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Characterization of the Moment Space of a Sequence of Exponentials

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Abstract

We consider the moment problem for the sequence $\left\{e^{-\lambda_i t}\right\}_{i\in N}$ in $L^2(0,T)$ $(0 < T \leq \infty)$, being $\{\lambda_i\}_{i\in N}$ a sequence of positive real numbers such that $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. We prove properties of the moment space M of that sequence. In **[K]** it is shown that M is a moment space. Our main result is that M is a Hilbert space and moreover, that is the image of ℓ^2 by the operator $G^{1/2}$, the square root of the Gram matrix G of the sequence. The operator $G^{1/2}$ is proved to be the limit in $B(\ell^2)$ of a sequence of simple operators of finite rank. We also obtain an upper bound for the norm of the operator G. We find different expressions for the solution of minimum norm of the stated moment problem, extending some results of **[Z]**.

1 Introduction

We consider the moment problem of the sequence:

$$\left\{e^{-\lambda_i t}\right\}_{i\in N} \tag{1}$$

in $L^2(0,T)$ $(0 < T \leq \infty)$, being $\{\lambda_i\}_{i\in N}$ a sequence of positive real numbers such that:

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$$

Remark: This condition implies that the sequence **(1)** is not dense in $L^2(0,T)$.

Our main goal is to characterize the moment space M of that sequence. In the first section we introduce the moment problem and recall some well known results about it. In the second section we prove the following properties of M:

*) M is a dense and proper subspace in ℓ^2.

*) M does not depend on T.

*) M is a Hilbert space, and there exists a continous inmersion in ℓ^2.

In the third section we obtain the operator G. It is defined by the Gram matrix of the sequence (1) as the limit in $B(\ell^2)$ of a sequence of simple operators of finite rank. This allows us to show that $G^{1/2}$ is a compact operator.

In section four we prove that M is the image of ℓ^2 by the operator $G^{1/2}$. In the last two sections we find different expressions for the solution of minimal norm of the moment problem of our interest.

2 The moment problem.

Let H be a real Hilbert space, provided by an inner product $(\cdot,\cdot)$. Let $\{f_k\}_{k\in N}$ a sequence of elements of H such that any finite subfamily of this sequence is linearly independent. We note by $\{c_k\}_{k\in N}$ an arbitrary real sequence. So, the inner product (f, f_k) , $k \in N$ is called *the nth. moment of* f, and the sequence $\{(f, f_k)\}_{k\in N}$ is *the moment sequence of* f. Then in the theory of moments the following problem arises:

Does there exist an element $f \in H$ *such that* : $(f, f_k) = c_k, k = 1, 2, ...?$

The *moment space* M of $\{f_k\}$ is then the collection of all the moment sequences $M = \{(f, f_k) : f \in H\}$. Thus a numerical sequence $\{c_k\}_{k\in N}$ belongs to M if and only if there exist $f \in H$ such that $c_k = (f, f_k)$, $k = 1, 2, ...$

M is a Banach space with the norm defined by:

$$\|c\|_M^2 = \sup_{n\in N} \sum_{k,l=1}^{n} \sigma_{l,k}^{(n)} c_k c_l = \lim_{n\to\infty} \sum_{k,l=1}^{n} \sigma_{l,k}^{(n)} c_k c_l$$

where $\sigma_{l,k}^{(n)}$ is the (l,k) element of the inverse of the Gram matrix G_n of $\{f_1, f_2, ..., f_n\}$. The last equality is valid because:

$$\sum_{k,l=1}^{n} \sigma_{k,l}^{(n)} c_k c_l$$

does not decrease as n increases **[K]**. It is easily proved that M is also a Hilbert space (cf. Lema 2).

Remark: To avoid confussion we use a subscript denoting the space we are refering to; for example $(\cdot,\cdot)_H$ or $\|\cdot\|_H$.

3 The moment space of a sequence of exponentials. Some properties.

Let $H = H(T) = L^2(0,T)$, $0 < T \leq \infty$ and let $f_k(t) = e^{-\lambda_k t}$, k = 1,2,..., being $\{\lambda_k\}_{k\in N}$ a sequence of positive real numbers such that $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$ and $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$. In what follows, we will call $M(T)$ the moment space of (1) if $0 < T < \infty$, and M if $T = \infty$. We will study properties of M and $M(T)$.

If $T < \infty$,let

$$G_n(T) = \left(\frac{1 - e^{-(\lambda_i+\lambda_j)T}}{\lambda_i + \lambda_j}\right)_{1\leq i,j\leq n}$$

be the Gram matrix of $\left\{e^{-\lambda_k t}\right\}_{1\leq k\leq n}$, $n \in N$, and

$$G(T) = \left(\frac{1 - e^{-(\lambda_i+\lambda_j)T}}{\lambda_i + \lambda_j}\right)_{i,j\in N}$$

be the Gram matrix of $\left\{e^{-\lambda_k t}\right\}_{k\in N}$.

If $T = \infty$,then

$$G_n = \left[\frac{1}{\lambda_i + \lambda_j}\right]_{1\leq i,j\leq n} \quad n \in N \qquad G(T) = \left[\frac{1}{\lambda_i + \lambda_j}\right]_{i,j\in N}$$

PROPOSITION:

a) $M(T) \subset \ell^2$, $M(T) \neq \ell^2, \forall T > 0$

b) $M(T) = M$, $\forall T > 0$

c) M *is dense in* ℓ^2 *,and the inmersion i:* $M \to \ell^2$ *is continuous.*

Proof:

a) Let $\gamma_1^{(n)}(T)$ be the greatest eingenvalue of $G_n(T)$, and $\gamma_n^{(n)}(T)$ be the smallest one. Then

$$\gamma_1^{(n)}(T) = \max_{x\in R, x\neq 0} \frac{(x, G_n(T)x)}{\|x\|^2} \qquad , x = (x_i)_{1\leq i\leq n}$$

and

$$(x, G_n(T)x) = \sum_{i,j=1}^{n} \frac{1 - e^{-(\lambda_i+\lambda_j)T}}{\lambda_i + \lambda_j} x_i x_j \leq \sum_{i,j=1}^{n} \frac{1}{\lambda_i + \lambda_j} |x_i| |x_j| =$$

$$\sum_{i,j=1}^{n} \frac{\left(\lambda_i \lambda_j\right)^{1/2}}{\lambda_i + \lambda_j} \frac{|x_i|}{(\lambda_i)^{1/2}} \frac{|x_j|}{(\lambda_j)^{1/2}} \leq \frac{1}{2} \left(\sum_{i=1}^{n} \frac{x_i}{(\lambda_i)^{1/2}}\right)^2 \leq Tr\, G_n \|x\|^2$$

where $Tr\ G_n$ is the trace of G_n. Then $\gamma_1^{(n)}(T) \leq Tr\ G_n$, $\forall n \in N$, (1) is a Bessel sequence **[Y]**, and $M(T) \subset \ell^2$.

Since

$$\gamma_n^{(n)}(T) \le \frac{1-e^{-2\lambda_n T}}{2\lambda_n}$$

then $\gamma_n^{(n)}(T) \to 0$ if $n \to \infty$, and (1) is not a Riesz-Fischer sequence. Then $M(T) \neq \ell^2$.

b) $(G_n - G_n(T))_{i,j} = \int_T^\infty e^{-\lambda_i t} e^{-\lambda_j t} dt$ then $G_n - G_n(T)$ is the Gram matrix of $\left\{e^{-\lambda_i t}\right\}_{1\le i\le n}$ in $L^2(T,\infty)$. So $G_n - G_n(T)$ is positive definite. It follows that $G_n \ge G_n(T)$.

In addition to this, the following result is valid

LEMMA 1: $G_n^{-1}(T) \ge G_n^{-1}$.

Proof:

Let L be a linear transformation such that [CH] $L^T G_n(T) L = Id$ and $L^T G_n L = D$ where $D = (d_{i,j})_{1\le i,j\le n}$ is the diagonal matrix of order n such that

$$d_{i,j} = \begin{cases} \rho_i & i = j \\ 0 & i \neq j \end{cases}$$

Then $G_n - G_n(T) \ge 0$ implies that $\rho_i \ge 1$, $1 \le i \le n$. Also $L^{-1} G_n^{-1}(T) (L^T)^{-1} = Id$ and $L^{-1} G_n^{-1}(T) (L^T)^{-1} = \widetilde{D}$, where $\widetilde{D} = (\widetilde{d}_{i,j})_{1\le i,j\le n}$ is the diagonal matrix of order n such that

$$\widetilde{d}_{i,j} = \begin{cases} 1/\rho_i & i = j \\ 0 & i \neq j \end{cases}$$

Then $Id - \widetilde{D} \ge 0$ and $G_n^{-1}(T) \ge G_n^{-1}$.

As a consequence of Lemma 1, $M(T) \subseteq M$. Also, there exists a constant $K = K(T)$ such that:

$$\frac{1}{K(T)} G_n \le G_n(T).$$

In fact, let $c = (c_j)_{j\in N} \in \omega$, and $c(n) = (c_1, c_2, ..., c_n) \in R^n$

$$(c(n), G_n c(n)) = \int_0^\infty \left(\sum_{i=1}^n c_i e^{-\lambda_i t}\right)\left(\sum_{j=1}^n c_j e^{-\lambda_j t}\right) dt = \|P(t)\|^2_{L^2(0,\infty)}$$

where $P(t) := \sum_{i=1}^n c_i e^{-\lambda_i t}$. In an analogous way,

$$(c(n), G_n(T) c(n)) = \|P(t)\|^2_{L^2(0,T)} .$$

Acording to a result proved by Scwartz [S] there exists a constant $K = K(T)$ such that

$$\|P(t)\|_{L^2(0,\infty)} \leq K(T)\,\|P(t)\|_{L^2(0,T)}\,.$$

Hence $\frac{1}{K(T)}G_n \leq G_n(T)$ and $G_n^{-1}(T) \leq K(T)G_n^{-1}$. Therefore $M \subseteq M(T)$.

c) Let $x \in \ell^2$ be such that $(x,c)_{\ell^2} = 0$, $\forall c \in M$. Since $c \in M$ there exists $\Psi(t) \in L^2(0,T)$ such that:

$$\int_0^T \Psi(t)\, e^{-\lambda_j t} dt = c_j, \forall j \in N.$$

Then $\sum_{i=1}^{\infty} x_i c_i = \sum_{i=1}^{\infty} x_i \int_0^T \Psi(t)\, e^{-\lambda_i t} dt = 0$, $\forall \Psi(t) \in L^2(0,T)$. By the continuity of the inner product

$$\lim_{N\to\infty} \int_0^T \left(\sum_{i=1}^{N} x_i e^{-\lambda_i t} \right) \Psi(t)\, dt = 0.$$

Since $\sum_{i=1}^{\infty} x_i e^{-\lambda_i t} \in L^2(0,T)$, it follows that $\int_o^T \left(\sum_{i=1}^{\infty} x_i e^{-\lambda_i t} \right) \Psi(t)\, dt = 0$.

The sequence (1) is minimal in the sense that each element of the sequence lies outside the closed linear span of the others. Then there exists a biorthogonal sequence [Y] $\{g_i(t)\}_{i\in N}$ such that taking $\Psi(t) = g_i(t)$ will give $x_i = 0$, $\forall i \in N$. Then $x \equiv 0$.

To show that the inmersion $i : M \to \ell^2$ is continuous, we shall show that:

$$\|c\|_{\ell^2}^2 \leq Tr\, G\, \|c\|_M^2\,.$$

This is inmediate since

$$\left(c(n), G_n^{-1}c(n)\right) = \|c(n)\|^2 \frac{(c(n), G_n^{-1}c(n))}{\|c(n)\|^2} \geq \|c(n)\|^2 \left(\gamma_1^{(n)}\right)^{-1} \geq$$

$$\|c(n)\|^2 \left(Tr\, G_n\right)^{-1} \qquad \bullet$$

LEMMA 2: $(M; \|\cdot\|_M)$ is a Hilbert space.

4 An approximation to the Gram matrix.

The Gram matrix:

$$G = \left(\frac{1}{\lambda_i + \lambda_j} \right)_{1\leq i,j<\infty}$$

generates a bounded operator on ℓ^2 because $\|G\| \leq Tr\, G$. This result is a particular case of the following one:

LEMMA 3: If $G = (g_{i,j})_{1\leq i,j<\infty}$ *is the Gram matrix of a system* $\{f_i\}_{i\in N}$ *such that* $\sum_{i=1}^{\infty} g_{i,i} < \infty$ *then* $|g_{i,j}| = |(f_i, f_j)| \leq \|f_i\|\,\|f_j\| \leq (g_{i,i})^{1/2}\,(g_{j,j})^{1/2}$, $1 \leq i,j < \infty$ and $\left|\sum_{i,j=1}^{\infty} g_{i,j} x_i x_j\right| \leq \left(\sum_{i=1}^{\infty} g_{i,i}\right)\left(\sum_{i=1}^{\infty} |x_i|^2\right)$. Hence $\|G\| \leq \sum_{i=1}^{\infty} g_{i,i} = Tr\, G$.

LEMMA 4: $\|G\| < TrG$.

Let G_n be the nth. section of G, $G_n = (g_{i,j})_{1\leq i,j\leq n}$.

Then the infinite matrix $\widetilde{G_n} = (\tilde{g}_{i,j})_{1\leq i,j<\infty} = \begin{cases} g_{i,j}\ ,1 \leq i,j \leq n \\ 0\ ,i > n \text{ or } j > n \end{cases}$ defines a bounded operator $\widetilde{G_n} : \ell^2 \to \ell^2, \forall n \in N$

LEMMA 5: $\widetilde{G_n} \to G$ *on* $B(\ell^2)$ *if* $n \to \infty$.

Proof:

Let $R_n := G - \widetilde{G_n}$ and let $x \in \ell^2$, $y = R_n x$. Then

$$y_i = \sum_{j=n+1}^{\infty} g_{i,j} x_j \quad i = 1, 2, ..., n \qquad y_{n+i} = \sum_{j=1}^{\infty} g_{n+i,j} x_j \quad i = 1, 2, ...$$

thus, if $1 \leq i \leq n$,

$$y_i^2 \leq \left(\sum_{j=n+1}^{\infty} g_{i,j}^2\right) \cdot \left(\sum_{j=n+1}^{\infty} x_j^2\right) \text{ and } y_{n+i}^2 \leq \left(\sum_{j=1}^{\infty} g_{n+i,j}^2\right) \cdot \left(\sum_{j=1}^{\infty} x_j^2\right).$$

Hence

$\sum_{i=1}^{n} y_i^2 \leq \left(\sum_{i=1}^{n}\sum_{j=n+1}^{\infty} g_{i,j}^2\right) \cdot \left(\sum_{j=n+1}^{\infty} x_j^2\right) \leq \frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{\lambda_i}\right)\left(\sum_{j=n+1}^{\infty}\frac{1}{\lambda_j}\right)\left(\sum_{j=1}^{\infty} x_j^2\right) =$
$= \frac{1}{2}\tau\left(\sum_{j=n+1}^{\infty}\frac{1}{\lambda_j}\right)\|x\|_{\ell^2}^2$ where $\tau := \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{\lambda_i}$. In an analogous way results

$$\sum_{i=n+1}^{\infty} y_i^2 \leq \frac{1}{2}\tau\left(\sum_{j=n+1}^{\infty}\frac{1}{\lambda_j}\right)\|x\|_{\ell^2}^2.$$

Hence $\left\|G - \widetilde{G_n}\right\|^2 \leq \tau\left(\sum_{j=n+1}^{\infty}\frac{1}{\lambda_j}\right)$ and $\widetilde{G_n} \to G$ on $B(\ell^2)$ if $n \to \infty$. •

Remark: It can be proved in a similar way that Lemma 5 is valid if $G = (g_{i,i})_{1\leq i,j<\infty}$is a Gram matrix such that $\sum_{i=1}^{\infty} g_{i,i} < \infty$

The operators $\widetilde{G_n}$ are of finite rank and positive (recall that a bounded linear operator T on a Hilbert space H is said to be *positive* if $(Tf, f) \geq 0$, $\forall f \in H$). Therefore G is a compact and positive operator. Since

$$\left(x, \widetilde{G_n} x\right) = \sum_{i,j=1}^{\infty} g_{i,j} x_i x_j \leq \left(\sum_{i=1}^{\infty} g_{i,i}\right) \cdot \left(\sum_{i=1}^{\infty} x_i^2\right) \leq \tau \|x\|_{\ell^2}^2$$

it follows that $0 \leq \widetilde{G_n} \leq \tau\ Id$, $\forall n \in N$, and $0 \leq G \leq \tau\ Id$. Hence for every natural number n there exists a unique operator T_n such that $T_n^2 = \widetilde{G_n}$ and a unique operator T such that $T^2 = G$. We will denote them by $\widetilde{G_n}^{1/2}$ and $G^{1/2}$ respectively. Now, because of the uniqueness, it follows that

$$\widetilde{G_n}^{1/2} = \begin{pmatrix} Q_n & \ldots & 0 & \ldots \\ \vdots & & \vdots & \\ 0 & \ldots & 0 & \ldots \\ \vdots & & \vdots & \end{pmatrix}$$

where Q_n is the only matrix such that $Q_n \geq 0$ and $Q_n^2 = G_n$.

LEMMA 6: $\widetilde{G_n}^{1/2} \to G^{1/2}$ *on* $B(\ell^2)$ *if* $n \to \infty$.

Proof:

Let $\{P_k(\lambda)\}_{k \in N}$ be a sequence of polinomials with real coefficients that converges uniformly to the function $\rho(\lambda) = \lambda^{1/2}$, $\lambda \in [0, \tau]$. Let T be a selfadjoint operator such that $0 \leq T \leq \tau.Id$. Then

$$\|P_m(T) - P_n(T)\| \leq \max_{\lambda \in [0,\tau]} \|P_m(\lambda) - P_n(\lambda)\|.$$

Therefore $\{P_k(T)\}_{k \in N}$ is a Cauchy sequence in $B(\ell^2)$. Accordingly, there exists an operator $\widetilde{T} \in B(\ell^2)$ satisfying:

i) $P_m(T) \to \widetilde{T}$, if $m \to \infty$

ii) $\widetilde{T}^2 = T$

iii) $\widetilde{T} \geq 0$

iv) $\widetilde{T}$ is the only operator with the properties *i)-iii)*.

We note $T^{1/2} = \widetilde{T}$.We choose an arbitrary positive small ϵ and find an index k such that

$$\sup_{\lambda \in [0,\tau]} \left|P_k(\lambda) - \lambda^{1/2}\right| < \frac{\epsilon}{3}.$$

For that k we have: $\left\|P_k(G) - G^{1/2}\right\| < \frac{\epsilon}{3}$ and $\left\|P_k(\widetilde{G_n}) - \widetilde{G_n}^{1/2}.\right\| < \frac{\epsilon}{3}$. Let $n_0 = n_0(\epsilon)$ be such that $\left\|P_k(\widetilde{G_n}) - P_k(G)\right\| < \frac{\epsilon}{3}$, $\forall n > n_0$. Hence

$$\left\|\widetilde{G_n}^{1/2} - G^{1/2}\right\| < \epsilon, \forall n > n_0. \qquad \bullet$$

5 A characterization of M.

THEOREM 1: $M = G^{1/2}(\ell^2)$.
Proof:

Let $c \in M$. Then $(c(n), G_n^{-1}c(n)) \leq K$, $\forall n \in N$. We denote

$$x(n) = \left(G_n^{-1}\right)^{1/2} c(n).$$

Hence $\|x(n)\| \leq K$, $\forall n \in N$, and $c(n) = G_n^{1/2}x(n)$. We define the elements

$$\widetilde{x}_{n,i} := \begin{cases} x_i(n) & if\ 1 \leq i \leq n \\ 0 & if\ i > n \end{cases}$$

and we denote $\widetilde{x}_n = (\widetilde{x}_{n,i})_{i \in N}$. As $\|\widetilde{x}_n\|_{\ell^2} = \|x(n)\|_{R^n} \leq K$, $\forall n \in N$,we can suppose that $\{\widetilde{x}_n\}_{n \in N}$ is weak convergent in ℓ^2 (if it is not the case, it is sufficient to consider a subsequence with this property). Then

$$(\widetilde{x}_n, y) \to (x, y)\ if\ n \to \infty, \forall y \in \ell^2.$$

Since $G^{1/2}$ is a compact operator $G^{1/2}\widetilde{x}_n \to G^{1/2}x$ if $n \to \infty$ and $G^{1/2}\widetilde{x}_n \to c$ if $n \to \infty$, then $c = G^{1/2}x$.

To show that $G^{1/2}(\ell^2) \subseteq M$, let c be an element of $G^{1/2}(\ell^2)$. Then there exists $x \in \ell^2$ such that $G^{1/2}x = c$. We now introduce the elements

$$u^{(s)} := \widetilde{G}_s^{1/2} x$$

We assume for an instant that $u^{(s)} \in M$, $\forall n \in N$.. Then we have

$$\left\| \left(G_n^{-1}\right)^{1/2} c(n) \right\| \leq \left\| \left(G_n^{-1}\right)^{1/2} \left(c(n) - u_{(n)}^{(s)}\right) \right\| + \sup_{n \in N} \left\| \left(G_n^{-1}\right)^{1/2} u_{(n)}^{(s)} \right\| \leq$$

$\leq \left\| (G_n^{-1})^{1/2} \left(c(n) - u_{(n)}^{(s)}\right) \right\| + K$,being K a constant. So

$$\left\| \left(G_n^{-1}\right)^{1/2} c(n) \right\| \leq \left\| \left(G_n^{-1}\right)^{1/2} \left(c(n) - \lim_{s \to \infty} u_{(n)}^{(s)}\right) \right\| + K = K$$

because $u^{(s)} \to c$ in ℓ^2 if $s \to \infty$. Thus $c \in M$.

To show that $u^{(s)} \in M$, $\forall n \in N$ let's introduce the set

$$\widetilde{R}_s = \left\{ \alpha = (\alpha_i)_{i \in N} \in \ell^2 : \alpha_i = 0\ \forall i > s \right\}$$

and consider $\{g_i\}_{i \in N}$ a biorthogonal sequence to the sequence (1). Next we define $g = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_s g_s$, $g \in L^2(0, \infty)$. Then

$$(g, e^{-\lambda_j t}) = \begin{cases} \alpha_i & i \leq s \\ 0 & i > s \end{cases}$$

and hence $\widetilde{R}_s \subseteq M$, $\forall s \in N$. •

Remark: Now the part *a)* of the proposition is obvious.

6 Solution of the moment problem.

If $\varphi_n(t)$ is the solution with minimun norm of the truncated moment problem

$$\left(\varphi_n(t), e^{-\lambda_j t}\right) = c_j \quad j = 1, 2, \ldots, n$$

then [K]

$$\varphi_n(t) = \sum_{i=1}^{n} \gamma_i e^{-\lambda_i t}$$

where $\gamma_i = \sum_{i=1}^{n} \sigma_{j,i}(n) c_j$ and $\sigma_{i,j}(n)$ is the (i,j)-element of G_n^{-1}. It can be proved that

$$\sigma_{i,j}(n) = \frac{4\lambda_i \lambda_j}{\lambda_i + \lambda_j} \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\lambda_k + \lambda_i}{\lambda_k - \lambda_i} \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{\lambda_k + \lambda_j}{\lambda_k - \lambda_j} .$$

If we call $\alpha_i(n) = 2\lambda_i \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\lambda_k + \lambda_i}{\lambda_k - \lambda_i}$ we can write $\sigma_{i,j}(n) = \frac{1}{\lambda_i + \lambda_j} \alpha_i(n) \alpha_j(n)$. The moment problem has a solution if and only if there exist a constant $K > 0$ such that [K] $\|\varphi_n(t)\| \leq K$, $\forall n \in N$. Let $D_n = (d_{i,j})_{1 \leq i,j \leq n}$ be a diagonal matrix of order n such that

$$d_{i,j} = \begin{cases} \alpha_i(n) & i = j \\ 0 & i \neq j \end{cases}$$

then $G_n^{-1} = D_n G_n D_n$ and

$$\varphi_n(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\alpha_i(n)}{\lambda_i + \lambda_j} c_i \right) \alpha_j(n) e^{-\lambda_j t} = \sum_{j=1}^{n} d_j(n) \alpha_j(n) e^{-\lambda_j t}$$

where $d(n) = \begin{pmatrix} d_1(n) \\ d_2(n) \\ \vdots \\ d_n(n) \end{pmatrix} = G_n D_n c(n).$

The condition $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$ implies convergence of the infinite products $\lim_{n \to \infty} \alpha_i(n) = \alpha_i$, $\forall i \in N$ [C] . For every $i \in N$ the sequence $\{d_i(n)\}_{n \in N}$ has also a finite limit when $n \to \infty$. Then we write $d_i = \lim_{n \to \infty} d_i(n)$.

In fact, let $P_n(t) = \sum_{i=1}^{n} c_i(n) \alpha_i(n) e^{-\lambda_i t}$; then $\|P_n(t)\| = \|\varphi_n(t)\| \leq K$, $\forall n \in N$ and $\left(P_n, e^{-\lambda_i t}\right) = d_i(n)$. This shows that $\{P_n(t)\}_{n \in N}$ is a sequence of elements in $L^2(0, \infty)$ such that the norms form a nondecreasing sequence of real numbers with K as an upper bound. Then there exists $P \in L^2(0, \infty)$ such that $P_n \to P$ if $n \to \infty$.

The following theorem is valid

THEOREM 2: If there exist a constant $\beta > 0$ such that $\lambda_{n+1} - \lambda_n \geq \beta$, $\forall n \in N$, *and* $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$ *then*

$$\varphi(t) = \sum_{j=1}^{\infty} d_j \alpha_j e^{-\lambda_j t}$$

is the solution with minimun norm of the moment problem

$$\int_0^{\infty} \varphi(t) e^{-\lambda_i t} = c_i \ , \ i \in N.$$

Proof:

First,

$$\sum_{j=1}^{\infty} d_j \alpha_j e^{-\lambda_j t} \in L^2(0, \infty),$$

is a consequence of a theorem of Schwartz [S]. In fact, as $\varphi_n(t) = \sum_{i=1}^{n} d_i(n) \alpha_i(n) e^{-\lambda_i t}$ is the solution of minimun norm of the problem of order n:

$$\int_0^{\infty} \varphi(t) e^{-\lambda_i t} = c_i \ , \ 1 \leq i \leq n,$$

there exists $\varphi(t) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i(n) d_i(n) e^{-\lambda_i t} \in L^2(0, \infty)$, being $\varphi(t)$ the solution of minimum norm of the moment problem [K]. Then $\varphi(t)$ belongs to the clausure of the subspace of $L^2(0, \infty)$ generated by $\left\{e^{-\lambda_i t}\right\}_{i \in N}$ and can be written as a Dirichlet series [S]

$$\varphi(t) = \sum_{i=1}^{\infty} k_i e^{-\lambda_i t}$$

As $\left\{e^{-\lambda_i t}\right\}_{i \in N}$ is a minimal system [S] it follows that $k_i = \alpha_i d_i$, $\forall i \in N$, i.e.:

$$\varphi(t) = \sum_{i=1}^{\infty} \alpha_i d_i e^{-\lambda_i t} \in L^2(0, \infty)$$

It remains to prove that $\sum_{j=1}^{\infty} d_j \alpha_j e^{-\lambda_j t}$ is a solution. As

$$\left(\sum_{i=1}^{\infty} d_i \alpha_i e^{-\lambda_i t} , e^{-\lambda_k t} \right) = \sum_{i=1}^{\infty} d_i \alpha_i \frac{1}{\lambda_i + \lambda_k}$$

then we must prove that:

$$\sum_{i=1}^{\infty} d_i \alpha_i \frac{1}{\lambda_i + \lambda_k} = c_k \,,\, \forall k \in N.$$

If $k \leq n$, $\sum_{i=1}^{n} d_i(n)\alpha_i(n)\frac{1}{\lambda_i+\lambda_k} = (G_n D_n G_n D_n c(n))_k = c_k$ then

$$\lim_{n\to\infty} \sum_{i=1}^{n} d_i(n)\alpha_i(n)\frac{1}{\lambda_i + \lambda_k} = c_k \,,\, \forall k \in N.$$

But $\sum_{i=1}^{\infty} \alpha_i d_i e^{-\lambda_i t} \in L^2(0,\infty)$ then

$$c_k = \sum_{i=1}^{\infty} d_i \alpha_i \frac{1}{\lambda_i + \lambda_k} \qquad \bullet$$

7 Another expression for the solution

The solution of minimun norm of the problem of order n $\varphi_n(t) = \sum_{j=1}^{\infty} d_j(n)\alpha_j(n)e^{-\lambda_j t}$ can be written as $\varphi_n(t) = \sum_{j=1}^{\infty} \gamma_j(n)e^{-\lambda_j t}$ with $\gamma(n) = (\gamma_i(n))_{1\leq i \leq n} = D_n G_n D_n c(n)$. But $D_n G_n D_n = G_n^{-1}$, then

$$\gamma(n) = (\gamma_i(n))_{1\leq i\leq n} = G_n^{-1}c(n).$$

The goal of this section is to find an analogue expression for the solution $\varphi(t)$. In section 5 we proved that there exists $P(t) \in L^2(0,\infty)$ such that

$$P(t) = \lim_{n\to\infty} P_n(t) = \lim_{n\to\infty} \sum_{i=1}^{n} c_i(n)\alpha_i(n)e^{-\lambda_i t}$$

Then $P(t)$ belongs to the clausure of the subespace of $L^2(0,\infty)$ generated by the system $\left\{e^{-\lambda_i t}\right\}_{i\in N}$ and $P(t)$ can be developed in a Dirichlet series

$$P(t) = \sum_{i=1}^{\infty} h_i e^{-\lambda_i t}.$$

But $\left\{e^{-\lambda_i t}\right\}_{i\in N}$ is a minimal system, then $h_i = \alpha_i c_i \,,\, \forall i \in N$,

$$P(t) = \sum_{i=1}^{\infty} c_i \alpha_i e^{-\lambda_i t} \in L^2(0,\infty).$$

Then $\left(P(t)\,,\, e^{-\lambda_j t}\right) = \sum_{i=1}^{\infty} \frac{c_i \alpha_i}{\lambda_i + \lambda_j}$ converges and

$$d_i = \lim_{n\to\infty} d_i(n) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{c_j \alpha_j(n)}{\lambda_i + \lambda_j} = \sum_{j=1}^{\infty} \frac{c_j \alpha_j}{\lambda_i + \lambda_j}.$$

Then $\varphi(t) = \sum_{i=1}^{\infty} d_i \alpha_i e^{-\lambda_i t} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} c_j e^{-\lambda_i t}$.

If we define the operator DGD as the one generated by the infinite matrix $\left(\frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j}\right)_{i,j}$ and the operator GD as the one generated by the infinite matrix $\left(\frac{\alpha_i}{\lambda_i + \lambda_j}\right)_{i,j}$ it follows that $\varphi(t) = \sum_{i=1}^{\infty} (DGDc)_i e^{-\lambda_i t}$.

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