Revista de la
Unión Matemática Argentina
Volumen 40, 1996.

ON THE STRUCTURE OF THE CLASSIFYING RING OF SO(n,1) AND SU(n,1)

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ABSTRACT. Let G_o be a non compact real semisimple Lie group with finite center, and let $U(\mathfrak{g})^K$ denote the centralizer in $U(\mathfrak{g})$ of a maximal compact subgroup K_o of G_o. By the fundamental work of Harish-Chandra it is known that many deep questions concerning the infinite dimensional representation theory of G_o reduce to questions about the structure and finite dimensional representation theory of the algebra $U(\mathfrak{g})^K$, called the classifying ring of G_o. To study the algebra $U(\mathfrak{g})^K$, B. Kostant suggested to consider the projection map $P: U(\mathfrak{g}) \to U(\mathfrak{k})\otimes U(\mathfrak{a})$, associated to an Iwasawa decomposition $G_o = K_o A_o N_o$ of G_o, adapted to K_o. When P is restricted to $U(\mathfrak{g})^K$ P becomes an injective anti-homomorphism of algebras. In this paper we use the characterization of the image of $U(\mathfrak{g})^K$, when G_o =SO(n,1) or SU(n,1) obtained in Tirao [11], to prove that $U(\mathfrak{g})^K \simeq Z(\mathfrak{g}) \otimes Z(\mathfrak{k})$, where $Z(\mathfrak{g})$ and $Z(\mathfrak{k})$ denote respectively the centers of $U(\mathfrak{g})$ and of $U(\mathfrak{k})$. By a well known theorem of Harish-Chandra these two centers are polynomial rings in rank($\mathfrak{g}$) and rank($\mathfrak{k}$) indeterminates, respectively. Thus the algebraic structure of $U(\mathfrak{g})^K$ is completely determined in this two cases.

1. INTRODUCTION

Let G_o be a non compact real semisimple Lie group with finite center, and let K_o denote a maximal compact subgroup of G_o. If $\mathfrak{k} \subset \mathfrak{g}$ denote the respective complexified Lie algebras, let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $U(\mathfrak{g})^K$ denote the centralizer of K_o in $U(\mathfrak{g})$.

By the fundamental work of Harish-Chandra it is known that many deep questions concerning the infinite dimensional representation theory of G_o reduce to questions about the structure and finite dimensional representaton theory of the algebra $U(\mathfrak{g})^K$, called the classifying ring of G_o (cf. Cooper [2]). Briefly, the reason for this is as follows: To any quasi-simple irreducible Banach space representation π of G_o there is associated an algebraically irreducible $U(\mathfrak{g})$-module V which is locally finite for K_o and which determines π up to infinitesimal equivalence. In fact one has a primary decomposition $V = \bigoplus V_\delta$, where the sum is taken over the set $\hat{K}_o$ of all equivalence classes δ of finite dimensional irreducible representations of K_o, and the multiplicity of δ is finite for any $\delta \in \hat{K}_o$. Then, in particular, any V_δ is finite dimensional and hence, a finite dimensional $U(\mathfrak{g})^K$-module. The point is that V itself as a $U(\mathfrak{g})$-module is completely determined by V_δ as a $U(\mathfrak{g})^K$-module

1991 *Mathematics Subject Classification.* Primary 22E46; Secondary 17B35.

Key words and phrases. Semisimple Lie group, universal enveloping algebra, classifying ring, Lepowsky anti-homomorphism.

for any fixed δ when $V_\delta \neq 0$. See Lepowsky and McCollum [10] and Lepowsky [9] for a nice exposition of this. See also Dixmier [3] and Wallach [12].

When $V_{\delta_0} \neq 0$, where δ_0 is the class of the trivial representation of K_0, then π is called spherical. The approach above has been quite successful in dealing with spherical irreducible representations of G_0 (see e.g. Kostant [7]). Indeed, we may take $\delta = \delta_0$ and thus we have only to consider a quotient $U(\mathfrak{g})^K/I$ instead of $U(\mathfrak{g})^K$. Here I is the intersection of $U(\mathfrak{g})^K$ with the left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{k}$. Now by a theorem of Harish-Chandra, $U(\mathfrak{g})^K/I$ is not only commutative but also isomorphic to a polynomial ring in r variables, where r is the split rank of G_0. More precisely one has an algebra exact sequence

$$0 \to I \to U(\mathfrak{g})^K/I \xrightarrow{\gamma} U(\mathfrak{a})^{\widetilde{W}} \to 0 \tag{1}$$

where $\mathfrak{a}$ is the complex abelian Lie algebra associated to an Iwasawa decomposition $G_0 = K_0A_0N_0$ of G_0 adapted to K_0, and $U(\mathfrak{a})^{\widetilde{W}}$ is the ring of $\widetilde{W}$-invariants in $U(\mathfrak{a})$, $\widetilde{W}$ being the translated Weyl group.

To investigate the general (not necessarily spherical) case along these lines one must look at $U(\mathfrak{g})^K$ itself, not just $U(\mathfrak{g})^K/I$. It is known (see e.g. Kostant and Tirao [8]) that the map (1) may be replaced by an exact sequence

$$0 \to U(\mathfrak{g})^K \xrightarrow{P} U(\mathfrak{k})^M \otimes U(\mathfrak{a})$$

where $U(\mathfrak{k})^M$ denote the centralizer of M_0 in $U(\mathfrak{k})$, M_0 being the centralizer of A_0 in K_0 and $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ is given the tensor product algebra structure. Moreover P is an antihomomorphism of algebras. In order to generalize (1) it is necessary to determine the image of P. Towards this end we introduced in Tirao [11] a subalgebra B of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ defined by a set of equations derived from certain imbeddings among Verma modules and the subalgebra $B^{\widetilde{W}}$ of all elements in B which commute with certain intertwining operators. Such operators are in a one to one correspondence with the elements of the Weyl group W and are rather closely related to the Kunze-Stein intertwining operators. In fact the relation of $B^{\widetilde{W}}$ to B may be taken as the generalization of the relation of $U(\mathfrak{a})^{\widetilde{W}}$ to $U(\mathfrak{a})$. In Tirao [11] it is proved that the image of P lies always in $B^{\widetilde{W}}$, and that when G_0 =SO(n,1) or SU(n,1) we have $P(U(\mathfrak{g})^K) = B^{\widetilde{W}}$.

In this paper we use this result to exhibit the structure of $U(\mathfrak{g})^K$ in this two cases. In fact we shall prove that $U(\mathfrak{g})^K \simeq Z(\mathfrak{g}) \otimes Z(\mathfrak{k})$, where $Z(\mathfrak{g})$ and $Z(\mathfrak{k})$ denote respectively the centers of $U(\mathfrak{g})$ and of $U(\mathfrak{k})$. By a well known theorem of Harish-Chandra these two centers are polynomial rings in rank($\mathfrak{g}$) and rank($\mathfrak{k}$) indeterminates, respectively. Thus our work is finished.

Nowadays there are several proofs that $U(\mathfrak{g})^K$ is a polynomial ring (Cooper [2], Benabdallah [1], Knop [6]), nevertheless our approach should prove to be useful to attack the general case, or at least the case when G_0 is any real rank one group.

2. The algebra B

Let $\mathfrak{t}_0$ be a Cartan subalgebra of the Lie algebra $\mathfrak{m}_0$ of M_0. Set $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the corresponding complexification. Then $\mathfrak{h}_0$ and $\mathfrak{h}$ are Cartan subalgebras of $\mathfrak{g}_o$ and $\mathfrak{g}$, respectively. Now we choose a Borel subalgebra $\mathfrak{t} \oplus \mathfrak{m}^+$ of the complexification $\mathfrak{m}$ of $\mathfrak{m}_o$ and take $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{n}$ as a Borel subalgebra of $\mathfrak{g}$. Let Δ^+ be the corresponding set of positive roots, put $\mathfrak{g}^+ = \mathfrak{m}^+ \oplus \mathfrak{n}$ and $\mathfrak{g}^- = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$. Also put $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. Set $\langle , \rangle$ denotes the Killing form of $\mathfrak{g}$ and $(\mu, \alpha) = 2\langle\mu, \alpha\rangle / \langle\alpha, \alpha\rangle$. For $\alpha \in \Delta^+$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $(\mu, \alpha) = \mu(H_\alpha)$ for all $\mu \in \mathfrak{h}^*$. Also set $H_\alpha = Y_\alpha + Z_\alpha$ where $Y_\alpha \in \mathfrak{t}$ and $Z_\alpha \in \mathfrak{a}$. Let $P^+ = \{\alpha \in \Delta^+ : Z_\alpha \neq 0\}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition, associated to K_o, and let θ denote the corresponding Cartan involution. Also let M_o' denote the normalizer of A_o in K_o. Let $\alpha \in P^+$ be a simple root such that $Y_\alpha \neq 0$. Set $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$ where $X_{-\alpha}$is a non zero root vector corrresponding to $-\alpha$.

When $G_o = \mathrm{SO}(n,1)_e$ $(n \neq 3)$ there is only one simple root $\alpha_1 \in P^+$ (if $n = 3$ there are two simple roots $\alpha_1, \alpha_2 \in P^+$). When $G_o = \mathrm{SU}(n,1)$ $(n \geq 2)$ there are exactly two simple roots α_1, α_n in P^+. Set $E_1 = E_{\alpha_1}$ $(n \neq 3)$ and $E_1 = E_{\alpha_1}, E_{\alpha_2}$ when $n = 3$ in the first case, and $E_2 = E_{\alpha_1}, E_3 = E_{\alpha_n}$ in the second case. We shall also use E to designate any one of the vectors E_1, E_2 or E_3 and α for α_1, (α_1 or α_2), α_1 or α_n, respectively. Moreover $Y_\alpha \neq 0$ if $G_o = \mathrm{SO}(n,1)_e$ $n \geq 3$ or $G_o = \mathrm{SU}(n,1)$ $n \geq 2$. From now on we shall take for granted that we are in one of these cases.

From (8) and (9) of Tirao [11] we know that the algebra B is the set of all $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ such that for all $n \in \mathbf{N}$

$$(2) \qquad E^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E^n \qquad \mod (U(\mathfrak{k})m^+)$$

holds for $(E, \alpha) = (E_1, \alpha_1)$ and $(E, \alpha) = (E_2, \alpha_1), (E_3, \alpha_n)$, respectively. Also

$$(3) \qquad B^{\widetilde{W}} = \{b \in B : \delta_w * b(\lambda - \rho) = b(w(\lambda) - \rho) * \delta_w \text{ for all } w \in M_o', \lambda \in \mathfrak{a}^*\}.$$

The algebraic structure of $U(\mathfrak{g})^K$ when G_o =SO(n,1) or SU(n,1) $n \geq 2$ will be determined by induction on n. The case SO(2,1) is quite simple and will be considered later. Thus we shall take up now the case $G_o = \mathrm{SU}(2,1)$. If $\mathfrak{u}$ is any Lie algebra $z(\mathfrak{u})$ will denote the center of $\mathfrak{u}$ and $Z(\mathfrak{u})$ will denote the center of $U(\mathfrak{u})$.

Lemma 1. *If $G_o = \mathrm{SU}(2,1)$ set $Y = Y_{\alpha_1} = -Y_{\alpha_2}$. Also let $0 \neq D \in z(\mathfrak{k})$ and let ζ denote the Casimir element of $[\mathfrak{k}, \mathfrak{k}]$. Then $\{\zeta^i D^j Y^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$. Moreover the canonical homomorphism $\mu : Z(\mathfrak{k}) \otimes Z(\mathfrak{m}) \to U(\mathfrak{k})^M$ is a surjective isomorphism.*

Proof. The set $\{E_2, E_3, D, Y\}$ is a basis of $\mathfrak{k}$. Therefore the monomials $E_2^i E_3^l D^j Y^k$ form a basis of $U(\mathfrak{k})$. Now $\mathfrak{m}$ is one-dimensional and $Y \in \mathfrak{m}$. From Lemma 29 of Tirao [11] it follows that $[Y, E_2] = -(3/2)E_2$ and $[Y, E_3] = (3/2)E_3$. Hence $\{E_2^i E_3^i D^j Y^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$. Now $\zeta = aE_2E_3 + bY^2 + cD^2 + dYD + eY + fD$, $a, b, c, d, e, f \in \mathbf{C}$, $a \neq 0$. Thus $\{\zeta^i D^j Y^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$.

Since $\{\zeta^i D^j\}_{i,j\geq 0}$ is a basis of $Z(\mathfrak{k})$ and $\{Y^k\}_{k\geq 0}$ is a basis of $U(\mathfrak{m}) = Z(\mathfrak{m})$ the first assertion of the lemma implies the second.

Proposition 2. *For $j = 2, 3$ let*

$$B_j = \{b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a}) : E_j^t b(t - (-1)^j Y - 1) = b(-t - (-1)^j Y - 1)E_j^t, t \in \mathbf{N}\}.$$

Then B_j, as an algebra over $\mathbf{C}$, is generated by the algebraically independent elements $\zeta \otimes 1, D \otimes 1, (Y \otimes 1 + (-1)^j \otimes Z + (-1)^j)^2, Y \otimes 1 - 3(-1)^j \otimes Z$ and 1.

Proof. If $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ then by Lemma 1 b can be written uniquely as $b = \sum a_{i,j,k,l}\zeta^i D^j Y^k \otimes Z^l$, $a_{i,j,k,l} \in \mathbf{C}$. Since $[(-1)^j Y, E_j] = -\frac{3}{2}E_j$ $(j = 2,3)$ from Lemma 18 (vi) of Tirao [11] we get

$$E_j^t b(t-(-1)^j Y - 1) = \sum_{i,j,k,l} a_{i,j,k,l}\zeta^i D^j E_j^t Y^k (t-(-1)^j Y - 1)^l$$
$$= \sum_{i,j,k,l} a_{i,j,k,l}\zeta^i D^j (Y + (-1)^j \tfrac{3}{2}t)^k (-\tfrac{t}{2} - (-1)^j Y - 1)^l E_j^t.$$

Thus $b \in B_j$ if and only if for all $i, j, t \in \mathbf{N}$ we have

$$\sum_{k,l} a_{i,j,k,l}(Y + (-1)^j \tfrac{3}{2}t)^k(-\tfrac{t}{2} - (-1)^j Y - 1)^l = \sum_{k,l} a_{i,j,k,l} Y^k(-t - (-1)^j Y - 1)^l.$$

Hence the problem of characterizing all $b \in B_j$ is equivalent to determine all $f \in \mathbf{C}[x_1, x_2]$ such that

$$f(y + (-1)^j \tfrac{3}{2}t, -\tfrac{t}{2} - (-1)^j y - 1) = f(y, -t - (-1)^j y - 1) \tag{4}$$

for all $t, y \in \mathbf{C}$.

For $j = 2, 3$ let $f_j \in \mathbf{C}[x_1, x_2]$ be defined by

$$f(x_1, x_2) = f_j(x_1 + (-1)^j(x_2+1), x_1 - 3(-1)^j(x_2+1)). \tag{5}$$

Then f satisfies (4) if and only if $f_j((-1)^j t, 4y + 3(-1)^j t) = f_j(-(-1)^j t, 4y + 3(-1)^j t)$ for all $t, y \in \mathbf{C}$. Equivalently if and only if

$$f = \sum_{k,l} a_{k,l}(x_1 + (-1)^j(x_2+1))^{2k}(x_1 - 3(-1)^j(x_2+1))^l. \tag{6}$$

From this it follows that B_j is generated by $\zeta \otimes 1, D \otimes 1, (Y \otimes 1 + (-1)^j \otimes Z + (-1)^j)^2, Y \otimes 1 - 3(-1)^j \otimes Z$ and 1. Clearly these elements are algebraically independent.

Now we want to determine the algebra $B = B_2 \cap B_3$. Given $f \in \mathbf{C}[x_1, x_2]$ let $a(f) \in \mathbf{C}[x_1, x_2]$ be defined by $a(f)(x_1, x_2) = f(\sqrt{3}x_1, x_2 - 1)$. Also let T_j $(j = 2,3)$ be the automorphism of $\mathbf{C}[x_1, x_2]$ induced by the linear map: $T_j(x_1) = -\frac{1}{2}(x_1 + (-1)^j\sqrt{3}x_2)$, $T_j(x_2) = -\frac{1}{2}((-1)^j\sqrt{3}x_1 - x_2)$.

Lemma 3. *An element $f \in \mathbf{C}[x_1, x_2]$ satisfies (4) if and only if $T_j(a(f)) = a(f)$ $(j = 2,3)$.*

Proof. First of all for $j = 2, 3$ we compute $T_j(\sqrt{3}x_1 + (-1)^j x_2) = -(\sqrt{3}x_1 + (-1)^j x_2)$ and $T_j(\sqrt{3}x_1 - 3(-1)^j x_2) = \sqrt{3}x_1 - 3(-1)^j x_2$. If we use the notation introduced in (5) we get

$$a(f)(x_1, x_2) = f_j(\sqrt{3}x_1 + (-1)^j x_2, \sqrt{3}x_1 - 3(-1)^j x_2),$$
$$T_j(a(f))(x_1, x_2) = f_j(-(\sqrt{3}x_1 + (-1)^j x_2), \sqrt{3}x_1 - 3(-1)^j x_2).$$

Therefore $T_j(a(f)) = a(f)$ if and only if f_j is even in the first variable. This is the same as saying that f has the form stated in (6), which was shown to be equivalent to (4).

Proposition 4. *Let W denote the group of automorphisms of $\mathbf{C}[x_1, x_2]$ generated by T_2, T_3. Then:*
(i) W is isomorphic to the Weyl group of $\mathfrak{su}(2,1)$.
(ii) The algebra $\mathbf{C}[x_1, x_2]^W$ of all W-invariants is generated by the algebraically independent polynomials $x_1^2 + x_2^2$, $x_1(x_1^2 - 3x_2^2)$ and 1.

Proof. Let us consider on $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ the inner product defined by requiring that x_1, x_2 be an orthonormal basis. Then the restriction of T_j to $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ is the reflection on the line generated by $\frac{1}{2}(x_1 - (-1)^j\sqrt{3}x_2)$ $(j = 2, 3)$. Moreover, if we identify $\mathfrak{h}_{\mathbf{R}}^*$ with $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ by the linear map $\iota\colon \mathfrak{h}_{\mathbf{R}}^* \to \mathbf{R}x_1 \oplus \mathbf{R}x_2$ defined by $\iota(\alpha_1) = \frac{1}{2}(\sqrt{3}x_1 + x_2)$, $\iota(\alpha_2) = \frac{1}{2}(-\sqrt{3}x_1 + x_2)$, then the simple reflections s_{α_1} and s_{α_2} correspond respectively to T_2 and T_3. This establishes (i).

To prove (ii) we just need to recall how one gets the Weyl group invariants on $\mathfrak{h}_{\mathbf{R}}$. Let e_1, e_2, e_3 be the canonical basis of $\mathbf{R}^3$ and let H be the orthogonal complement of $\mathbf{R}(e_1 + e_2 + e_3)$. Then the inclusion map $j\colon \mathfrak{h}_{\mathbf{R}}^* \to \mathbf{R}^3$ defined by $j(\alpha_1) = e_1 - e_2$, $j(\alpha_2) = e_2 - e_3$ identifies $h_{\mathbf{R}}^*$ with H. Also the action of the Weyl group on $\mathfrak{h}_{\mathbf{R}}^*$ corresponds to the restriction to H of the action of the symmetric group S_3 on $\mathbf{R}^3$ defined by $\sigma(e_i) = e_{\sigma(i)}$, $\sigma \in S_3$; $i = 1, 2, 3$. If y_1, y_2, y_3 denote the coordinate functions on $\mathbf{R}^3$ then it is well known that the S_3-invariants on $\mathbf{R}^3$ are generated by the elementary symmetric polynomials $p_1 = y_1 + y_2 + y_3$, $p_2 = y_1^2 + y_2^2 + y_3^2$, $p_3 = y_1^3 + y_2^3 + y_3^3$ and 1. Moreover the restrictions of p_2 and p_3 to H together with 1 generates all S_3-invariants on H. Since $j(x_1) = (e_1 - 2e_2 + e_3)/\sqrt{3}$ and $j(x_2) = e_1 - e_3$ we get

$$(p_2 \circ j)(ux_1 + vx_2) = 2(u^2 + v^2), \quad (p_3 \circ j)(ux_1 + vx_2) = -2u(u^2 - 3v^2)/\sqrt{3}.$$

But W is contained in the orthogonal group of $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ therefore $x_1^2 + x_2^2$, $x_1(x_1^2 - 3x_2^2)$ and 1 generate $\mathbf{C}[x_1, x_2]^W$.

Theorem 5. *If $G_o = \mathrm{SU}(2,1)$ then the algebra B is generated by the algebraically independent elements $\zeta \otimes 1, D \otimes 1, Y^2 \otimes 1 + 3 \otimes (Z+1)^2, Y^3 \otimes 1 - 9Y \otimes (Z+1)^2$ and 1. Moreover $B^{\widetilde{W}} = B$.*

Proof. From Proposition 2 and Lemma 3 we know that all elements b of B are precisely of the form $b = \sum_{i,j} (\zeta^i D^j \otimes 1) f_{i,j}(Y \otimes 1, 1 \otimes Z)$ where $a(f_{i,j}) \in \mathbf{C}[x_1, x_2]^W$. Now Proposition 4 tells us that $a(x_1^2 + 3(x_2+1)^2) = 3(x_1^2 + x_2^2)$, $a(x_1^3 - 9x_1(x_2+1)^2) = 3\sqrt{3}x_1(x_1^2 - 3x_2^2)$ and 1 generates $\mathbf{C}[x_1, x_2]^W$. The first assertion is proved.

It is well known that there is an element w in the center of K_o such that $Ad(w)|_{\mathfrak{a}} = -I$. Then (3) implies that $B^{\widetilde{W}} = \{b \in B : b(\lambda - \rho) = b(-\lambda - \rho) \text{ for all } \lambda \in \mathfrak{a}^*\}$. Using Lemma 29 of Tirao [11] we obtain: $\alpha_1(Z_{\alpha_1}) = \alpha_1(H_{\alpha_1}) - \alpha_1(Y_{\alpha_1}) = 2 - 3/2 = 1/2$, thus $\rho(Z) = 2\alpha_1(Z_{\alpha_1}) = 1$. If $b = \sum b_j \otimes Z^j \in U(\mathfrak{k}) \otimes U(\mathfrak{a})$ let $\tilde{b} = \sum b_j \otimes (Z-1)^j$. Then $b(\lambda - \rho) = b(-\lambda - \rho)$ if and only if $\tilde{b}(\lambda) = \tilde{b}(-\lambda)$ $(\lambda \in \mathfrak{a}^*)$. Now $B = B^{\widetilde{W}}$ is a direct consequence of the first assertion. The theorem is proved.

3. The structure of $U(\mathfrak{g})^K$

Proposition 6. *If $u \in Z(\mathfrak{g})$ then $P(u) \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$.*

Proof. Let $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \sum_{\lambda>0} \mathfrak{g}_\lambda$ and $\bar{\mathfrak{n}} = \sum_{\lambda>0} \mathfrak{g}_{-\lambda}$. We enumerate $\Delta(\mathfrak{g}, \mathfrak{a})^+$ as $\{\lambda_1, \ldots, \lambda_p\}$. Let $X_{j,1}, \ldots, X_{j,m(j)}$ (resp. $Y_{j,1}, \ldots, Y_{j,m(j)}$) be a basis of $\mathfrak{g}_{\lambda_j}$ (resp. $\mathfrak{g}_{-\lambda_j}$). Then set $X_j^K = (X_{j,1})^{k_1} \cdots (X_{j,m(j)})^{k_{m(j)}}$ and $Y_j^I = (Y_{j,1})^{i_1} \ldots (Y_{j,m(j)})^{i_{m(j)}}$, where $K = (k_1, \ldots k_{m(j)})$ and $I = (i_1, \ldots i_{m(j)})$. Then the Poincaré-Birkhoff-Witt Theorem implies that $u \in U(\mathfrak{g})$ can be written in a unique way as

$$u = \sum_{\tilde{I},\tilde{K}} (Y_1)^{I_1} \cdots (Y_p)^{I_p} u_{\tilde{I},\tilde{K}} (X_1)^{K_1} \cdots (X_p)^{K_p}, \quad u_{\tilde{I},\tilde{K}} \in U(\mathfrak{m} \oplus \mathfrak{a}), \tag{7}$$

where $\tilde{I} = (I_1, \ldots, I_p)$ and $\tilde{K} = (K_1, \ldots, K_p)$. If $u \in Z(\mathfrak{g})$ then $Hu - uH = 0$ for all $H \in \mathfrak{a}$, therefore the sum (51) is restricted to all pairs $\tilde{I}, \tilde{K}$ such that $\sum |I_j|\lambda_j = \sum |K_j|\lambda_j$, which clearly implies that $P(u) = u_{\tilde{0},\tilde{0}} \in U(\mathfrak{m} \oplus \mathfrak{a})$ or more precisely that $P(u) \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$. The proposition is proved.

Since $\mathfrak{m} = \mathfrak{m}^- \oplus \mathfrak{t} \oplus \mathfrak{m}^+$ we have

$$U(\mathfrak{m}) = U(\mathfrak{t}) \oplus \left(\mathfrak{m}^- U(\mathfrak{m}) \oplus U(\mathfrak{m})\mathfrak{m}^+\right).$$

Let q denote the projection of $U(\mathfrak{m})$ onto $U(\mathfrak{t})$ corresponding to this direct sum decomposition and set $Q = q \otimes id : U(\mathfrak{m}) \otimes U(\mathfrak{a}) \to U(\mathfrak{t}) \otimes U(\mathfrak{a})$. Since $\mathfrak{t} \oplus \mathfrak{a}$ is abelian, we shall use $U(\mathfrak{t}) \otimes U(\mathfrak{a})$ and $S(\mathfrak{t}) \otimes S(\mathfrak{a}) = S(\mathfrak{t} \oplus \mathfrak{a})$ interchangeably.

Recall the following notation: if $\alpha \in P^+$ is a simple root such that $Y_\alpha \neq 0$ ($H_\alpha = Y_\alpha + Z_\alpha$, $Y_\alpha \in \mathfrak{t}$, $Z_\alpha \in \mathfrak{a}$) set $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$ where $X_{-\alpha} \neq 0$ in $\mathfrak{g}_{-\alpha}$. Also we put

$$B_\alpha = \{b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a}) : E_\alpha^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E_\alpha^n, n \in \mathbf{N}\}.$$

Let $\tilde{\nu}, \sigma \in (\mathfrak{t} \oplus \mathfrak{a})^*$ be defined by $\tilde{\nu}|_{\mathfrak{t}} = \alpha|_{\mathfrak{t}}$, $\tilde{\nu}(Z_\alpha) = -\alpha(Y_\alpha)$ and $\sigma|_{\mathfrak{t}} = 0$, $\sigma(Z_\alpha) = 1$.

Lemma 7. *An element $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ belongs to B_α if and only if*

$$Q(b)(t\sigma + \tilde{\mu} + t\tilde{\nu} - \sigma) = Q(b)(-t\sigma + \tilde{\mu} - \sigma) \tag{8}$$

for all $\tilde{\mu} \in (\mathfrak{t} \oplus \mathfrak{a})^*$ such that $\tilde{\mu}(Z_\alpha) = -\tilde{\mu}(Y_\alpha)$ and all $t \in \mathbf{N}$.

Proof. We enumerate $\Delta(\mathfrak{m}, \mathfrak{t})^+ = \{\beta_1, \ldots, \beta_q\}$ and choose a basis $X_1, \ldots, X_q$ of $\mathfrak{m}^+$ with $X_j \in \mathfrak{m}_{\beta_j}$. Also let $Y_1, \ldots, Y_q$ be a basis of $\mathfrak{m}^-$ with $Y_j \in \mathfrak{m}_{-\beta_j}$. Moreover let $H_1, \ldots, H_l$ be a basis of $\mathfrak{t}$. If $I, K \in \mathbf{N}_0^q$ then set $X^K = (X_1)^{k_1} \cdots (X_q)^{k_q}$, $Y^I = (Y_1)^{i_1} \cdots (Y_q)^{i_q}$. If $J \in \mathbf{N}_0^l$ then put $H^J = (H_1)^{j_1} \cdots (H_l)^{j_l}$. Then the Poincaré-Birkhoff-Witt Theorem implies that the elements $Y^I H^J X^K \otimes Z_\alpha^s$ form a basis of $U(\mathfrak{m}) \otimes U(\mathfrak{a})$.

Now if $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$, $b = \sum a_{I,J,K,s} Y^I H^J X^K \otimes Z_\alpha^s$ then $a_{I,J,K,s} \neq 0$ and $I \neq 0$ imply $K \neq 0$. Therefore $b \in B_\alpha$ if and only if for all $t \in \mathbf{N}$

$$\sum a_{I,J,K,s} E_\alpha^t Y^I H^J X^K (t - Y_\alpha - 1)^s \equiv \sum a_{I,J,K,s} Y^I H^J X^K (-t - Y_\alpha - 1)^s E_\alpha^t$$

which is equivalent to

$$\sum a_{0,J,0,s} E_\alpha^t H^J (t - Y_\alpha - 1)^s \equiv \sum a_{0,J,0,s} H^J (-t - Y_\alpha - 1)^s E_\alpha^t, \tag{9}$$

because $[\mathfrak{m}^+, E_\alpha] = 0$. Using Lemma 18 (vi) of Tirao [11] repeately (9) can be written as

(10)

$$E_\alpha^t \sum a_{0,J,0,s} H^J (t - Y_\alpha - 1)^s \equiv E_\alpha^t \sum a_{0,J,0,s} \times (H_1 - t\alpha(H_1))^{j_1} \cdots (H_l - t\alpha(H_l))^{j_l} \big(-t - Y_\alpha + t\alpha(Y_\alpha) - 1\big)^s.$$

By Lemma 20 of Tirao [11] E_α^t can be cancelled in both sides of (10) and then clearly the equivalence sign can be replaced by an equal sign. Thus

(11)

$$\sum a_{0,J,0,s} H^J (t - Y_\alpha - 1)^s = \sum a_{0,J,0,s} (H_1 - t\alpha(H_1))^{j_1} \cdots (H_l - t\alpha(H_l))^{j_l} \times \big(-t - Y_\alpha + t\alpha(Y_\alpha) - 1\big)^s.$$

If we evaluate both sides of (11) at $\mu \in \mathfrak{t}^*$ we get

$$\sum a_{0,J,0,s} H^J(\mu) \big(t - \mu(Y_\alpha) - 1\big)^s = \sum a_{0,J,0,s} H^J(\mu - t\alpha) \times \big(-t - \mu(Y_\alpha) + t\alpha(Y_\alpha) - 1\big)^s. \tag{12}$$

Let $\tilde\mu \in (\mathfrak{t} \oplus \mathfrak{a})^*$ be defined by $\tilde\mu|_{\mathfrak{t}} = \mu$ and $\tilde\mu(Z_\alpha) = -\mu(Y_\alpha)$. Then $t - \mu(Y_\alpha) - 1 = (t\sigma + \tilde\mu - \sigma)(Z_\alpha)$ and $-t - \mu(Y_\alpha) + t\alpha(Y_\alpha) - 1 = (-t\sigma + \tilde\mu - t\tilde\nu - \sigma)(Z_\alpha)$. Therefore (12) is equivalent to

$$\sum a_{0,J,0,s} (H^J \otimes Z_\alpha^s)(t\sigma + \tilde\mu - \sigma) = \sum a_{0,J,0,s} (H^J \otimes Z_\alpha^s)(-t\sigma + \tilde\mu - t\tilde\nu - \sigma).$$

If we change $\tilde\mu$ by $\tilde\mu + t\tilde\nu$ and since $Q(b) = \sum a_{0,J,0,s} H^J \otimes Z_\alpha^s$ we get that $b \in B_\alpha$ if and only if (8) holds for all $\tilde\mu \in (\mathfrak{t} \oplus \mathfrak{a})^*$ such that $\tilde\mu(Z_\alpha) = -\tilde\mu(Y_\alpha)$. This completes the proof of the lemma.

To make things more transparent we recall some basic facts about the structure of $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. Let $\mathbf{F}$ denote either the reals $\mathbf{R}$ or the complexes $\mathbf{C}$ and let $x \mapsto \bar x$ be the standard involution. For $x \in \mathbf{F}$ set $|x|^2 = x\bar x$.

Consider on $\mathbf{F}^{n+1}$ the quadratic form $q(x_1, \dots, x_{n+1}) = |x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$. Then G_o is the connected component of the identity in the group of all $\mathbf{F}$-linear transformations g of $\mathbf{F}^{n+1}$ preserving q and such that $\det(g) = 1$. Then $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. If we set

$$Q = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix},$$

where I denotes the $n \times n$ identity matrix, we have

$$G_o = \{A \in \mathrm{GL}(n+1, \mathbf{F}) : {}^t\bar A Q A = Q, \det(A) = 1\}_o.$$

Here the subindex "o" in the right hand side denotes the connected componet of the identity. We also have

$$\mathfrak{g}_o = \{X \in \mathfrak{gl}(n+1, \mathbf{F}) : {}^t\bar X Q + QX = 0, \mathrm{Tr}(X) = 0\}.$$

The Lie algebra $\mathfrak{g}_o$ has a Cartan decomposition $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$ where

$$\mathfrak{k}_o = \left\{ \begin{pmatrix} X & 0 \\ 0 & w \end{pmatrix} : {}^t\bar{X} + X = 0, w + \mathrm{Tr}(X) = 0 \right\}$$

and

$$\mathfrak{p}_o = \left\{ \begin{pmatrix} 0 & u \\ {}^t\bar{u} & 0 \end{pmatrix} : u \in \mathbf{F}^n \right\}.$$

In each case the Cartan involution θ is given by $\theta(X) = -{}^t\bar{X}$.

Let $E_{i,j} \in \mathfrak{gl}(n+1, \mathbf{F})$ denote the matrix with a one in the (i, j) entry and zero otherwise. Set $H_o = E_{1,n+1} + E_{n+1,1}$ and let $\mathfrak{a}_o = \{tH_o : t \in \mathbf{R}\}$ in both cases. As we know $\mathfrak{a}_o$ is a maximal abelian subspace of $\mathfrak{p}_o$. Let λ be the complex linear functional on $\mathfrak{a}$ defined by $\lambda(H_o) = 1$. Then, we have $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda\}$ if $\mathbf{F} = \mathbf{R}$ and $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda, \pm 2\lambda\}$ if $\mathbf{F} = \mathbf{C}$. In both cases we choose $\Pi = \{\lambda\}$ as a set of simple roots. Now consider the following Cartan subalgebra of $\mathfrak{m}$:
if $\mathbf{F} = \mathbf{R}$

$$\mathfrak{t} = \{T = \sum_{j=1}^{p-1} i t_{j+1}(E_{2j,2j+1} - E_{2j+1,2j}) : t_j \in \mathbf{C}\}, \tag{13}$$

if $\mathbf{F} = \mathbf{C}$

$$\mathfrak{t} = \{T = t_1(E_{1,1} + E_{n+1,n+1}) + \sum_{j=2}^{n} t_j E_{j,j} : \mathrm{Tr}(T) = 0, t_j \in \mathbf{C}\}, \tag{14}$$

where $p - 1 = [(n-1)/2]$. Then as we know $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Now according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$ we define linear functionals λ_j on $\mathfrak{h}$ as follows,

$$\lambda_j(H) = \begin{cases} t, & j = 1 \\ t_j, & j = 2, \ldots, p \end{cases} \quad \text{and } \lambda_j(H) = \begin{cases} t_1 + t, & j = 1 \\ t_j, & j = 2, \ldots, n \\ t_1 - t, & j = n+1, \end{cases} \tag{15}$$

respectively. Here $H = T + tH_o$ where T is as in (13) and (14). Now a positive root system of $\mathfrak{m}$ with respect to $\mathfrak{t}$ can be discribed as follows:
if $\mathbf{F} = \mathbf{R}$

$$\Delta(\mathfrak{m}, \mathfrak{t})^+ = \begin{cases} \{\lambda_i \pm \lambda_j : 2 \le i < j \le p\} \cup \{\lambda_i : 2 \le i \le p\}, & n = 2p \\ \{\lambda_i \pm \lambda_j : 2 \le i < j \le p\}, & n = 2p - 1, \end{cases} \tag{16}$$

if $\mathbf{F} = \mathbf{C}$

$$\Delta(\mathfrak{m}, \mathfrak{t})^+ = \{\lambda_i - \lambda_j : 2 \le i < j \le n\}.$$

If $\Delta(\mathfrak{g}, \mathfrak{h})$ denotes the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, we define a positive root system $\Delta(\mathfrak{g}, \mathfrak{h})^+$ compatible with $\Delta(\mathfrak{g}, \mathfrak{a})^+$ and $\Delta(\mathfrak{m}, \mathfrak{t})^+$, as follows: we say that $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is positive if, whenever $\alpha|_{\mathfrak{a}} \ne 0$ then $\alpha|_{\mathfrak{a}} \in \Delta(\mathfrak{g}, \mathfrak{a})^+$ and if α is such that $\alpha|_{\mathfrak{a}} = 0$ then $\alpha|_{\mathfrak{t}} \in \Delta(\mathfrak{m}, \mathfrak{t})^+$. A straightforward computation shows that:

if $\mathbf{F}=\mathbf{R}$

$$\Delta(\mathfrak{g},\mathfrak{h})^+ = \begin{cases} \{\lambda_i \pm \lambda_j : 1 \le i < j \le p\} \cup \{\lambda_i : 1 \le i \le p\}, & n=2p \\ \{\lambda_i \pm \lambda_j : 1 \le i < j \le p\}, & n=2p-1, \end{cases}$$

if $\mathbf{F}=\mathbf{C}$

$$\Delta(\mathfrak{g},\mathfrak{h})^+ = \{\lambda_i - \lambda_j : 1 \le i < j \le n+1\}.$$

The corresponding sets of simple roots are:
if $\mathbf{F}=\mathbf{R}$

$$\Pi(\mathfrak{g},\mathfrak{h}) = \begin{cases} \{\alpha_i\}, & \alpha_i = \lambda_i - \lambda_{i+1} (1 \le i \le p-1), \alpha_p = \lambda_p, n=2p \\ \{\alpha_i\}, & \alpha_i = \lambda_i - \lambda_{i+1} (1 \le i \le p-1), \alpha_p = \lambda_{p-1} + \lambda_p, n=2p-1, \end{cases}$$

$$\Pi(\mathfrak{m},\mathfrak{t}) = \begin{cases} \{\alpha_1,\dots,\alpha_p\}, & n=2p, p\ge 2 \\ \{\alpha_1,\dots,\alpha_p\}, & n=2p-1, p \ge 3 \\ \emptyset, & n=3; \end{cases}$$

if $\mathbf{F}=\mathbf{C}$

$$\Pi(\mathfrak{g},\mathfrak{h}) = \{\alpha_1,\dots,\alpha_n\}, \quad \alpha_i = \lambda_i - \lambda_{i+1} (i=1,\dots,n),$$

$$\Pi(\mathfrak{m},\mathfrak{t}) = \begin{cases} \{\alpha_2,\dots,\alpha_{n-1}\}, & n \ge 3 \\ \emptyset, & n=2. \end{cases}$$

In what follows we shall consider Q as a linear map from $U(\mathfrak{m}) \otimes U(\mathfrak{a})$ onto $S(\mathfrak{h})$. Also if $w \in W(\mathfrak{g},\mathfrak{h})$ we set

$$S(\mathfrak{h})^{\tilde{w}} = \{p \in S(\mathfrak{h}) : p(w(\mu)-\rho) = p(\mu-\rho), \text{ for all } \mu \in \mathfrak{h}^*\}.$$

Proposition 8. *Let $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. If $\alpha \in P^+$ is a simple root then an element $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ belongs to B_α if and only if $Q(b) \in S(\mathfrak{h})^{\tilde{s}_\alpha}$.*

Proof. We shall consider three cases according to: (i) $G_o = \mathrm{SO}(2p-1,1)$, $p \ge 2$, (ii) $G_o = \mathrm{SO}(2p,1)$, $p \ge 2$ and (iii) $G_o = \mathrm{SU}(n,1)$ $n \ge 2$.

(i) If $p \ge 3$ then $\alpha_1 = \lambda_1 - \lambda_2$ is the unique simple root in P^+. When $p=2$, $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_2 = \lambda_1 + \lambda_2$ are both in P^+. We shall only consider the case $\alpha = \alpha_1$, leaving the other to the reader. A simple computation gives: $H_{\alpha_1} = H_o - i(E_{2,3} - E_{3,2})$; hence $Y_{\alpha_1} = -i(E_{2,3}-E_{3,2})$ and $Z_{\alpha_1} = H_o$. Now $\tilde{\mu} \in \mathfrak{h}^*$ satisfies $\tilde{\mu}(Z_{\alpha_1}) = -\tilde{\mu}(Y_{\alpha_1})$ if and only if $\tilde{\mu} = x(\lambda_1+\lambda_2) + x_3\lambda_3 + \dots + x_p\lambda_p$, $x, x_3, \dots, x_p \in \mathbf{C}$. We have $\tilde{\nu} = -\lambda_1 - \lambda_2$ and $\sigma = \lambda_1$ (see the definitions given right before Lemma 7). Also $\rho = (p-1)\lambda_1 + (p-2)\lambda_2 + \dots + \lambda_{p-1}$.

We shall identify $p \in S(\mathfrak{h})$ with the polynomial function on $\mathbf{C}^p$ defined by $p(x_1,\dots,x_p) = p(x_1\lambda_1 + \dots + x_p\lambda_p)$. Then (see (8)) the following equation

$$p(t\sigma + \tilde{\mu} + t\tilde{\nu} - \sigma) = p(-t\sigma + \tilde{\mu} - \sigma) \tag{17}$$

for all $\tilde{\mu} \in \mathfrak{h}^*$ such that $\tilde{\mu}(Z_{\alpha_1}) = -\tilde{\mu}(Y_{\alpha_1})$ and all $t \in \mathbf{N}$, can be rewritten as

$$p(x-1, x-t, x_3, \dots, x_p) = p(x-t-1, x, x_3, \dots, x_p) \tag{18}$$

for all $x, x_3, \ldots, x_p \in \mathbf{C}$ and all $t \in \mathbf{N}$. For $p \in S(\mathfrak{h})$ let $\tilde{p} \in S(\mathfrak{h})$ be defined by $\tilde{p}(\mu) = p(\mu - \rho)$, $\mu \in \mathfrak{h}^*$. Then it can be easily seen that (18) is equivalent to

$$(19) \qquad \tilde{p}(x, x+t, x_3, \ldots, x_p) = \tilde{p}(x+t, x, x_3, \ldots, x_p)$$

for all $x, x_3, \ldots, x_p \in \mathbf{C}$ and all $t \in \mathbf{Z}$. Let $s \colon \mathbf{C}^p \to \mathbf{C}^p$ be the symmetry given by $s(x_1, x_2, x_3, \ldots, x_p) = (x_2, x_1, x_3, \ldots, x_p)$. If $\tilde{p}$ satisfies (19) then the zero set of $\tilde{p} \circ s - \tilde{p}$ contains an infinite number of parallel hyperplanes. Hence p satisfies (17) if and only if $\tilde{p} \circ s = \tilde{p}$. But $s_{\alpha_1}(\lambda_1) = \lambda_2$ and $s_{\alpha_1}(\lambda_j) = \lambda_j$ for $3 \leq j \leq p$. Therefore s corresponds precisely to s_{α_1} under the identification of $\mathfrak{h}^*$ with $\mathbf{C}^p$ defined above. Thus if $p = Q(b)$, $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$, then $b \in B_{\alpha_1}$ if and only if (Lemma 7) $\tilde{p} \in S(\mathfrak{h})^{s_{\alpha_1}}$ as we wanted to prove.

(ii) The cases $G_o = \mathrm{SO}(2p, 1)$ $p \geq 2$, are completely similar to those considered in (i) and are left to the reader.

(iii) Now we take $G_o = \mathrm{SU}(n, 1)$ $n \geq 2$. In this case there are two simple roots $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_n = \lambda_n - \lambda_{n+1}$ in P^+: $H_{\alpha_1} = \frac{1}{2}(E_{1,1} + E_{n+1,n+1}) - E_{2,2} + \frac{1}{2}H_o$ and $H_{\alpha_n} = -\frac{1}{2}(E_{1,1} + E_{n+1,n+1}) + E_{n,n} + \frac{1}{2}H_o$; hence $Y_{\alpha_1} = \frac{1}{2}(E_{1,1} + E_{n+1,n+1}) - E_{2,2}$, $Y_{\alpha_n} = -\frac{1}{2}(E_{1,1} + E_{n+1,n+1}) + E_{n,n}$, $Z_{\alpha_1} = Z_{\alpha_2} = \frac{1}{2}H_o$, $\rho = \frac{1}{2}\sum_{j=1}^{n+1}(n - 2j + 2)\lambda_j$.

Any $\mu \in \mathfrak{h}^*$ can be written in a unique way as $\mu = x_1\lambda_1 + \cdots + x_{n+1}\lambda_{n+1}$ with $x_j \in \mathbf{C}$ and $\sum x_j = 0$. We shall identify μ with $(x_1, \ldots, x_{n+1}) \in \mathbf{C}^{n+1}$ and $\mathfrak{h}^*$ with the corresponding subspace of $\mathbf{C}^{n+1}$.

Let us consider the case $\alpha = \alpha_1$. Then $\tilde{\mu} \in \mathfrak{h}^*$ satisfies $\tilde{\mu}(Z_{\alpha_1}) = -\tilde{\mu}(Y_{\alpha_1})$ if and only if $\tilde{\mu} = x(\lambda_1 + \lambda_2) + x_3\lambda_3 + \cdots + x_{n+1}\lambda_{n+1}$. We have $\tilde{\nu} = -\lambda_1 - \lambda_2 + 2\lambda_{n+1}$ and $\sigma = \lambda_1 - \lambda_{n+1}$. We shall identify the restriction to $\mathfrak{h}^*$ of an element $p \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ with the corresponding $p \in S(\mathfrak{h})$ by setting $p(x_1, \ldots, x_{n+1}) = p(x_1\lambda_1 + \cdots + x_{n+1}\lambda_{n+1})$, $x_j \in \mathbf{C}$ and $\sum x_j = 0$. Then the equation (17) can be written as

$$(20) \quad p(x-1, x-t, x_3, \ldots, x_n, x_{n+1}+t+1) = p(x-t-1, x, x_3, \ldots, x_n, x_{n+1}+t+1)$$

for all $x, x_3, \ldots, x_{n+1} \in \mathbf{C}$ such that $2x + \sum_{j=3}^{n+1} x_j = 0$ and all $t \in \mathbf{N}$. For $p \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ let $\tilde{p} \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ be defined by $\tilde{p}(x_1, \ldots, x_{n+1}) = p(x_1 - n/2, x_2 - (n-2)/2, \ldots, x_{n+1} - (-n)/2)$; in this way $\tilde{p}(\mu) = p(\mu - \rho)$ for all $\mu \in \mathfrak{h}^*$. Then it can be easily seen that (20) is equivalent to

$$(21) \qquad \tilde{p}(x, x+t, x_3, \ldots, x_{n+1}) = \tilde{p}(x+t, x, x_3, \ldots, x_{n+1})$$

for all $x, x_3, \ldots, x_{n+1} \in \mathbf{C}$, $t \in \mathbf{Z}$ such that $2x + t + x_3 + \cdots + x_{n+1} = 0$. As before this implies that

$$\tilde{p}(x_1, x_2, x_3, \ldots, x_{n+1}) = \tilde{p}(x_2, x_1, x_3, \ldots, x_{n+1})$$

for all $x_1, \ldots, x_{n+1} \in \mathbf{C}$ such that $\sum x_j = 0$. But the symmetry $(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_2, x_1, \ldots, x_{n+1})$ of $\mathbf{C}^{n+1}$ when restricted to $\mathfrak{h}^*$ coincide with s_{α_1}. Therefore if $p = Q(b)$, $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$, then $b \in B_{\alpha_1}$ if and only if (Lemma 7) $\tilde{p} \in S(\mathfrak{h})^{s_{\alpha_1}}$.

When $\alpha = \alpha_n$ the proof is exactly the same. The proof of the proposition is now complete.

The following choice of a representative in M_o' of the non-trivial element in $W = W(\mathfrak{g}, \mathfrak{a})$ will be convenient. Let

$$w = \begin{cases} \mathrm{Diag}(-1,1,\ldots,1,-1,1), & \text{for } G_o = \mathrm{SO}(2p-1,1), p \geq 2 \\ \mathrm{Diag}(-1,\ldots,-1,1), & \text{for } G_o = \mathrm{SO}(2p,1), p \geq 2 \\ \mathrm{Diag}(\xi,\ldots,\xi,-\xi), & \text{for } G_o = \mathrm{SU}(n,1), n \geq 2, \xi^{n+1} = -1. \end{cases}$$

Then $w \in M_o'$ and $Ad(w)H_o = -H_o$. Moreover in the first case we have

$$Ad(w)\sum_{j=1}^{p-1} it_{j+1}(E_{2j,2j+1} - E_{2j+1,2j}) = \sum_{j=1}^{p-2} it_{j+1}(E_{2j,2j+1} - E_{2j+1,2j}) - it_p(E_{2p-2,2p-1} - E_{2p-1,2p-2}).$$

Therefore (see (13)) the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$ is $Ad(w)$- stable, $w(\lambda_j) = \lambda_j$ $(j = 2,\ldots,p-1)$ and $w(\lambda_p) = -\lambda_p$ (see (15)). Hence $\Delta(\mathfrak{m},\mathfrak{t})^+$ is also stable under the action of w (see (16)). In the other two cases it is clear that $Ad(w)$ restricts to the identity on $\mathfrak{t}$. Thus in all cases $Ad(w)|_{\mathfrak{h}}$ defines an element in $W(\mathfrak{g},\mathfrak{h})$, which we shall also denote by w.

Proposition 9. *Let $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. An element $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ belongs to $\left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}}$ if and only if $Q(b) \in S(\mathfrak{h})^{\tilde{w}}$.*

Proof. When $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ we have

$$\left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}} = \{b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a}) : Ad(w)(b(\lambda-\rho)) = b(-\lambda-\rho), \lambda \in \mathfrak{a}^*\}.$$

(See (3), also Kostant, Tirao [15, Corollary 3.3].) If $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ let $b^w \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ be defined by $b^w(\lambda-\rho) = Ad(w^{-1})(b(-\lambda-\rho))$ for all $\lambda \in \mathfrak{a}^*$. Then $b \in \left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}}$ if and only if $b^w = b$. The projection $q\colon U(\mathfrak{m}) \to U(\mathfrak{t})$ commutes with $Ad(w)$ because $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are $Ad(w)$-stable. Therefore if $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$

$$(22) \qquad Q(b^w)(\nu, \lambda-\rho) = Q(b)(w(\nu), \lambda-\rho)$$

for all $\nu \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*$. If we replace in (22) ν by $\nu - \rho_{\mathfrak{m}}$ and take into account that $w(\rho_{\mathfrak{m}}) = \rho_{\mathfrak{m}}$ we see that

$$(23) \qquad Q(b^w)(\nu-\rho_{\mathfrak{m}}, \lambda-\rho) = Q(b)(w(\nu)-\rho_{\mathfrak{m}}, \lambda-\rho)$$

for all $\nu \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*$. Now from the explicit description of $\Delta(\mathfrak{g},\mathfrak{h})^+$ and of $\Delta(\mathfrak{m},\mathfrak{t})^+$ it follows that $\rho|_{\mathfrak{t}} = \rho_{\mathfrak{m}}$. Then (23) is equivalent to

$$(24) \qquad Q(b^w)(\mu-\rho) = Q(b)(w(\mu)-\rho)$$

for all $\mu \in \mathfrak{h}^*$. Therefore $Q(b) \in S(\mathfrak{h})^{\tilde{w}}$ if and only if $Q(b) = Q(b^w)$. Since $Q\colon U(\mathfrak{m})^M \otimes U(\mathfrak{a}) \to S(\mathfrak{h})$ is one-to-one (cf. Wallach [22, Theorem 3.2.3]) we finally have: $b \in \left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}} \iff b = b^w \iff Q(b) = Q(b^w) \iff Q(b) \in S(\mathfrak{h})^{\tilde{w}}$, for all $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$.

Proposition 10. *If* $G_\circ = \mathrm{SO}(n,1)_e$ *or* $\mathrm{SU}(n,1)$. *Then* $\big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B^{\widetilde{W}} = \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$ *and* $Q\big(\big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B\big) = S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$.

Proof. If $c \in U(\mathfrak{m})^M$ it is well known (cf. Wallach [22, Theorem 3.2.3]) that $q(c)(\nu - \rho_\mathfrak{m}) = q(c)(\omega(\nu) - \rho_\mathfrak{m})$ for all $\nu \in \mathfrak{t}^*, \omega \in W(\mathfrak{m},\mathfrak{t})$. By extending each $\omega \in W(\mathfrak{m},\mathfrak{t})$ to $\mathfrak{h}$ by making it trivial on $\mathfrak{a}$ we can consider $W(\mathfrak{m},\mathfrak{t})$ as a subgroup of $W(\mathfrak{g},\mathfrak{h})$. Then for all $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ we have

$$Q(b)(\nu - \rho_\mathfrak{m}, \lambda - \rho) = Q(b)(\omega(\nu) - \rho_\mathfrak{m}, \lambda - \rho) = Q(b)(\omega(\nu) - \rho_\mathfrak{m}, \omega(\lambda) - \rho)$$

for all $\nu \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*, \omega \in W(\mathfrak{m},\mathfrak{t})$. Equivalently

$$Q(b)(\mu - \rho) = Q(b)(\omega(\mu) - \rho)$$

for all $\mu \in \mathfrak{h}^*, \omega \in W(\mathfrak{m},\mathfrak{t})$. Hence $Q\big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \subset S(\mathfrak{h})^{\widetilde{W(\mathfrak{m},\mathfrak{t})}}$.

From the explicit description of the corresponding sets of simple roots given before we see that:

$$W(\mathfrak{g},\mathfrak{h}) = \begin{cases} \langle s_1, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p \\ \langle s_1, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p-1 \\ \langle s_1, \dots, s_n \rangle, & \text{for } \mathbf{F} = \mathbf{C}; \end{cases}$$

$$W(\mathfrak{m},\mathfrak{t}) = \begin{cases} \langle s_2, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p, p \geq 2 \\ \langle s_2, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p-1, p \geq 3 \\ \langle e \rangle & \text{for } \mathbf{F} = \mathbf{R}, n = 3 \\ \langle s_2, \dots, s_{n-1} \rangle, & \text{for } \mathbf{F} = \mathbf{C}, n \geq 3 \\ \langle e \rangle & \text{for } \mathbf{F} = \mathbf{C}, n = 2, \end{cases}$$

where $s_i = s_{\alpha_i}$ in all cases.

If $G_\circ = \mathrm{SO}(2p,1)_e, p \geq 2$ or $G_\circ = \mathrm{SO}(2p-1,1)_e, p \geq 3$, then α_1 is the unique simple root in P^+. If $G_\circ = \mathrm{SU}(n,1), n \geq 2$, then α_1 and α_n are the unique simple roots in P^+. While if $G_\circ = \mathrm{SO}(3,1)_e$ then α_1 and α_2 are in P^+. In any case we see that $W(\mathfrak{g},\mathfrak{h})$ is generated by $W(\mathfrak{m},\mathfrak{t})$ and $\{s_\alpha : \alpha \in P^+ \text{ is a simple root}\}$. Thus from Proposition 8 and from what was observed above it follows that $Q(b) \in S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ for all $b \in \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$.

Conversely if $p \in S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ there exists a unique $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ such that $Q(b) = p$ (see Wallach [22, Theorem 3.2.3]). Now Propositions 8 and 9 imply that $b \in \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B^{\widetilde{W}}$. This completes the proof of our proposition.

Theorem 11. *If* $G_\circ = \mathrm{SO}(n,1)_e$ *or* $\mathrm{SU}(n,1)$ *then* $P(Z(\mathfrak{g})) = \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$.

Proof. From Theorem 37 of Tirao [11] and Proposition 6 it follows that $P(Z(\mathfrak{g})) \subset \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$. If $b \in \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$ then $Q(b) \in S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ (Proposition 10). Now $Q \circ P \colon Z(\mathfrak{g}) \colon \to S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ is the Harish-Chandra isomorphism (see Wallach [22, Theorem 3.2.3]). Hence there exists $u \in Z(\mathfrak{g})$ such that

$Q(P(u)) = Q(b)$. Since $Q\colon U(\mathfrak{m})^M \otimes U(\mathfrak{a}) \to S(\mathfrak{h})$ is injective we get $P(u) = b$, proving what we wanted.

To prove that when $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ we have $U(\mathfrak{g})^K \simeq Z(\mathfrak{g}) \otimes Z(\mathfrak{k})$ it will be convenient to begin discussing the following concept.

Let $\Delta(\mathfrak{k},\mathfrak{j})^+$ be a choice of a positive root system of $\mathfrak{k}$ and let Λ be the corresponding set of all dominant integral linear functions on $\mathfrak{j}$. Also let Ω be the set of all dominant integral linear functions on $\mathfrak{t}$, with respect to $\Delta(\mathfrak{m},\mathfrak{t})^+$. A subset $X \subset \mathfrak{j}^*$ ($X \subset \mathfrak{t}^*$) is a *separating set* of $S(\mathfrak{j})_l$ ($S(\mathfrak{t})_l$) if for any $f \in S(\mathfrak{j})_l$ ($f \in S(\mathfrak{t})_l$) $f|_X = 0$ implies $f = 0$. ($S(\mathfrak{h})_l$ denotes the subspace of $S(\mathfrak{h})$ of all elements of degree $\leq l$.) For $\lambda \in \Lambda$ ($\omega \in \Omega$) let V_λ (W_ω) be a finite dimensional irreducible $\mathfrak{k}$-module ($\mathfrak{m}$-module) with highest weight λ (ω). If $\omega \in \Omega$ set

$$\Lambda(\omega) = \{\lambda \in \Lambda : \mathrm{Hom}_{\mathfrak{m}}(W_\omega, V_\lambda) \neq 0\}.$$

When $G_o = \mathrm{SO}(n,1)_e$ ($\mathrm{SU}(n,1)$) the algebra $\mathfrak{k} \simeq \mathfrak{so}(n,\mathbf{C})$ ($\mathfrak{gl}(n,\mathbf{C})$) and $\mathfrak{m} \simeq \mathfrak{so}(n-1,\mathbf{C})$ ($\mathfrak{gl}(n-1,\mathbf{C})$) corresponds to the subalgebra of all matrices in $\mathfrak{so}(n,\mathbf{C})$ ($\mathfrak{gl}(n,\mathbf{C})$) with all zeros in the first row and in the first column. Let Λ' (Ω') be the set of all $\lambda \in \Lambda$ ($\omega \in \Omega$) such that there exists a representation of $\mathrm{SO}(n,\mathbf{C})$ or $\mathrm{GL}(n,\mathbf{C})$ ($\mathrm{SO}(n-1,\mathbf{C})$ or $\mathrm{GL}(n-1,\mathbf{C})$) of highest weight λ (ω), according to $G_o = \mathrm{SO}(n,1)_e$ or $G_o = \mathrm{SU}(n,1)$.

For the proof of the following proposition we need to recall how a representation V_λ of $\mathrm{SO}(n,\mathbf{C})$ or $\mathrm{GL}(n,\mathbf{C})$ decomposes as a representation of $\mathrm{SO}(n-1,\mathbf{C})$ or $\mathrm{GL}(n-1,\mathbf{C})$, respectively. We need to distinguish three cases: $\mathrm{SO}(2\nu+1,\mathbf{C})$, $\mathrm{SO}(2\nu,\mathbf{C})$ or $\mathrm{GL}(\nu,\mathbf{C})$. In any of these cases a basis $\lambda_1,\ldots,\lambda_\nu$ of $\mathfrak{j}$ can be chosen in such a way that any $\lambda \in \Lambda'$ can be writen as $\lambda = m_1\lambda_1 + \cdots + m_\nu\lambda_\nu$ where

$$\begin{cases} m_1 \geq \cdots \geq m_\nu \geq 0, \ m_i \text{ all integers or half-integers,} & \text{for } \mathrm{SO}(2\nu+1,\mathbf{C}) \\ m_1 \geq \cdots \geq m_{\nu-1} \geq |m_\nu|, \ m_i \text{ all integers or half-integers,} & \text{for } \mathrm{SO}(2\nu,\mathbf{C}) \\ m_1 \geq \cdots \geq m_\nu \geq 0, \ m_i \text{ all integers,} & \text{for } \mathrm{GL}(\nu,\mathbf{C}). \end{cases}$$

Now the following branching formulas hold (see Foulton, Harris [4,§25.3]).

The restriction from $\mathrm{SO}(2\nu+1,\mathbf{C})$ to $\mathrm{SO}(2\nu,\mathbf{C})$ is determined by the following spectral formula

$$V_{(m_1,\ldots,m_\nu)} = \sum W_{(p_1,\ldots,p_\nu)} \tag{25}$$

the sum over all $(p_1,\ldots,p_\nu)$ with $m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq m_\nu \geq |p_\nu|$, with the p_i and m_i simultaneously all integers or all half-integers.

When we restrict from $\mathrm{SO}(2\nu,\mathbf{C})$ to $\mathrm{SO}(2\nu-1,\mathbf{C})$ we have

$$V_{(m_1,\ldots,m_\nu)} = \sum W_{(p_1,\ldots,p_{\nu-1})}$$

the sum over all $(p_1,\ldots,p_{\nu-1})$ with $m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq |m_\nu|$, with the p_i and m_i simultaneously all integers or all half-integers.

For $\mathrm{GL}(\nu-1,\mathbf{C}) \subset \mathrm{GL}(\nu,\mathbf{C})$ the restriction of V_λ $\lambda = (m_1,\ldots,m_\nu)$ from $\mathrm{GL}(\nu,\mathbf{C})$ to $\mathrm{GL}(\nu-1,\mathbf{C})$ is given by

$$V_{(m_1,\ldots,m_\nu)} = \sum W_{(p_1,\ldots,p_{\nu-1})}$$

the sum over all $(p_1,\ldots,p_{\nu-1})$ with $m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq m_\nu \geq 0$, with the p_i and m_i all integers.

Proposition 12. *Let $G_\circ = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. The set Y_l of all $\omega \in \Omega$ such that $\Lambda(\omega)$ is a separating set of $S(\mathfrak{j})_l$ is a separating set of $S(\mathfrak{t})_n$ for all $n \in \mathbf{N}$.*

Proof. If $\omega \in \Omega'$ let $\Lambda'(\omega) = \{\lambda \in \Lambda' : \mathrm{Hom}_{\mathfrak{m}}(W_\omega, V_\lambda) \neq 0\}$ and $Y_l' = \{\omega \in \Omega' : \Lambda'(\omega)$ is a separating set of $S(\mathfrak{j})_l\}$. Then clearly $\Lambda'(\omega) \subset \Lambda(\omega)$ and $Y_l' \subset Y_l$ for all $\omega \in \Omega', l \in \mathbf{N}$. Thus it will be enough to prove that Y_l' is a separating set of $S(\mathfrak{t})$.

If $G_\circ = \mathrm{SO}(2\nu+1,1)_e$ and $\omega = (p_1, \ldots, p_\nu), p_1 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq |p_\nu|$, p_i simultaneously all integers or all half-integers, then from (25) it follows that $\Lambda'(p_1, \ldots, p_\nu) = \{\lambda = (m_1, \ldots, m_\nu) : m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq m_\nu \geq |p_\nu|, p_i$ and m_i all integers or all half-integers$\}$. Now we claim that $\Lambda'(p_1, \ldots, p_\nu)$ is a separating set of $S(\mathfrak{j})_l$ if and only if $l(p_1, \ldots, p_\nu) = \min\{p_1 - p_2, p_2 - p_3, \ldots, p_{\nu-1} - |p_\nu|\} \geq l$. In fact, if $x_1, \ldots, x_\nu$ is the dual basis of $\lambda_1, \ldots, \lambda_\nu$ then any element of $S(\mathfrak{j})$ can be viewed as a polynomial in $x_1, \ldots, x_\nu$. Thus if $l(p_1, \ldots, p_\nu) \geq l$, $f = f(x_1, \ldots, x_\nu) \in S(\mathfrak{j})_l$ and $f(m_1, \ldots, m_\nu) = 0$ for all $(m_1, \ldots, m_\nu) \in \Lambda'(p_1, \ldots, p_\nu)$ then clearly $f = 0$, i.e. $\Lambda'(p_1, \ldots, p_\nu)$ is a separating set of $S(\mathfrak{j})_l$. Conversely, if $p_{i-1} - |p_i| < l$ for some $i = 2, \ldots, \nu$ then $f(x_1, \ldots, x_\nu) = \prod(x_i - m_i)$ (the product over all m_i such that $p_{i-1} \geq m_i \geq |p_i|$ p_i and m_i both integers or both half-integers) is a nonzero element in $S(\mathfrak{j})_l$ which vanishes on $\Lambda'(p_1, \ldots, p_\nu)$. Therefore $Y_l' = \{\omega = (p_1, \ldots, p_\nu) \in \Omega' : l(p_1, \ldots, p_\nu) \geq l\}$ which obviously implies that Y_l' is a separating set of $S(\mathfrak{t})$.

In a completely similar way the proposition is proved when $G_\circ = \mathrm{SO}(2\nu, 1)$ or $G_\circ = \mathrm{SU}(\nu, 1)$.

Corollary 13. *Let $a_1, \ldots, a_m$ be a linearly independent subset of $Z(\mathfrak{k})$ and let $p_1, \ldots, p_m \in U(\mathfrak{t})$. Then $\sum_i a_i p_i \equiv 0 \mod (U(\mathfrak{k})\mathfrak{m}^+)$ implies $p_i = 0$, $i = 1, \ldots, m$.*

Proof. Let $l = \max\{\deg(a_i), \deg(p_i) : i = 1, \ldots, m\}$. Given $\omega \in Y_l$ and $\lambda \in \Lambda(\omega)$ let $0 \neq v \in V_\lambda$ be a highest weight vector of $\mathfrak{m}$ of weight ω. Let $\gamma\colon U(\mathfrak{k}) \to U(\mathfrak{j})$ be the Harish-Chandra projection defined by the direct sum decomposition $U(\mathfrak{k}) = U(\mathfrak{j}) \oplus (\mathfrak{k}^- U(\mathfrak{k}) \oplus U(\mathfrak{k})\mathfrak{k}^+)$. Then an element $a \in Z(\mathfrak{k})$ acts on V_λ by multiplication by $\gamma(a)(\lambda)$. Therefore

$$\left(\sum_{i=1}^m \gamma(a_i)(\lambda)p_i(\omega)\right)v = \sum_{i=1}^m a_i p_i \cdot v = 0,$$

hence $\sum_i \gamma(a_i)(\lambda)p_i(\omega) = 0$ for all $\lambda \in \Lambda(\omega), \omega \in Y_l$. Now we claim that the linear span L of $\{(\gamma(a_1)(\lambda), \ldots, \gamma_m(\lambda)) : \lambda \in \Lambda(\omega)\}$ is $\mathbf{C}^m$. In fact, let $\xi = (\xi_1, \ldots, \xi_m)$ be an element in the annihilator of L. Thus $\xi_1\gamma(a_1)(\lambda) + \cdots + \xi_m\gamma(a_m)(\lambda) = 0$ for all $\lambda \in \Lambda(\omega)$. Since $\Lambda(\omega)$ is a separating set of $S(\mathfrak{j})_l$ it follows that $\xi_1\gamma(a_1) + \cdots + \xi_m\gamma(a_m) = 0$. But $\gamma\colon Z(\mathfrak{k}) \to U(\mathfrak{j})$ is injective, therefore $\xi_1 a_1 + \cdots + \xi_m a_m = 0$ which implies that $\xi = 0$, because by assumption $a_1, \ldots, a_m$ are linearly independent. From this we get that $p_i(\omega) = 0$ for all $\omega \in Y_l, i = 1, \ldots, m$. Since Y_l is a separating set of $S(\mathfrak{t})_l$ we finally get that $p_i = 0$, $i = 1, \ldots, m$, as we wanted to prove.

Proposition 14. *Let $G_\circ = \mathrm{S0}(n,1)_e$ or $\mathrm{SU}(n,1)$. Take a linearly independent subset $a_1, \ldots, a_m$ of $Z(\mathfrak{k})$ and elements $c_i \in Z(\mathfrak{m}) \otimes U(\mathfrak{a})$ for $i = 1, \ldots, m$.*
(i) If $\sum_i a_i c_i \in B$ then $c_i \in B$ for $i = 1, \ldots, m$.
(ii) If $\sum_i a_i c_i \in B^{\widetilde{W}}$ then $c_i \in B^{\widetilde{W}}$ for $i = 1, \ldots, m$.

Proof. We enumerate $\Delta(\mathfrak{m},\mathfrak{t})^+ = \{\beta_1,\dots,\beta_q\}$ and choose bases $Y_1,\dots,Y_q$ of $\mathfrak{m}^-$, $X_1,\dots,X_q$ of $\mathfrak{m}^+$ with $Y_j \in \mathfrak{m}_{-\beta_j}$, $X_j \in \mathfrak{m}_{\beta_j}$. Also let $H_1,\dots,H_l$ be a basis of $\mathfrak{t}$. Let $E = E_\alpha, Y = Y_\alpha, Z = Z_\alpha$, where $\alpha \in P^+$ is a simple root. If $I, K \in \mathbb{N}_0^q$ set $Y^I = (Y_1)^{i_1}\cdots(Y_q)^{i_q}$, $X^K = (X_1)^{k_1}\cdots(X_q)^{k_q}$. If $J \in \mathbb{N}_0^l$ put $H^J = (H_1)^{j_1}\cdots(H_l)^{j_l}$. Then the Poincaré-Birkhoff-Witt Theorem implies that the elements $Y^I H^J X^K \otimes Z^s$ form a basis of $U(\mathfrak{m})\otimes U(\mathfrak{a})$. Let $c_i = \sum_{i,s,I,J,K} c_{i,s,I,J,K} Y^I H^J X^K \otimes Z^s$.

The element $b = \sum_i a_i c_i \in B$ if and only if (see (2))

$$E^n b(n-Y-1) \equiv b(-n-Y-1)E^n \qquad \mod (U(\mathfrak{k})\mathfrak{m}^+).$$

Now, using Lemma 18 (vi) of Tirao [11] and the hypothesis, we obtain

$$\begin{aligned} E^n b(n-Y-1) &= E^n \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J X^K (n-Y-1)^s \\ &= E^n \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J \\ &\quad \times \big(n-Y-1+(k_1\beta_1+\cdots+k_q\beta_q)(Y)\big)^s X^K \\ &\equiv E^n \sum_{i,s,J} a_i c_{i,s,0,J,0} H^J (n-Y-1)^s. \end{aligned} \tag{26}$$

Similarly, and taking into account that $[\mathfrak{m}^+, E] = 0$, we get

$$\begin{aligned} b(-n-Y-1)E^n &= \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J X^K (-n-Y-1)^s E^n \\ &= \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J \\ &\quad \times \big(-n-Y-1+(k_1\beta_1+\cdots+k_q\beta_q)(Y)\big)^s E^n X^K \\ &\equiv \sum_{i,s,J} a_i c_{i,s,0,J,0} H^J (-n-Y-1)^s E^n \\ &= E^n \sum_{i,s,J} a_i c_{i,s,0,J,0} \big(H-n\alpha(H)\big)^J \big(-n-Y-1+n\alpha(Y)\big)^s. \end{aligned} \tag{27}$$

Hence if $b \in B$, from (26) and (27) and using Lemma 20 of Tirao [11], we get

$$\sum_{i,s,J} a_i c_{i,s,0,J,0} H^J (n-Y-1)^s \equiv \sum_{i,s,J} a_i c_{i,s,0,J,0} \big(H-n\alpha(H)\big)^J \big(-n-Y-1+n\alpha(Y)\big)^s.$$

If we set $p_i = \sum_{s,J} c_{i,s,0,J,0}\big[H^J(n-Y-1)^s - (H-n\alpha(H))^J(-n-Y-1+n\alpha(Y))^s\big] \in U(\mathfrak{t})$ and apply Corollary 13 to $\sum_i a_i p_i \equiv 0$ we get that $p_i = 0$ for $i = 1,\dots,m$. Therefore

$$\sum_{s,J} c_{i,s,0,J,0} H^J (n-Y-1)^s = \sum_{s,J} c_{i,s,0,J,0} \big(H-n\alpha(H)\big)^J \big(-n-Y-1+n\alpha(Y)\big)^s \tag{28}$$

for $i = 1, \dots, m$. If we multiply (28) on the left by E^n and follow the steps leading to (26) and (27) backwards, we see that

$$E^n c_i(n - Y - 1) \equiv c_i(-n - Y - 1)E^n,$$

i.e. $c_i \in B$ for $i = 1, \dots, m$, proving (i).

To prove (ii) we just need to observe that for $w \in M_o' - M_o$, $\lambda \in \mathfrak{a}^*$ (see (3)) $Ad(w)(b(\lambda-\rho)) = b(-\lambda-\rho)$ is equivalent to $\sum_i a_i Ad(w)(c_i(\lambda-\rho)) = \sum_i a_i c_i(\lambda-\rho)$ which implies that $Ad(w)(c_i(\lambda - \rho))$ for all $i = 1, \dots, m$, because $Z(\mathfrak{k})Z(\mathfrak{m}) \simeq Z(\mathfrak{k}) \otimes Z(\mathfrak{m})$. This finishes the proof of our proposition.

Theorem 15. *If $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ then $\mu\colon Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \to U(\mathfrak{g})^K$ is a surjective isomorphism.*

Proof. Let us first consider the case $G_o = \mathrm{SO}(n,1)_e$. The proof will be by induction on $n \geq 2$. For $n = 2$ an s-triple $\{H, X, Y\}$ can be chosen in $\mathfrak{g}$ with $H \in \mathfrak{k}$. Set $\zeta = H^2 - 2H + 4XY$. Then $Z(\mathfrak{g}) = \mathbf{C}[\zeta]$, $Z(\mathfrak{k}) = \mathbf{C}[H]$ and $\{X^iY^iH^j\}$ is a basis of $U(\mathfrak{g})^K$. From this it is clear that $\mu\colon Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \to U(\mathfrak{g})^K$ is a surjective isomorphism. For $n \geq 2$ let $K_n = \mathrm{SO}(n) \times \mathrm{SO}(1) \simeq \mathrm{SO}(n)$, $M_n = \mathrm{SO}(1)\times\mathrm{SO}(n-1)\times\mathrm{SO}(1) \simeq \mathrm{SO}(1)\times\mathrm{SO}(n-1)$ and let $\mathfrak{g}_n$, $\mathfrak{k}_n$, $\mathfrak{m}_n$ denote respectively the complexifications of the Lie algebras of $\mathrm{SO}(n,1)_e$, K_n and M_n. Also let η be the automorphism of $\mathfrak{gl}(n, \mathbf{C})$ which interchanges the first and the last row and the first and the last column of a matrix. Since η is given by conjugation by an orthogonal matrix it clearly restricts to an automorphism of $\mathfrak{k}_n$.

Now assume the theorem has been already proved for $G_o = \mathrm{SO}(n-1,1)_e$, $n \geq 3$. Then

$$(29) \qquad U(\mathfrak{k}_n)^{M_n} = \eta\big(U(\mathfrak{g}_{n-1})^{K_{n-1}}\big) = \eta\big(Z(\mathfrak{g}_{n-1})\big)\eta\big(Z(\mathfrak{k}_{n-1})\big) = Z(\mathfrak{k}_n)Z(\mathfrak{m}_n).$$

Let us return to our old notation for $G_o = \mathrm{SO}(n,1)_e$. Given $u \in U(\mathfrak{g})^K$ set $b = P(u) \in B^{\widetilde{W}} \subset U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. Then we can write (see (29)) $b = \sum_{i=1}^m a_i c_i$ where $a_1, \dots, a_m$ are linearly independent in $Z(\mathfrak{k})$ and $c_i \in Z(\mathfrak{m}) \otimes U(\mathfrak{a})$ for $i = 1, \dots, m$. From Proposition 14 we know that $c_i \in B^{\widetilde{W}}$. Now by Theorem 13 there exist $u_i \in Z(\mathfrak{g})$ such that $c_i = P(u_i)$. Then $\sum_i a_i u_i \in U(\mathfrak{g})^K$ and $P(\sum_i a_i u_i) = P(u)$, hence $u = \sum_i a_i u_i \in Z(\mathfrak{k})Z(\mathfrak{g})$. This proves that $\mu\colon Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \to U(\mathfrak{g})^K$ is surjective. As we pointed out in the introduction this establishes the theorem for $G_o = \mathrm{SO}(n,1)_e$.

The proof for $\mathrm{SU}(n,1)$ will be also by induction on $n \geq 2$. For $n = 2$ we have $U(\mathfrak{k})^M = Z(\mathfrak{k})Z(\mathfrak{m})$ (Lemma 1). Given $u \in U(\mathfrak{g})^K$ set $b = P(u) \in B^{\widetilde{W}} \subset U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. Then $b = \sum_{i=1}^m a_i c_i$ where $a_1, \dots, a_m$ are linearly independent in $Z(\mathfrak{k})$ and $c_i \in Z(\mathfrak{m}) \otimes U(\mathfrak{a})$ for $i = 1, \dots, m$. As before from Proposition 14 and Theorem 11 it follows that $u \in Z(\mathfrak{k})Z(\mathfrak{g})$, proving the theorem for $\mathrm{SU}(2,1)$. For $n \geq 2$ let $K_n = \mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big)$ and

$$M_n = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} : a \in U(1), A \in U(n-1), a^2 \det A = 1 \right\}.$$

Also set $\mathfrak{g}_n$, $\mathfrak{k}_n$, $\mathfrak{m}_n$ denote respectively the complexifications of the Lie algebras of $\mathrm{SU}(n,1)$, K_n and M_n. Now take $n \geq 3$ and assume the theorem has been proved for $G_o = \mathrm{SU}(n-1,1)$. Then $\mathfrak{k}_n \simeq \mathfrak{gl}(n,\mathbf{C}) = \mathfrak{z}(\mathfrak{gl}(n,\mathbf{C})) \oplus \mathfrak{sl}(n,\mathbf{C}) = \mathfrak{z}(\mathfrak{gl}(n,\mathbf{C})) \oplus \mathfrak{g}_{n-1}$. Let

$$\bar{M}_n = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} : a \in U(1), A \in U(n-1), a^2 \det A = 1 \right\}.$$

$$\bar{K}_{n-1} = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : a \in U(1), A \in U(n-1), a \det A = 1 \right\}$$

and observe that

$$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in \bar{M}_n \text{ if and only if } a^{1/n} \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in \bar{K}_{n-1}.$$

Thus $U(\mathfrak{g}_{n-1})^{\bar{M}_n} = \eta\big(U(\mathfrak{g}_{n-1})^{\bar{K}_{n-1}}\big)$. Therefore

$$\begin{aligned} U(\mathfrak{k}_n)^{M_n} &\simeq U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big)U(\mathfrak{g}_{n-1})^{\bar{M}_n} \\ &= U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big)\eta\big(U(\mathfrak{g}_{n-1})^{\bar{K}_{n-1}}\big) \\ &= U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big)\eta\big(Z(\mathfrak{g}_{n-1})\big)\eta\big(Z(\mathfrak{k}_{n-1})\big) \\ &= U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big)Z(\mathfrak{g}_{n-1})Z(\mathfrak{m}_n) \\ &\simeq Z(\mathfrak{k}_n)Z(\mathfrak{m}_n). \end{aligned}$$

From this the proof is completed in the same way as in the case of $G_o = \mathrm{SO}(n,1)_e$.

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Recibido en Febrero 1995