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A REMARK ON A NONLINEAR INTEGRAL EQUATION

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ABSTRACT. It is shown that a unique solution to a nonlinear integral equation obtained from the heat equation with nonlinear boundary conditions, can be obtained by successive approximations in the spaces $L^r(0, \infty)$. A natural extension is then considered in the final section.

1. INTRODUCTION

We consider initially, a problem involving the temperature of a semi-infinite heat-conducting solid occupying the half-space $x \geq 0$ and satisfying a nonlinear boundary condition at $x = 0$. By letting $T(x, t)$ be the temperature, the following initial value problem results.

$$T_t(x, t) = T_{xx}(x, t), \qquad x > 0,\ t > 0 \tag{1.1}$$

$$T_x(0, t) = \alpha T^n(0, t) - f(t), \qquad t > 0 \tag{1.2}$$

$$T(x, 0) = 0 \qquad x \geq 0 \tag{1.3}$$

$$T(x, t) \to 0 \text{ as } x \to \infty \qquad t \geq 0 \tag{1.4}$$

where α is a given positive constant and n is a positive integer.

It could be observed that the case where $n = 1$ is equivalent to Newton's Law of Cooling and the case where $n = 4$ is equivalent to the Stefan-Boltzmann Radiation Law. This problem was considered by Keller and Olmstead [2], Mann and Wolf [4], Padmavally[5], and Roberts and Mann [6].

2. ESTIMATES

A solution $T(x, t)$ of equations (1.1) to (1.4) can be obtained in the form

$$T(x, t) = c_1 \int_0^t \frac{\exp[-x/4(t-s)]}{\sqrt{(t-s)}} T^n(0, s)\, ds + c_2 \int_0^t \frac{\exp[-x/4(t-s)]}{\sqrt{(t-s)}} f(s)\, ds \tag{2.1}$$

where c_1 and c_2 are constants.

By setting $x = 0$ in equation (2.1), we obtain the integral equation

$$T(0, t) = c_1 \int_0^t \frac{1}{\sqrt{(t-s)}} T^n(0, s)\, ds + c_2 \int_0^t \frac{1}{\sqrt{(t-s)}} f(s)\, ds \tag{2.2}$$

A solution $T(0, t)$ to equation (2.2) will yield the solution $T(x, t)$ of equations (1.1) to (1.4). Keller and Olmstead [2] showed the existence of this solution by successive approximations. In this note, we show that a contraction mapping does exist in an appropriate $L^p(0, \infty)$ space, namely, f; $\|f\|_p = \left(\int_0^\infty |f(t)|^p\, dt \right]^{1/p} < \infty$ where $1 \leq p < \infty$.

This consequently implies the existence of a unique solution achievable by successive approximations. More precisely, we state the following theorem.

Theorem 1

Equation (2.2) has a unique solution in the space $L^r(0, \infty)$, provided that the function $F_2(t)$ $= c_2 \int_0^t (t-s)^{-1/2} f(s)\, ds$ satisfies the relation $\|F_2\|_r = \left[\int_0^\infty |F(s)|^r\, ds \right]^{1/r} < \varepsilon_0$ where

$\varepsilon_0 = \varepsilon_0(n, c_1, c_2)$ and $r = 2(n - 1)$. The solution is obtainable through the iterations,

$$\phi_{k+1}(t) = c_1 \int_0^t \frac{1}{\sqrt{(t-s)}} \phi_k^n(s)\, ds + c_2 \int_0^t \frac{1}{\sqrt{(t-s)}} f(s)\, ds .$$

Proof (A Version of the Contraction Mapping):

Let us call $\phi(t) = T(0, t)$ and define the mapping

$A(\phi) = c_1 \int_0^t \frac{1}{\sqrt{(t-s)}} \phi^n(s)\, ds + c_2 \int_0^t \frac{1}{\sqrt{(t-s)}} f(s)\, ds$ From the potential theorem

(pages 119 and 120 of [7]), we obtain $\|A(\phi)\|_r \leq C_1 \|\phi\|_r^n + C_2 \|F\|_r$ where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$,

$p = \frac{r}{n}$, $r = 2(n - 1)$, and $F = c_2 \int_0^t \frac{1}{\sqrt{(t-s)}} f(s)\, ds$ Now, consider the convex function

$y = c_1 x^n + y_0$ where $y_0 > 0$ and $n > 1$. The equation $x = c_1 x^n + y_0$ possesses two positive roots if y_0 is small. Call x_1 the smallest positive root. If $0 < x \leq x_1$, then we have

$C_1x^n + y_0 \le x_1$. Hence $\|A(\phi)\|_r \le x_1$ if $\|\phi\|_r \le x_1$. Secondly, if $y_0 \to 0^+$ then $x_1 \to 0^+$. On the other hand, it can be readily seen that

$|A(\phi_1) - A(\phi_2)| \le nc_1 \int_0^t \frac{1}{\sqrt{(t-s)}}\left(|\phi_1| + |\phi_2|\right)^{n-1} |\phi_1 - \phi_2| ds$. On applying the potential

theorem [7] again with $\frac{1}{r} = \frac{n-1}{r} + \frac{1}{r} - \frac{1}{2}$ and $r = 2(n - 1)$, we obtain

$\|A(\phi_1) - A(\phi_2)\|_r \le nC_1 \left\||\phi_1| + |\phi_2|\right\|_r^{n-1} \|\phi_1 - \phi_2\|_r$ and if $\|\phi\|_r \le x_1$ then

$\|A(\phi_1) - A(\phi_2)\|_r \le 2^{n-1} nC_1 x_1^{n-1} \|\phi_1 - \phi_2\|_r$. By selecting $2^{n-1} nC_1 x_1^{n-1} < 1$, the mapping $\phi \to A(\phi)$ becomes a contraction mapping in the ball $\left\{\phi;\ \|\phi\|_r \le x_1\right\}$. Here it should be noted that x_1 is a function of y_1 and moreover, $x_1 \to 0^+$ if $y_0 \to 0^+$. Thus with the establishment of a contraction mapping, we note that if $f \in L^q(0, \infty)$ and $\phi \in L^r(0, \infty)$ with $r = 2(n - 1)$ and $q = \frac{2(n-1)}{n}$, then the solution will always be meaningful for $n > 2$.

In the particular case of Stefan's Radiation Law, where $n = 4$, we obtain $r = 6$ implying that $\|A(\phi)\|_6 \le C_1\|\phi\|_6^4 + C_2\|F\|_6$. The equation $y = c_1x^n + y_0$ then becomes $x = c_1x^4 + y_0$ and x_1 is its smallest positive root. Then for any ϕ_1 and ϕ_2, we can obtain $\|A(\phi_1) - A(\phi_2)\|_6 \le c(2x_1)^3 \|\phi_1 - \phi_2\|_6$. By choosing $c(2x_1)^3 < 1$, we obtain a contraction mapping.

3. A NATURAL EXTENSION

The problem in Section 1 can be stated in a more general context. Namely, to solve the integral equation;

$$u(x) = c \int \frac{1}{\rho(x-y)^{|a|-\beta}} \left(u(y)\right)^k dy + f(x) \tag{3.1}$$

where ρ is the parabolic distance, that is, ρ is the unique root of the equation

$\sum_1^n \left[\frac{x_i}{\rho^{a_i}}\right]^2 = 1$. $a_i \ge 1$, $|a| = a_1 + a_2 + \cdots + a_n$, and $0 < \beta < |a|$. Kernels of the type shown in equation (3.1) arise in the modeling of nonisotropic diffusion. The case where $k = 4$, as indicated earlier, relates to the Stefan-Boltzmann Radiation Law. Equation (3.1) can be solved by using the method of successive approximations.

Due to the appearance of a parabolic potential operator in equation (3.1) we invoke the following theorem, the proof of which can be found in [1].

Theorem 2

Let $u(x)=\int \frac{1}{\rho(x-y)^{|a|-\beta}} v(y)\,dy$, if $v \in L^p(R^n)$ then $u \in L^r(R^n)$ where $\frac{1}{r}=\frac{1}{p}-\frac{\beta}{|a|}$,

$1>\frac{1}{p}-\frac{\beta}{|a|}>0$, and $\|u\|_r \le c\|v\|_p$ (c does not depend on v).

A consequence of this theorem is the following:

Theorem 3

$$\text{Let } u(x)=\int \frac{1}{\rho(x-y)^{|a|-\beta}} v^k(y)\,dy + f(x) \tag{3.2}$$

If $\frac{|a|}{\beta}>\frac{k}{k-1}$, then for $p=(k-1)\frac{|a|}{\beta}$ we have that $\|u\|_p \le c\|v\|_p^k + \|f\|_p$.

The proof of this theorem is a direct consequence of the following theorem, the proof of which follows the same lines used in proving theorem 1.

Theorem 4

Let $f_0=\|f\|_p$ and let x_1 be the smallest positive root of the equation $y = cx^k + f_0$ when $y = x$. If v; $\|v\|_p \le x_1$ then $\|u\|_p \le x_1$. Furthermore, $x_1 \to 0^+$ as $f_0 \to 0^+$ and if $\|f\|_p$ is small enough; then the mapping described by equation (3.2) is a contraction mapping in $L^p(R^n)$ for $p=(k-1)\frac{|a|}{\beta}$.

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