8), \mathcal{Z}(5,13)$, among others, in the same Fig. Let us take the corresponding associated pairs (1,3), (2,5), (3,8), (5,13) ... etc. in the same Fig., and let us denote them by $\{(x_h, y_h)\}_{h\in I\!N}$. Let us recall that (see the end of section 3.1) $x = \alpha_{-1}$; $y = \alpha_0$; $x_h = \alpha_{h-2}$ and $y_h = \alpha_h$ where the α_h are the h-Fibonacci-type numbers derived from the initial pair (x, y), i.e. $\alpha_h = F_h x + F_{h+1} y$. Replacing this in (8) we obtain a sequence of irrational numbers $\{i(x_h, y_h)\}_{h\in I\!N}$. We affirm that this sequence is ordered as an H-P. For the quotient of the amplitudes in a half turn is

$$\frac{|i(x_{h+1}, y_{h+1}) - i(x_{h+3}, y_{h+3})|}{|i(x_h, y_h) - i(x_{h+2}, y_{h+2})|} = \frac{1}{\phi^2}.$$

Therefore, in order to calculate the centre of this H-P, it is enough to take the limit $\lim_{h\to\infty} i(x_h, y_h) = \frac{2}{\sqrt{5}} \cdot \frac{2}{\phi^2(\phi+2)} \phi^{-k}(x + \phi y)$, k as before. We will denote these irrationals by $i_\Omega^{(1)}(x, y)$, i.e. the points around which coil the original H-P's. Notice that we can write $i_\Omega^{(1)}(x, y) = \frac{2}{\sqrt{5}} i_\Omega^{(0)}(x, y)$.

SECTION 6.2. The irrational numbers $i_\Omega^{(n)}$ in any generation n have an expression identical to that of $i_\Omega^{(0)}$, except for a change of scale. Indeed, all we have to do is to iterate the last formula in section 6.1. The result:

$$i_\Omega^{(n)}(x, y) = \left(\frac{2}{\sqrt{5}}\right)^n i_\Omega^{(0)}(x, y),$$

where $\left(\frac{2}{\sqrt{5}}\right)^n$ is the change of scale referred to above.

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