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ON THE SUFFICIENT CONDITIONS OF MONOGENEITY FOR FONCTIONS OF COMPLEX-TYPE VARIABLE

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ABSTRACT. The theories of functions of hyperbolic and dual complex variable were deeply investigated between 1935 and 1941 as parallel theories with the classical complex analysis (see e.g. [2-6], [13-20]).

In some recent papers [7-8], [10-11], these theories present interest by some applications in the interpretation of physical phenomenoms.

In this spirit of ideas, the purpose of this paper is firstly to prove by counter-examples that the sufficient conditions of monogeneity in [5, p.148] and in [14, Theorem V, p.258] are false and secondly, to consider new correct conditions of monogeneity which moreover have the advantage of an unitary presentation.

1. INTRODUCTION

It is well known that a two-component number system forming an algebraic ring can be written in the form $z=a+qb,\ a,b \in R$, where q satisfies the equation $q^2 = \alpha q + \beta$ with fixed $\alpha, \beta \in R$. An important result states that all the systems $C_q = \{z=a+qb; a,b \in R\}$ are ring isomorphic with one of the following three types (see e.g. [9]):

(i) C_q with $q^2 = -1$, called the system of usual complex number, if $\alpha^2/4+\beta<0$;

(ii) C_q with $q^2=0$, called the system of dual complex numbers, if $\alpha^2/4+\beta=0$;

(iii) C_q with $q^2=+1$, if $\alpha^2/4+\beta>0$. In this case, a number in C_q is called binary [9], or double [21], or perplex [7-8], or anormal complex [1], or hyperbolic complex [4-6], [13].

While the theory of functions of usual complex variable is well known and does not represents the aim of the present note, the teory of functions of hyperbolic complex and dual complex variable was deeply investigated between 1935 and 1941 in e.g. [2-6], [13-20] (see also the more recent monograph [12] for generalisations to functions of hypercomplex variables).

In some recent papers (see e.g. [7-8], [10-11]), these theories were been taken in attention by some applications in the interpretation of physical phenomenoms.

In this spirit of ideas we firstly prove by counter-examples that the sufficient conditions of monogeneity in [5, p.148] and that the Theorem V in [14, p.258] are false and secondly, we consider new correct conditions of monogeneity which present the advantage that all the three cases $q^2= -1$, $q^2=0$ and $q^2=+1$ can be more unitaryly treated.

Throughout in this paper we will consider $q^2=+1$, or $q^2=0$, or $q^2= -1$ and a number $z=a+qb$ will be called q - complex number.

2. CONDITIONS OF MONOGENEITY

Keeping the notations in Introduction we can consider the following

DEFINITION 2.1 ([13], [14]). *If* $z=a+bq \in C_q$ *then* $|z|=\sqrt{a^2+b^2}$ *represents the modulus of the q - complex number z, in all the three cases* $q^2=+1$, $q^2=0$ *and* $q^2= -1$. *Also,* $N_q(z)=a^2-q^2b^2$ *represents the q - norm of the q - complex number z.*

THEOREM 2.2 ([13], [14]). *If* $q^2=0$ or $q^2= +1$ *then the set of all divisors of 0 in* C_q *is given by* $Z_q=\{z=a+qb; N_q(z)=0\}$. *Also, if* $z \in C_q \setminus Z_q$ *then z is invertible.*

REMARK. If $q^2= -1$ then $Z_q=\{0\}$ and C_q is even a field.

Let $D \subset C_q$ be and $f: D \to C_q$. Then we can write: $f(z)=u(x,y)+qv(x,y)$, for all $z=x+qy \in D$, where u and v are real functions of two real variables.

DEFINITION 2.3 ([5], [14]). *f is called q-monogenic in* $z_0 \in D$ *if there exists the limit*

$$\lim_{\substack{z\to z_0 \\ z-z_0\notin Z_q}} [f(z)-f(z_0)]/(z-z_0)=f'(z_0)$$

Concerning this concept, the following results are known.

THEOREM 2.4 ([5, p. 147]). *Let* $q^2=+1$. *If f is q-monogenic in* $z_0=x_0+qy_0 \in D$, *then u and v have partial derivatives of order one in* (x_0,y_0) *and the equalities*

(1) $[\partial u/\partial x](x_0,y_0)=[\partial v/\partial y](x_0,y_0)$, $[\partial u/\partial y](x_0,y_0)=[\partial v/\partial x](x_0,y_0)$

hold.

THEOREM 2.5 ([5, p. 148]). *Let* $q^2=+1$. *If u and v have continuous partial derivatives of order one in* (x_0,y_0) *which satisfy* (1), *then is q-monogenic in* $z_0=x_0+qy_0$.

THEOREM 2.6 ([14, Theorem V, p.258]). *Let* $q^2=0$. *The function f is q-monogenic in* $z_0=x_0+qy_0 \in D$ *if and only if u and v are differentiable in* (x_0,y_0) *and satisfy*

(2) $[\partial u/\partial y](x_0,y_0)=0$, $[\partial u/\partial x](x_0,y_0)=[\partial v/\partial y](x_0,y_0)$.

Firstly, we will prove by counter - examples that the Theorems 2.5 and 2.6 are false.

Indeed, let us define $u(x,y)=x^2+y^2$, $v(x,y)=0$ and $f(z)=u(x,y)+qv(x,y)=u(x,y)$, for all $z=x+qy$.

Obviously u and v have continuous partial derivative of order one in (0,0), which implies that u is differentiable in (0,0). Also, we immediately get

$$[\partial u/\partial x](0,0)=[\partial v/\partial y](0,0)=0,\ [\partial u/\partial y](0,0)=[\partial v/\partial x](0,0)=0.$$

Let $q^2=+1$. We have

$$\lim_{\substack{z\to z_0\\ z\notin Z_q}} [f(z)-f(0)]/z = \lim_{\substack{x,y\to 0\\ |x|\neq|y|}} u(x,y)/(x+qy) = \lim_{\substack{x,y\to 0\\ |x|\neq|y|}} (x^2+y^2)(x-qy)/(x^2-y^2) =$$

$$\lim_{\substack{x,y\to 0\\ |x|\neq|y|}} x(x^2+y^2)/(x^2-y^2) - q \lim_{\substack{x,y\to 0\\ |x|\neq|y|}} y(x^2+y^2)/(x^2-y^2).$$

But if we choose, for example, $x_n = 1/\sqrt{n}$, $y_n = 1/\sqrt{n+1}$ we get $x_n \to 0$, $y_n \to 0$, $|x_n| \neq |y_n|$ and

$$x_n\left(x_n^2+y_n^2\right)/\left(x_n^2-y_n^2\right) = \left(1/\sqrt{n}\right)\cdot\left(1/n+1/(n+1)\right)/\left[1/n-1/(n+1)\right] =$$

$$n(n+1)\cdot(2n+1)/\left[n(n+1)\sqrt{n}\right] = (2n+1)/\sqrt{n} \to +\infty, \text{ for } n\to+\infty.$$

Analogously,

$$y_n(x_n{}^2+y_n{}^2)/(x_n{}^2-y_n{}^2) = (2n+1)\sqrt{n+1} \to +\infty, \text{ for } n\to+\infty.$$

As conclusion, f is not monogenic in $z=0$ although u and v satisfy the conditions in Theorem 2.5. This means that Theorem 2.5 is false.

Now, let $q^2=0$. We get

$$\lim_{\substack{z\to 0\\ z\notin Z_q}} \left[f(z)-f(0)\right]/z = \lim_{\substack{x,y\to 0\\ x\neq 0}} u(x,y)/(x+qy) =$$

$$\lim_{\substack{x,y\to 0\\ x\neq 0}} \left(x^2+y^2\right)\left(x-qy\right)/x^2 = \lim_{\substack{x,y\to 0\\ x\neq 0}} (x^2+y^2)/x - q\cdot \lim_{\substack{x,y\to 0\\ x\neq 0}} \left(x^2+y^2\right)/x^2$$

But choosing $x=y^3$, $y\neq 0$, we obtain

$(x^2+y^2)/x = y^3+y^2/y^3 = y^3 + 1/y \to +\infty$, for $y\to 0$

and

$y(x^2+y^2)/x^2 = y+y^3/y^6 = y + 1/y^3 \to +\infty$, for $y\to 0$.

As conclusion, f is not monogenic in $z=0$, although u and v are differentiable in (0,0) and satisfy the relations (2) in Theorem 2.6. This means that the sufficient conditions in Theorem 2.6. are false.

Now, let $f(z) = u(x,y)+qv(x,y)$, $z = x+qy$, $q^2=0$, where

$$u(x,y) = \begin{cases} x, x\neq 0, y\in R \\ |y|, x=0, y\in R \end{cases}, \quad v(x,y) = \begin{cases} y, x\neq 0, y\in R \\ 0, x=0, y\in R \end{cases}$$

We have $u(0,0) = v(0,0) = f(0) = 0$ and

$$\lim_{\substack{z\to 0\\ z\notin Z_q}} [f(z)-f(0)]/z = \lim_{\substack{x,y\to 0\\ x\neq 0}} [u(x,y)+qv(x,y)]/(x+qy) =$$

$$\lim_{\substack{x,y\to 0\\ x\neq 0}} (x+qy)/(x+qy) = 1 = f'(0)$$

i.e. f is monogenic in z=0.
On the other hand, $(\partial u/\partial y)(0,0)$ does not exists because

$$\lim_{\substack{y\to 0\\ y\neq 0}}[u(0,y)-u(0,0)]/y = \lim_{\substack{y\to 0\\ y\neq 0}}|y|/y$$

As conclusion the necessary conditions in Theorem 2.6. also are false.
In the sequel we will give correct versions for the above Theorems 2.5. and 2.6.
Firstly, we will introduce the following.

DEFINITION 2.7. *Let* $u: M \to R, M \subset R^2$ *be and* $(x_0, y_0) \in M$. *We say that u is q-differentiable in* (x_0,y_0) *if there exist* $A, B \in R$ *and* $\omega = \omega(x,y)$ *with*

$$\lim_{\substack{x\to x_0\\ y\to y_0\\ z-z_0\notin Z_q}} \omega(x,y) = \omega(x_0,y_0) = 0 \text{ where } z = x+qy,\ z_0 = x_0+qy_0 \text{ such that}$$

$$u(x,y) - u(x_0,y_0) = A(x-x_0) + B(y-y_0) + \omega(x,y)\cdot N_q(z-z_0)/|z-z_0|, \text{ for all } (x,y)\in M$$

with $z - z_0 \notin Z_q$.

REMARKS. 1). Obviously we have

$$N_q(z-z_0)/|z-z_0| = \left[(x-x_0)^2 - q^2(y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2}$$

2). If $q^2 = -1$ then the Definition 2.7 becomes the usual definition of differentiability in (x_0,y_0). Concerning the q - differentiability we can prove the following.

LEMMA 2.8. (i) *Let* $q^2 = +1$. *If u is q - differentiable in* (x_0,y_0) *then there exists* $[\partial u/\partial x](x_0,y_0) = A$ *and* $[\partial u/\partial y](x_0,y_0) = B$.

(ii) *Let* $q^2 = 0$. *If u is q - differentiable in* (x_0,y_0) *then there exists* $[\partial u/\partial x](x_0,y_0) = A$. *If moreover there exist* $\delta > 0$ *such that* $F(x) = u(x,y)$ *is continuous as function of x* **in** $x_0, \forall |y-y_0| < \delta$, *then there exists* $[\partial u/\partial y](x_0,y_0) = B$.

PROOF. (i) Taking in Definition 2.7 $x = x_0$ and $y \neq y_0$ (which implies $z - z_0 \notin Z_q$), we obtain

$$u(x_0,y) - u(x_0,y_0) = B(y-y_0) + \omega(x_0,y)\cdot\left[-(y-y_0)^2\right]/|y-y_0|.$$

Dividing by $y - y_0 \neq 0$ and then passing to limit with $y \to y_0$ we get

$$\lim_{\substack{y\to y_0\\ y\neq y_0}} [u(x_0,y)-u(x_0,y_0)]/(y-y_0) = B - \lim_{\substack{y\to y_0\\ y\neq y_0}} \omega(x_0,y)(y-y_0)/|y-y_0| = B$$

since

$$\lim_{\substack{x\to x_0\\ y\to y_0\\ z-z_0\notin Z_q}} \omega(x,y)=\lim_{\substack{y\to y_0\\ y\neq y_0}} \omega(x_0,y)=0$$

Analogously, taking in Definition 2.7 $y=y_0$ and $x\neq x_0$ and reasoning as above, we get that there exists $[\partial u/\partial y](x_0,y_0)=B$.

(ii) Taking in Definition 2.7 $y=y_0$ and $x\neq x_0$ (which implies $z-z_0\notin Z_q$), we obtain

$u(x,y_0)-u(x_0,y_0) = A(x-x_0)+\omega(x,y_0)\cdot|x-x_0|, \forall x\neq x_0$.

Dividing by $x-x_0\neq 0$ and passing to limit with $x\to x_0$ we immediately get $[\partial u/\partial x](x_0,y_0)=A$.

Now, let $|y-y_0|<\delta$, $y\neq y_0$ be fixed. Passing to limit with $x\to x_0$, $x\neq x_0$ in Definition 2.7, we obtain

$$u(x_0,y)-u(x_0,y_0)=B(y-y_0)+\lim_{\substack{x\to x_0\\ x\neq x_0}} \omega(x,y)(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}$$

for all $|y-y_0|<\delta$, $y\neq y_0$.

But by $\lim_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}} \omega(x,y)=0$ follows that for $\varepsilon>0$, there exists $\delta_1>0$ such that $|\omega(x,y)|<\varepsilon$, for all $|x-x_0|<\delta_1$, $x\neq x_0$ and all $|y-y_0|<\delta_1$.

Denote $\delta_0=\min\{\delta,\delta_1\}$ and let $|y-y_0|<\delta_0$, $y\neq y_0$.

We get $\lim_{\substack{x\to x_0\\ x\neq x_0}} |\omega(x,y)|\leq\varepsilon$, for all $|y-y_0|<\delta_0$, $y\neq y_0$. Since

$(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}=|x-x_0|\cdot|x-x_0|/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq|x-x_0|$, we obtain

$$\lim_{\substack{x\to x_0\\ x\neq x_0}} |\omega(x,y)|\cdot(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq\varepsilon\cdot\lim_{\substack{x\to x_0\\ x\neq x_0}} |x-x_0|=0 \text{ for all } |y-y_0|<\delta_0,\ y\neq y_0$$

As conclusion,

$u(x_0,y)-u(x_0,y_0)=B(y-y_0)$, $\forall\ y\neq y_0$, $|y-y_0|<\delta_0$.

Therefore, dividing with $y-y_0\neq 0$ and then passing to limit with $y\to y_0$ we obtain $[\partial u/\partial y](x_0,y_0)=B$, which proves the lemma.

A correct version of Theorem 2.5 is the

THEOREM 2.9 *Let* $q^2=+1$ *be and* $f:D\to C_q; D\subset C_q$, $f(z)=u(x,y)+qv(x,y)$,

$z = x + qy \in D$, $z_0 = x_0 + qy_0 \in D$.

If u and v are q - differentiable in (x_0, y_0) *and satisfy the relations* (1) *in Theorem 2.5, then f is q-monogenic in* z_0.

PROOF. By hypothesis and by Lemma 2.8, (i) we get

$$u(x,y) - u(x_0,y_0) = a(x-x_0) + b(y-y_0) + \omega_1(x,y)\cdot\left[(x-x_0)^2 - (y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2},$$

$$v(x,y) - v(x_0,y_0) = b(x-x_0) + a(y-y_0) + \omega_2(x,y)\cdot\left[(x-x_0)^2 - (y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2},$$

for all $x - x_0 + q(y - y_0) = z - z_0 \notin Z_q$, where $\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ z-z_0\notin Z_q}} \omega_j(x,y) = 0$, j=1,2.

By simple calculus we obtain

$$f(z) - f(z_0) = (a+bq)(z-z_0) + \left[\omega_1(x,y) + q\omega_2(x,y)\right]\cdot\left[(x-x_0)^2-(y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Dividind by $z - z_0 \notin Z_q$ and then multiplying by $1=[(x-x_0)-q(y-y_0)]/[(x-x_0)-q(y-y_0)]$ on the right hand, the above equality becomes

$$[f(z)-f(z_0)]/(z-z_0) = a+bq+[(x-x_0)-q(y-y_0)]\cdot[\omega_1(x,y)+q\omega_2(x,y)]/\sqrt{(x-x_0)^2+(x-x_0)^2}$$

$$= a+bq+(x-x_0)\cdot\omega_1(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2} - (y-y_0)\cdot\omega_2(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2} +$$

$$q\left[(x-x_0)\cdot\omega_2(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2} - (y-y_0)\cdot\omega_1(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2}\right]$$

By $|x-x_0|/\sqrt{(x-x_0)^2+(y-y_0)^2} \le 1$ and $|y-y_0|/\sqrt{(x-x_0)^2+(y-y_0)^2} \le 1$, passing to limit with $z \to z_0$, $z - z_0 \notin Z_q$ (which is equivalent with $x \to x_0, y \to y_0$, $|x-x_0| \ne |y-y_0|$), we immediately get that there exists $\lim\limits_{\substack{z\to z_0\\ z-z_0\notin Z_q}} [f(z)-f(z_0)]/(z-z_0) = a+qb$ which proves the theorem.

Now, a correct version of Theorem 2.6 is the

THEOREM 2.10. *Let* $q^2=0$ *and* $f: D \to C_q$, $D \subset C_q$, $f(z) = u(x,y) + qv(x,y)$, $z = x + qy \in D$, $z_0 = x_0 + qy_0 \in D$, *such that* $F(x) = u(x,y)$ *and* $G(x) = v(x,y)$ *are continuous as functions of x in* x_0, *for all y belonging to a neighbourhood of* y_0, *denoted by* $V(y_0)$.

If f is q-monogenic in z_0, *then u and v satisfy the relations* (2) *in Theorem 2.6.*

Conversely, if u and v are q-differentiable in (x_0, y_0) *and satisfy the relations* (2) *in Theorem 2.6, then f is q-monogenic in* z_0.

PROOF. Let suppose that f is q-monogenic in z_0.

Let us denote $h(z) = [f(z) - f(z_0)]/(z - z_0) - f'(z_0) =$

$$=[f(z)-f(z_0)]/(z-z_0)-(a+bq)=h_1(x,y)+qh_2(x,y),\ \ z-z_0\notin Z_q(i.e.\ x\neq x_0).$$

By hypothesis we get $\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}} h_i(x,y)=0,\ i=\overline{1,2}$ and

$$h_1(x,y)+qh_2(x,y)=[u(x,y)-u(x_0,y_0)+q(v(x,y)-v(x_0,y_0))]/[(x-x_0)+q(y-y_0)]-(a+qb), x\neq x_0.$$

By simple calculus, for all $x\neq x_0$ and all y with $z,z_0,z-z_0\in D$, we obtain

(3) $u(x,y)-u(x_0,y_0)=a(x-x_0)+h_1(x,y)(x-x_0)$,

(4) $v(x,y)-v(x_0,y_0)=b(x-x_0)+a(y-y_0)+h_2(x,y)(x-x_0)+h_1(x,y)(y-y_0)$

Taking $y=y_0$ in (3), dividing with $x-x_0\neq 0$ and then passing to limit with $x\to x_0, x\neq x_0$, it follows that $[\partial u/\partial x](x_0,y_0)=a$, since $\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}} h_1(x,y)=\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y_0)=0$.

Then, passing with $x\to x_0$ in (3) and taking into account that $F(x)=u(x,y)$ is continuous in x_0, we get

(5) $u(x_0,y)-u(x_0,y_0)=\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)\cdot 0, \forall y\in V(y_0)$.

But reasoning exactly as in the proof of Lemma 2.8, (ii), (for $\omega(x,y)\equiv h_1(x,y)$), there exists a neighbourhood $V_1(y_0)$ such that $|\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)|=\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} |h_1(x,y)|\leq\varepsilon,\ for\ all\ y\in V_1(y_0)$

Combining with (5) we obtain

$$u(x_0,y)-u(x_0,y_0)=0, \forall y\in V(y_0)\bigcap V_1(y_0).$$

This obviously implies $(\partial u/\partial y)(x_0,y_0)=0$.

Analogously, taking $y=y_0$ in (4) as above we have $[\partial v/\partial x](x_0,y_0)=b$.

Then passing to limit with $x\to x_0$ in (4) and taking into account that $G(x)=v(x,y)$ is continuous in x_0, it follows

$$v(x_0,y)-v(x_0,y_0)=a(y-y_0)+\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_2(x,y)\cdot 0+\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)\cdot(y-y_0),\ \ \text{for all}$$

$y\in V(y_0)$.

Reasoning as above, there exists $V_1(y_0)$ such that

$$v(x_0,y)-v(x_0,y_0)=a(y-y_0)+\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)\cdot(y-y_0), \forall\ y\in V(y_0)\bigcap V_1(y_0).$$

Dividing by $y-y_0\neq 0$ and then passing to limit with $y\to y_0$ we get

$$[\partial v/\partial y](x_0,y_0)=a+\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0, y\neq y_0}} h_1(x,y)=a+0=a$$

As conclusion, $[\partial u/\partial y](x_0,y_0)=0$ and $[\partial u/\partial x](x_0,y_0)=[\partial v/\partial y](x_0,y_0)$.

Now let suposse that u and v are q-differentiable in (x_0,y_0) and satisfy the relations (2) in Theorem2.6.

By Lemma 2.8, (ii) and by hypothesis we get

$$u(x,y)-u(x_0,y_0)=a(x-x_0)+\omega_1(x,y)\cdot(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2},$$

$$v(x,y)-v(x_0,y_0)=A(x-x_0)+a(y-y_0)+\omega_2(x,y)\cdot(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2},$$ for all $x\neq x_0$, y, such that $z,z_0,z-z_0\in D$, where

$$\lim_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}}\omega_i(x,y)=0,\ i=1,2 \text{ and } a=[\partial u/\partial x](x_0,y_0), A=[\partial v,\partial x](x_0,y_0)$$

By simple calculus we get

$$f(z)-f(z_0)=(a+qA)\cdot(z-z_0)+(x-x_0)^2\cdot[\omega_1(x,y)+q\omega_2(x,y)]/\sqrt{(x-x_0)^2+(y-y_0)^2}$$

Dividing by $z-z_0\notin Z_q$ and then multiplying with $1=[(x-x_0)-q(y-y_0)]/[(x-x_0)-q(y-y_0)]$ on the right hand, we arrive at

$$[f(z)-f(z_0)]/(z-z_0)=a+qA+\omega_1(x,y)\cdot(x-x_0)/\sqrt{(x-x_0)^2+(y-y_0)^2}+$$
$$q\cdot[\omega_2(x,y)\cdot(x-x_0)/\sqrt{(x-x_0)^2+(y-y_0)^2}-\omega_1(x,y)\cdot(y-y_0)/\sqrt{(x-x_0)^2+(y-y_0)^2}]$$

Passing to limit with $z\to z_0, z-z_0\notin Z_q$ (which is equivalent with $x\to x_0, y\to y_0, x\neq x_0$) by

$|x-x_0|/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq 1, |y-y_0|/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq 1$, and by the hypothesis on $\omega_i(x,y)$ we immediately get

$$\lim_{\substack{z\to z_0\\ z-z_0\notin Z_q}}[f(z)-f(z_0)]/(z-z_0)=a+qA,$$ which proves the theorem.

REMARKS. 1). If $q^2=-1$ it is known that the q-differentiability of u and v in (x_0,y_0) together with the Cauchy-Riemann conditions in (x_0,y_0) is even equivalent with the monogeneity of f in $z_0=x_0+qy_0$.

2). In the cases when $q^2=+1$ or $q^2=0$, there exist functions $f=u+qv$ with u and v q-differentiable in (x_0,y_0) and satisfying (1) or (2), respectively.

Indeed, for $q^2=+1$ let us define

$$u(x,y)=\begin{cases}0, |x|=|y|\\ |x^2-y^2|, |x|\neq|y|\end{cases}, v(x,y)\equiv 0, f(z)=u(x,y), z=x+qy.$$ We have

$$[\partial u/\partial x](0,0)=\lim_{\substack{x\to 0\\ x\neq 0}}[u(x,0)-u(0,0)]/x=\lim_{\substack{x\to 0\\ x\neq 0}}x^2/x=0,$$

$$[\partial u/\partial y](0,0)=\lim_{\substack{y\to 0\\ y\neq 0}}[u(0,y)-u(0,0)]/y=\lim_{\substack{y\to 0\\ y\neq 0}}y^2/y=0,$$

$[\partial v / \partial x](0,0) = [\partial v / \partial y](0,0) = 0.$

Also, $u(x,y) - u(0,0) = 0 \cdot x + 0 \cdot y + \omega(x,y) \cdot |x^2 - y^2| / \sqrt{x^2 + y^2}$ for all $|x| \neq |y|$, where $\omega(x,y) = \sqrt{x^2 + y^2}$ satisfies $\lim_{\substack{x \to 0 \\ y \to 0 \\ |x| \neq |y|}} \omega(x,y) = 0$, i.e. u is q-differentiable in (0,0).

Analougously, for $q^2 = 0$ we define

$u(x,y) = \begin{cases} x^2, x \neq 0, y \in R \\ 0, x = 0, y \in R \end{cases}, v(x,y) \equiv 0, f(z) = u(x,y), z = x + qy$. It is easy to check that $[\partial u / \partial x](0,0) = [\partial u / \partial y](0,0) = 0$ and u is q-differentiable in (0,0) with $\omega(x,y) = \sqrt{x^2 + y^2}$.

3). Let $q^2 = +1$. The sufficient conditions of q-monogenity in Theorem 2.9 however are not necessary. Indeed, let us define $f(z) = u(x,y) + q(x,y), z = x + qy, z_0 = 0,$

$$u(x,y) = \begin{cases} x(x^2 + y^2), |x| \neq |y| \\ 0, |x| = |y| \end{cases}, v(x,y) = \begin{cases} y(x^2 + y^2), |x| \neq |y| \\ 0, |x| = |y| \end{cases}$$

We have

$$f'(0) = \lim_{\substack{z \to 0 \\ |x| \neq |y|}} [f(z) - f(0)] / z = \lim_{\substack{x,y \to 0 \\ |x| \neq |y|}} [u(x,y) + qv(x,y)] / (x + qy) =$$

$$\lim_{\substack{x,y \to 0 \\ |x| \neq |y|}} (x^2 + y^2) \cdot (x + qy) / (x + qy) = \lim_{\substack{x,y \to 0 \\ |x| \neq |y|}} (x^2 + y^2) = 0,$$

wich means that f is monogenic in $z_0 = 0$.

On the other hand u is not q-differentiable in (0,0). Indeed, let suppose that u is q-differentiable. We easily get $[\partial u / \partial x](0,0) = [\partial u / \partial y](0,0) = 0$ and therefore by Lemma 2.8, (i) we get

$$u(x,y) = \omega(x,y) \cdot [x^2 - y^2] / \sqrt{x^2 + y^2}, \text{ for all } |x| \neq |y|, \text{ with } \lim_{\substack{x,y \to 0 \\ |x| \neq |y|}} \omega(x,y) = 0.$$

It follows $x(x^2 + y^2) = \omega(x,y) \cdot [x^2 - y^2] / \sqrt{x^2 + y^2}$, which implies $\omega(x,y) = x(x^2 + y^2)^{3/2} / [x^2 - y^2]$, for all $|x| \neq |y|$.

Now, choosing for example $x_n = 1/\sqrt{n}, y_n = 1/\sqrt{n+1} \to 0, |x_n| \neq |y_n|$, by simple calculus we obtain

$$\omega(x_n, y_n) = (2n+1)^{3/2} / [n\sqrt{n+1}] \xrightarrow{n \to +\infty} 2, \text{ contradiction.}$$

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