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SUBGROUPS OF THE GALILEO GROUP AND MEASURABLE FAMILIES OF CURVES

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ABSTRACT. The one and two-parameter subgroups of the Galileo group of actions on space-time of spaces dimension one are fully determined. Then, the measurable families of curves having the Galileo group or one of its subgroups as maximal invariance group are found.

1. INTRODUCTION**

The search for the Lie subgroups of a Lie group was initiated by Sophus Lie when he determined all subgroups of the projective group P_n. We intend in this work to obtain all the subgroups of the Galileo group of transformations of space-time of space dimension one. The one, two and three parameters families of measurable submanifolds (curves) of space-time will be also determined. The following result ([5]) will be used throughout:

Theorem 1.1. *Let $Y_1, \dots, Y_r$ be vectors fields on a manifold M, such that*

$$[Y_i,\ Y_j] = \sum_{k=1}^{r} C_{ij}^{k}\ Y_k; \qquad i,\ j = 1, \dots, r \tag{1.1}$$

where the C_{ij}^{k}. are constants. Then, there is a Lie group G whose Lie algebra has the C_{ij}^{k} as structure constants for some basis $X_1, \dots, X_r$, and a local action ϕ of G on M such that $X_{i_M} = Y_i$, $i = 1, \dots, r$

We also mention ([4]) that

Theorem 1.2. *Let G be a Lie group. If H is a Lie subgroup of G then the Lie algebra $\mathbb{H}$ of H is a subalgebra of $\mathbb{G}$, the Lie algebra of G. Each subalgebra of $\mathbb{G}$ is the Lie algebra of exactly one connected Lie subgroup of G.*

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2. The Galileo Group and its Subgroups

2.1 The Galileo Group.

As mentioned above, we restrict overselves to the simplest case of the Galileo group G of actions on space-time of space dimension one.
This group is determined by the equations

$$\begin{cases} r^* = r + vt + c \\ t^* = t + s \end{cases} \tag{2.1}$$

Where v, c, s are the group parameters. Thus G is a Lie group of dimension three. Its infinitesimal transformations are

$$X_1 = t\,\frac{\partial}{\partial r}, \qquad X_2 = \frac{\partial}{\partial r}, \qquad X_3 = \frac{\partial}{\partial t} \tag{2.2}$$

with structure equations

$$\begin{aligned} [X_1, X_1] &= [X_1, X_2] = [X_2, X_2] \\ &= [X_2, X_3] = [X_3, X_3] = 0, \\ [X_1, X_3] &= -X_2 \end{aligned} \tag{2.3}$$

2.2 Two-parameter subgroups of G.

These are the subgroups determined by two linearly independent vector fields Y_1, Y_2 in the linear span of X_1, X_2, X_3 such that their Lie bracket is a linear combination of Y_1, Y_2. Thus, equations

$$\begin{aligned} Y_1 &= \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \\ Y_2 &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \\ [Y_1, Y_2] &= (\alpha_3\beta_1 - \alpha_1\beta_3)\, X_2 \end{aligned} \tag{2.4}$$

with X_i, $i = 1, 2, 3$ as in (2.2), fully determine the two-parameters subgroups of the Lie group G. The following possibilities arise:

(i) $\alpha_1 \neq 0$. We may assume $\alpha_1 = 1$, $\beta_1 = 0$, to get

$$\begin{aligned} Y_1 &= X_1 + \alpha_2 X_2 + \alpha_3 X_3 \\ Y_2 &= \beta_2 X_2 + \beta_3 X_3 \\ [Y_1, Y_2] &= -\beta_3 X_2 = \theta Y_1 + \phi Y_2 \\ &= \theta X_1 + (\theta\alpha_2 + \phi\beta_2)\, X_2 + (\theta\alpha_3 + \phi\beta_3)\, X_3 \end{aligned} \tag{2.5}$$

which together with (2.4) ensures that

$$\theta = 0, \qquad \phi\beta_2 = -\beta_3, \qquad \phi\beta_3 = 0$$

Now, condition $\phi\,\beta_3 = 0$ opens the alternative

$$\begin{cases} \phi \neq 0 & \text{This lead to } \beta_3 = \beta_2 = 0 \text{ which is absurd} \\ \phi = 0 & \text{Then } \beta_3 = 0 \text{ and } \beta_2 \neq 0 \end{cases}$$

So, we may assume $\beta_2 = 1,\ \alpha_2 = 0$ Then

$$\begin{aligned} Y_1 &= X_1 + \alpha_3 X_3 = t\frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t} \\ Y_2 &= X_2 = \frac{\partial}{\partial r} \end{aligned} \tag{2.6}$$

Now if $Y = aY_1 + bY_2$ is in the span of Y_1 y Y_2 its integral curves are determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aY_1 r + bY_2 r = at + b \\ \dfrac{dt}{d\eta} = aY_1 t + bY_2 t = a\alpha_3 \end{cases} \tag{2.7}$$

Integrating this system and determining the transformations sending $(r(0), t(0))$ into $(r(1), t(1))$, we obtain (see [2]) the subgroup H_2^1 given by the system of equations

$$\begin{cases} r^* = r + vt + c \\ t^* = t + kv \qquad k \text{ a constant} \end{cases} \tag{2.8}$$

and having

$$t\frac{\partial}{\partial r} + k\frac{\partial}{\partial t}\,, \qquad \frac{\partial}{\partial r} \tag{2.9}$$

as infinitesimal trasformations

(ii) $\alpha_2 \neq 0$ We may take $\alpha_2 = 1,\ \beta_2 = \alpha_1 = 0$. Then:

$$\begin{aligned} Y_1 &= X_2 + \alpha_3 X_3 \\ Y_2 &= \beta_1 X_1 + \beta_3 X_3 \\ [Y_1, Y_2] &= \alpha_3 \beta_1 X_2 = \theta Y_1 + \phi Y_2 \\ &= \phi\beta_1 X_1 + \theta X_2 + (\theta\alpha_3 + \phi\beta_3) X_3 \end{aligned} \tag{2.10}$$

which together with (2.4) and (2.5) yields

$$\phi\,\beta_1 = 0, \qquad \theta = \alpha_3\,\beta_1, \qquad \theta\,\alpha_3 + \phi\,\beta_3 = 0$$

so that

$$\alpha_3^2\,\beta_1 + \phi\,\beta_3 = 0, \quad \phi\,\beta_1 = 0$$

Condition $\phi\beta_1 = 0$ raises the alternative

$$\begin{cases} \phi \neq 0 \text{ then } \beta_1 = \beta_3 = 0, \text{ This is absurd} \\ \phi = 0, \text{ so that } \alpha_3^2\,\beta_1 = 0. \text{ Then } \begin{cases} \alpha_3 \neq 0 \text{ and } \beta_1 = 0\,,\ \beta_3 \neq 0 \\ \text{or} \\ \alpha_3 = 0 \end{cases} \end{cases}$$

Assuming $\alpha_3,\ \beta_3 \neq 0,\ \beta_1 = 0$ and letting, as we may, $\beta_3 = 1$, produces the two fields

$$\begin{aligned} Y_1 &= X_2 + \alpha_3 X_3 = \frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t} \\ Y_2 &= X_3 \ = \frac{\partial}{\partial t} \end{aligned} \tag{2.11}$$

and if $Y = aY_1 + bY_2$ its integral curves are determined by

$$\frac{dr}{d\eta} = a, \qquad \frac{dt}{d\eta} = a\alpha_3 + b \tag{2.12}$$

Integrating this system and determining the transformations sending $(r(0),\ t(0))$ into $(r(1),\ t(1))$, we obtain the subgroup H_2^2 of equations

$$\begin{cases} r^* = r + c \\ t^* = t + s \end{cases} \tag{2.13}$$

with infinitesimal transformation

$$\frac{\partial}{\partial r} \quad , \quad \frac{\partial}{\partial t} \tag{2.14}$$

If $\alpha_3 = 0$ and $\beta_1 = 0$ the group H_2^2 above is obtained. However, if $\beta_1 \neq 0$, and we assume as we may $\beta_1 = 1$, the vectors fields are

$$\begin{aligned} Y_1 &= \ X_2 \ = \frac{\partial}{\partial r} \\ Y_2 &= X_1 + \beta_3 X_3 = t\frac{\partial}{\partial r} + \beta_3 \frac{\partial}{\partial t} \end{aligned} \tag{2.15}$$

and the corresponding group is H_2^1

(iii) $\alpha_3 \neq 0$. We take $\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 1,\ \ \beta_3 = 0$, to get

$$\begin{aligned} Y_1 &= X_3 \\ Y_2 &= \beta_1 X_1 + \beta_2 X_2 \\ [Y_1, Y_2] &= \beta_1 X_2 = \theta Y_1 + \phi Y_2 \\ &= \phi\beta_1 X_1 + \phi\beta_2 X_2 + \theta X_3 \end{aligned} \tag{2.16}$$

Comparing with (2.4) and (2.5), we get

$$\begin{cases} \theta = 0 \\ \phi\beta_2 = \beta_1 \\ \phi\beta_1 = 0, \text{so that } \begin{cases} \beta_1 = \beta_2 = 0 \ \text{ if } \phi \neq 0 \text{ Absurd} \\ \beta_1 = 0 \ \text{ if } \ \phi = 0 \end{cases} \end{cases}$$

Thus

$$\begin{aligned} Y_1 &= \ X_3 \ = \frac{\partial}{\partial t} \\ Y_2 &= \ \beta_2 X_2 \ = \ \beta_2 \frac{\partial}{\partial r} \end{aligned} \tag{2.17}$$

and the group is H_2^2.

We have proved that

Theorem 2.1. . *The Galileo group G determined by the infinitesimal transformations (2.2) with structure equation (2.3) has the two two-parameter subgroups*

$$H_2^1 = \left\{ t\frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \,,\, \frac{\partial}{\partial r} \right\}$$
$$H_2^2 = \left\{ \frac{\partial}{\partial r} \,,\, \frac{\partial}{\partial t} \right\} \tag{2.18}$$

2.3 One-parameter subgroups.

These are determined by the fields

$$Y = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3,$$

where X_i, $i = 1, 2, 3$ are the infinitesimal transformations of the Galileo group and α_1, α_2, α_3 are constants.
The different possibilities are:

(i) $\alpha_1 \neq 0$. Changing variables , if necessary , we may assume $\alpha_1 = 1$, so that

$$Y = X_1 + \alpha_2 X_2 + \alpha_3 X_3 = (t + \alpha_2)\frac{\partial}{\partial r} + \alpha_3\frac{\partial}{\partial t},$$

i.e, with t instead of $t + \alpha_2$,

$$Y = t\frac{\partial}{\partial r} + \alpha_3\frac{\partial}{\partial t} \tag{2.19}$$

Its integral curve is determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aYr = at \\ \dfrac{dt}{d\eta} = aYt = a\alpha_3 \end{cases} \tag{2.20}$$

Upon integration of this system, the group is defined by the transformations sending $(r(0), t(0))$ into $(r(1), t(1))$ and thus we obtain the group H_1^1 determined by

$$\begin{cases} r^* = r + vt + v^2 k \\ t^* = t + 2vk \end{cases} \tag{2.21}$$

or by infinitesimal transformation

$$t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t} \tag{2.22}$$

(ii) $\alpha_2 \neq 0$. We may assume $\alpha_2 = 1$ and $\alpha_1 = 0$, so that

$$Y = X_2 + \alpha_3 X_3 = \frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t} \tag{2.23}$$

with integral curve determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aYr = a, \\ \dfrac{dt}{d\eta} = aYt = a\alpha_3, \end{cases} \tag{2.24}$$

which yields the group H_1^2 defined by

$$\begin{cases} r^* = r + ca \\ t^* = t + ck, \quad k \text{ a constant} \end{cases} \tag{2.25}$$

or by infinitesimal transformatcion

$$\frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \tag{2.26}$$

(iii) $\alpha_3 \neq 0$. We may assume $\alpha_1 = \alpha_2 = 0$, so that

$$Y = X_3 = \frac{\partial}{\partial t}, \tag{2.27}$$

and its integral curve is determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aYr = 0, \\ \dfrac{dt}{d\eta} = aYt = a \end{cases} \tag{2.28}$$

which upon integration yields the group H_1^3 defined by

$$\begin{cases} r* = r \\ t^* = t + c \end{cases} \tag{2.29}$$

or by infinitesimal transformation

$$\frac{\partial}{\partial t} \tag{2.30}$$

Hence

Theorem 2.2.. *The Galileo group G determined by the infinitesimal transformations (2.2) with structure equations (2.3) has the three one-parameter subgroups*

$$\begin{aligned} H_1^1 &= \left\{ t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t} \right\} \\ H_1^2 &= \left\{ \frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \right\} \\ H_1^3 &= \left\{ \frac{\partial}{\partial t} \right\} \end{aligned} \tag{2.31}$$

3. Some Basic Notions

3.1 Maximal Invariance Subgroup of a family of manifolds.

Let G be a group acting on a manifold M and let F be a q-parameter family of p-dimensional submanifolds of M. If G^* is the subgroup of G leaving globally invariant the family F, (i.e. $s \in G^*$ *and* $v \in F$, implies $s(v) \in F$) and H^* is the subgroup of G^* fixing every submanifold in F, (i.e. $s \in H^*$, and$v \in F$ grants $s(v) = v$), the quotient group $K = G^* / H^*$ is called the **maximal invariance subgroup of F**. The subgroups of K are called the invariance subgroups of F. The group K leaves invariant the family F and has no other transformations but the identity fixing all submanifolds in F.

3.2 Associated Group.

If to each s in an invariance group G of F we associate a transformation β on the parameter of F, the set H of all such transformations is a group isomorphic to G which acts on the parameter space of F. The group H is called the **associated group of G relative to F.**

In [6] it is shown that

Teorem 3.1. *Let G, and H, be isomorphic groups. A necessary and sufficient condition for the existence of a q-parameter family F_q of p-dimensional submanifolds V_p having G as an invariance subgroup is that the matrix*

$$(\xi_h^1(x), \dots, \xi_h^n(x),\ \eta_h^1(\alpha), \dots, \eta_h^q(\alpha)), \quad h = 1, \dots, r \tag{3.1}$$

be of range $r_1 < n + q$. Where $x = (x^1, \dots, x^n)$, $\alpha = (\alpha^1, \dots, \alpha^q)$, $r \geq 1$ and , $\xi_h^i(x)$, $\eta_k^j(\alpha)$ are respectively the coefficients in the infinitesimal transformations of the groups G and H.

Under the above circumstances, the family F_q is determined by the equations

$$\Phi^\lambda\left(\phi^1(x,\alpha), \dots, \phi^{n+q-r_1}(x,\alpha)\right) = 0, \qquad \lambda = 1, \dots, n-p \tag{3.2}$$

where $\phi^k(x,\alpha)$, $k = 1, \dots, n+q-r_1$, are the independent integrals of the system

$$\xi_k^i(x)\,\frac{\partial F^\lambda(x,\alpha)}{\partial x^i} + \eta_k^j(\alpha)\,\frac{\partial F^\lambda(x,\alpha)}{\partial \alpha^j} = 0 \quad k = 1, \dots, r_1 \tag{3.3}$$

and

$$F^\lambda(x,\alpha) = \Phi^\lambda\left(\phi^1(x,\alpha), \dots, \phi^{n+q-r_1}(x,\alpha)\right)$$

Remark 3.1. For our purposes, $\lambda = 1$ and $r_1 = 1, 2$

3.3 Integral invariants of a Lie group.

Let G be an r-parameter group of transformations $\phi(x^1,\ldots,x^n,\ \alpha^1,\ldots,\alpha^r)$ of $\mathbb{R}^n$. A differentiable function $\psi:\mathbb{R}^n\to\mathbb{R}$ is an **integral invariant** of G if

$$\int_{\phi(U)} \psi(x^1,\ldots,x^n)dx^1\ldots dx^n = \int_U \psi(y^1,\ldots,y^n)dy^1\ldots dy^n \tag{3.4}$$

for any transformation ϕ of G where $y^k = \phi^k(x^1,\ldots,x^n;\alpha^1,\ldots,\alpha^r)$ and U is any subset of $\mathbb{R}^n$ where the right hand integral exists.

3.4 Families of measurable submanifolds.

Definition 3.1. A Lie group of transformations of $\mathbb{R}^n$ is measurable if it has a unique integral invariant, except for constant multiples.
A necessary condition for the measurability of a Lie group G is that G be transitive (see [6]).
Let F be a q-parameter family of p-dimensional submanifolds of $\mathbb{R}^n$ and let G be an invariance subgroup of F. Let H be the associated group of G relative to F. If H is measurable, F is said to be **measurable relative to** H **(or** G **)**; and if ψ is the essentially unique integral invariant of H,

$$\int_F \psi(\alpha^1,\ldots,\alpha^q)d\alpha^1\ldots d\alpha^q \tag{3.5}$$

is called the **measure of the family** F **for** H **(or** G**)**, and the q-form

$$\psi(\alpha^1,\ldots,\alpha^q)d\alpha^1\ldots d\alpha^q$$

is called the **invariant density** of F for H

Deltheil [3] has shown that

Theorem 3.2. *The integral invariants of a Lie transformation group are the solutions of the system of partial differential equations*

$$\sum_{i=1}^{n}\frac{\partial}{\partial x^i}[\xi_h^i(x)\ \psi(x)] = 0, \qquad h=1,\ldots,r \tag{3.6}$$

where the ξ_h^i *are the coefficients in the infinitesimal transformations of the group.*

A sufficient condition for the measurability of a family F of submanifolds of $\mathbb{R}^n$ is the measurability of the associated group of the maximal invariance group of F. This condition is also necessary for one, two and three-parameter families.

4. Measurable Families of Curves for the Galileo Group

4.1 Families of one-parameter curves which are measurable for the action of the Galileo group or of one of its subgroups.

Let F be a family of one-parameter curves determined in space-time (r,t) by the equation

$$\psi(r,t,\alpha) = 0, \tag{4.1}$$

α a parameter. If G is the maximal invariance group of F, its associated group H acts on $\mathbb{R}$, and F is measurable if and only if H is.
As Lie proved, the groups acting on $\mathbb{R}$ are the translation, the afin and the projective group, and only the first of these groups is measurable. Thus, H above is $\{\partial/\partial\alpha\}$, the translation group, and G is one of the group H_1^1, H_1^2, H_1^3. We examine the three posibilities:

(1) If the maximal invariance group G of F is

$$H_1^1 = \left\{t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t}\right\} \text{ and its associated group is } H = \left\{\frac{\partial}{\partial \alpha}\right\}$$

then, in (3.6),

$$\xi^1 = t,\ \xi^2 = 2k,\ \eta^1 = 1$$

i.e. ψ must satisfy

$$t\frac{\partial\psi}{\partial r} + 2k\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\alpha} = 0 \tag{4.2}$$

whose solutions are of the form

$$\psi(r - t\alpha + k\alpha^2,\ t - 2k\alpha) = 0, \quad \text{or,} \quad r = t\alpha - k\alpha^2 + \phi(t - 2k\alpha) \tag{4.3}$$

(2) If $G = H_1^2$,

$$H_1^2 = \left\{\frac{\partial}{\partial r} + k\frac{\partial}{\partial t}\right\}, \qquad \text{and} \qquad H = \left\{\frac{\partial}{\partial\alpha}\right\},$$

then $\xi^1 = 1$, $\xi^2 = k$, $\eta^1 = 1$ in (3.3), and $\psi(r,t,\alpha)$ must satisfy

$$\frac{\partial\psi}{\partial r} + k\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\alpha} = 0, \tag{4.4}$$

whose solutions have the form

$$\psi(r-\alpha, t - k\alpha) = 0, \qquad \text{or,} \qquad r = \alpha + \phi(t - k\alpha) \tag{4.5}$$

(3) If $G = H_1^3 = \left\{\frac{\partial}{\partial t}\right\}$ and $H = \left\{\frac{\partial}{\partial \alpha}\right\}$, then $\xi^1 = 0$ $\xi^2 = 1$ $\eta^1 = 1$
so that

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \alpha} = 0 \tag{4.6}$$

whose solutions are of the form

$$\psi(r,\ t-\alpha) \ = \ 0, \qquad \text{or,} \quad r \ = \ \phi(t-\alpha) \tag{4.7}$$

summing up, we have shown that

Theorem 4.1. *The one-parameter families of curves*

$$\begin{aligned} \psi(r - t\alpha + k\alpha^2,\ t - 2k\alpha) &= 0, \\ \psi(r-\alpha,\ t-k\alpha) &= 0, \\ \psi(r,\ t-\alpha) &= 0 \end{aligned} \tag{4.8}$$

have, respectively, as their maximal invariance groups, the one-parameter subgroups

$$\begin{aligned} H_1^1 &= \left\{t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t}\right\} \\ H_1^2 &= \left\{\frac{\partial}{\partial r} + k\frac{\partial}{\partial t}\right\} \\ H_1^3 &= \left\{\frac{\partial}{\partial t}\right\} \end{aligned}$$

of the Galileo group in space-time of space of dimension one.

4.2 Two- parameters families of curves which are measurable for the action of the Galileo group or one of its subgroups.

Let F be the two-parameters family of curves

$$\psi(r,t,\alpha,\beta) \ = \ 0 \tag{4.9}$$

where α, β are the parameters and (r,t) the space-time coordinates.
Let G be the maximal invariance group of F and H its associated group. Since F is measurable, also H is measurable, and therefore transitive. Then dim $H =$ dim $G \geq 2$ and G has to be one of the groups H_2^1, H_2^2, or the full three-parameters Galileo group.

(1) If $G = H_2^1 = \{Y_1, Y_2\}$ where

$$\begin{cases} Y_1 = \ t\dfrac{\partial}{\partial r} + k\dfrac{\partial}{\partial t}, \\ Y_2 = \ \dfrac{\partial}{\partial r} \end{cases} \tag{4.10}$$

with structure determined by $[Y_1,\ Y_2]=0$, then H is a two parameters group with infinitesimal transformations

$$\begin{aligned} A_1 &= \eta_1^1(\alpha,\beta)\frac{\partial}{\partial\alpha}+\eta_1^2(\alpha,\beta)\frac{\partial}{\partial\beta} \\ A_2 &= \eta_2^1(\alpha,\beta)\frac{\partial}{\partial\alpha}+\eta_2^2(\alpha,\beta)\frac{\partial}{\partial\beta} \end{aligned} \tag{4.11}$$

and structure equation $[A_1,\ A_2] = 0$. Then, the corresponding coefficients for equation (3.6) are

$$\begin{cases} \xi_1^1=t,\ \xi_1^2=k,\ \xi_2^1=1,\ \xi_2^2=0 \\ \eta_1^1=\alpha,\ \eta_1^2=0,\ \eta_2^1=0,\ \eta_2^2=\beta \end{cases}$$

and F is determined by the equations

$$\begin{cases} (i)\ t\dfrac{\partial\psi}{\partial r}+k\dfrac{\partial\psi}{\partial t}+\alpha\dfrac{\partial\psi}{\partial\alpha} = 0 \\ (\text{ii})\ \dfrac{\partial\psi}{\partial r}+\beta\dfrac{\partial\psi}{\partial\beta} = 0 \end{cases} \tag{4.12}$$

The solution to $(4.12),(i))$ is the family of curves implicitly given by

$$\psi(\alpha e^{-t/k},\ \beta,\ r-t^2/2k) = 0 \tag{4.13}$$

i.e by $\psi(r,\ t,\ \alpha,\ \beta)=0$ where

$$\psi(r,\ t,\ \alpha,\ \beta)= r-t^2/2k-\phi(\alpha e^{-t/k},\ \beta) \tag{4.14}$$

since $\psi(r,t,\alpha)$ must verify $(4.12),(ii)$, and

$$\frac{\partial\psi}{\partial r}=1;\qquad \frac{\partial\psi}{\partial\beta}=-\frac{\partial\phi}{\partial\beta} \tag{4.15}$$

we get

$$\phi = \ln\beta + f(\alpha e^{-t/k}) \tag{4.16}$$

and

$$\psi(\ r,\ t,\ \alpha,\ \beta) = r-t^2/2k-\ln\beta-f(\alpha\, e^{-t/k})=0 \tag{4.17}$$

is the measurable family of two-parameter curves having H_2^1 as maximal invariance group.

(2) If $G=H_2^2$

$$H_2^2=\left\{\frac{\partial}{\partial r},\ \frac{\partial}{\partial t}\right\} \qquad \text{then} \qquad H=\left\{\alpha\frac{\partial}{\partial\alpha},\ \beta\frac{\partial}{\partial\beta}\right\}$$

is as before the associated group of G, and the corresponding coefficients for (3.6) are

$$\begin{cases} \xi_1^1 = 1, & \xi_1^2 = 0, \quad \eta_1^1 = \alpha, \quad \eta_1^2 = 0 \\ \xi_2^1 = 0, & \xi_2^2 = 1, \quad \eta_2^1 = 0, \quad \eta_2^2 = \beta \end{cases}$$

so that (3.3) becomes

$$\begin{cases} \dfrac{\partial\psi}{\partial r} + \alpha\dfrac{\partial\psi}{\partial\alpha} = 0 \\ \dfrac{\partial\psi}{\partial t} + \beta\dfrac{\partial\psi}{\partial\beta} = 0 \end{cases} \tag{4.18}$$

whose solutions are implicitly given by

$$\psi(\alpha\, e^{-r},\ \beta\, e^{-t}) = 0 \tag{4.19}$$

i.e, F is the family of curves determined by

$$\psi(r,t,\alpha,\beta) = r - \ln\alpha - \phi(t - \ln\beta) \tag{4.20}$$

So far, we have proved that

Theorem 4.2. *The two parameters families of curves :*

$$\begin{aligned} r &= t^2/2k + \ln\beta + \phi(\alpha e^{-t/k}) \\ r &= \ln\alpha + \phi(t - \ln\beta) \end{aligned} \tag{4.21}$$

have, repectively as their maximal invariance groups, the two parameter subgroups

$$\begin{aligned} H_2^1 &= \left\{t\frac{\partial}{\partial r} + k\frac{\partial}{\partial t},\ \frac{\partial}{\partial r}\right\} \\ H_2^2 &= \left\{\frac{\partial}{\partial r},\ \frac{\partial}{\partial t}\right\} \end{aligned} \tag{4.22}$$

of the Galileo group in space-time of space dimension one.

M. Stoka [6] shows that the two-parameter family of measurable curves having as invariance group G the full Galileo group (2.1), are family of straight lines. He also proves that the measurable three-parameter family having G as maximal invariance group is given by

$$\psi(t - \gamma r - \beta, r - \alpha) = 0 \tag{4.23}$$

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