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CROWNS.
A UNIFIED APPROACH TO STARSHAPEDNESS

Fausto A. Toranzos
Departamento de Matemática
Universidad de Buenos Aires

ABSTRACT: *It is observed that many papers concerning starshaped sets have similar structure and objectives. Those papers usually deal with construction of the convex kernel, dimension of the kernel and Krasnoselsky-type theorems. Furthermore, the logical connections among these different topics are almost the same in the different papers. The aims of the present note are to exhibit these logical connections and to sketch a unified theory of starshapedness. A third implicit aim is the development of a brief survey of some aspects of this part of Convexity Theory. The main tool to obtain these objectives is the notion of crown of a starshaped set.*

1.- INTRODUCTION.

More than thirty years ago, F. A. Valentine, in his classical book [15] on Convexity, posed several problems regarding starshaped sets. The first and more important two problems were :

(P_0) *Characterize the starshapedness of* S *in terms of the maximal convex subsets of* S. (Problem 9.3 of [5]).

(P_1) *Determine neccessary and sufficient conditions that the convex kernel of* S *have dimension* α *, where* $0 \leq \alpha \leq d$ *, and* d *is the space dimension.* (Problem 1.1 of [5]).

Problem (P_0) was completely solved in [11] , but its solution provoked a similar and more general type of problem :

(P_2) *Describe the convex kernel of a starshaped set* S *as the intersection of a certain family of subsets of* S.

In 1946 Krasnoselsky [8] proved that a compact set $S \subset \mathbf{R}^n$ is starshaped if and only if for each subset of n+1 points of S there exists a point of S than can see via S all these points. This theorem, perhaps the most important result in the theory of starshapedness, suggested a new angle of research about starshaped sets and visibility. The results of this new approach are usually labelled as *Krasnoselsky-type theorems* , and provide answers to the following problem :

(P_3) *Describe properties (related to visibility and starshapedness) of the set* S *by means of conditions upon each subset of* k *points of* S *, where* k *is an integer related to the space dimension.*

The literature on starshapedness and related matters includes scores of particular solutions of problems (P_1) , (P_2) and (P_3) . We will mention some of those solutions in Paragraph 3. The main purpose of this note is to exhibit the logical connections among these problems. We intend to show that a solution to any of these problems can produce solutions to the remaining ones.

2.- BASIC DEFINITIONS.

Unless otherwise stated, all the points and sets considered here are included in a real locally convex linear topological space E. The interior, closure, boundary, convex hull and affine hull of a set S are denoted by int S , cl S , bdry S, conv S and aff S , respectively. The open segment joining x and y is denoted (x y) . The substitution of one or both parentheses by square ones indicates the adjunction of the correspondig extremes. The ray issuing from x and going through y is denoted $R(x \to y)$, while $R(y\ x \to)$ is the ray issuing from x and going in the opposite direction. All rays are considered closed. We say that x *sees* y *via* S if $[x\ y] \subset S$. The *star* of x in S is the set st(x,S) of all the points of S that see x via S. A *star-center* of S is a point $x \in S$ such that st(x,S) = S . The *kernel* (*convex kernel*, *mirador*) of S is the set ker S of all the star-centers of S . Finally, S is *starshaped* if ker S is not empty.

A *crown* of the starshaped set S is a collection $\mathfrak{R}$ of subsets of S whose intersection is ker S. If S is a starshaped set and $\mathfrak{R}$ is a crown of S, a *subcrown* is a subfamily $\mathfrak{I} \subset \mathfrak{R}$ such that $\mathfrak{I}$ itself be a crown of S. A *minimal crown* of S is a crown that admits no proper subcrown. A *covering crown* of S is a crown whose union is S. A *finite crown* is one with a finite number of members. Any other qualification of the word *"crown"* (e.g.: *convex crown, closed crown,* etc.) indicates that the same adjective applies to each of the members of the crown. That is, $\mathfrak{R}$ is a convex crown if and only if it is a crown and each of its members is convex. We are naturally inclined to try to prove, by means of a nonconstructive approach (i.e. Zorn's Lemma, well ordering principle, or the like), a theorem that assures that every crown admits a minimal subcrown. Unfortunately, such a theorem would be false, as a counterexample given below shows.

3.- EXAMPLES OF CROWNS.

In this paragraph we consider seven examples of crowns already in the literature. We shall restrict our exposition to the basic definition in each case, and the statement that identifies the crown considered.

THEOREM 3.1 *If* S *is a starshaped set, the family* $\mathfrak{R} = \{st(x,S) \mid x \in S\}$ *is a crown of* S .

No proof is needed here. This is just a different way to state the definition of the convex kernel of S. An interesting type of problem is to describe, in different environments and settings, a minimal subcrown of the crown just defined. Theorem 3.3 and Theorem 3.6, stated below, present two different approaches in this direction. A *convex component* of S is a maximal convex subset of S.

THEOREM 3.2 (Toranzos, [11]) *If* S *is a starshaped subset, a covering family of convex components of* S *is a covering and convex crown of* S.

The original statement of this result refers to the family of all convex components of S, but the proof applies to the present statement. It is important to remark that both previous theorems omit any topological and/or dimensional requirement, either on the space or on the starshaped set S.

The *relative interior* of a set M, denoted 'relint M', is the interior of M in the relative topology of aff M . A k-*simplex* is the convex hull of k+1 affinely independent points. A point $x \in S$ is a k-*extreme point* if no k-simplex $\Delta \subset S$ exists such that $x \in \text{relint } S$. Of course, in these two definitions k is not larger than the space dimension. The set of all the k-extreme points of S is denoted by $\text{ext}_k S$.

THEOREM 3.3 ([6], [10]) *Let* S *be a compact starshaped subset of* $\mathbf{R}^d$. *The family* $\Re = \{\text{st}(x,S) \mid x \in \text{ext}_{d-1}S\}$ *is a crown of* S .

This statement was proved simultaneously and independently by Tidmore [10] and by Kenelly et al. [6] . It is easy to construct, even in $\mathbf{R}^3$, counterexamples to show that the set of regular extreme points of S , that is $\text{ext}_1 S$ in the previous definition, is not enough to describe the convex kernel as intersection of its stars.

The point y *sees clearly* x *via* S if there exists a neighborhood $\mathcal{U}_x$ of x such that $\mathcal{U}_x \subset \text{st}(y,S)$. The *nova* (or *clear star*) of x in S is the set nova(x,S) of all points of S that see clearly x via S . A point $x \in S$ is a point of *local convexity* of S if there exists a neighborhood $\mathcal{U}_x$ of x such that $\mathcal{U}_x \cap S$ be convex. Otherwise, x is a *point of local nonconvexity* of S. The set of all points of local nonconvexity [local convexity] of S is denoted by lnc S [lc S].

THEOREM 3.4 (Stavrakas, [9]) *Let* S *be a compact connected subset of* $\mathbf{R}^d$. *Then, the family of novae of points of local nonconvexity of S is a crown of* S.

This theorem has recently been generalized in Theorem 2.2 of [14] where the requirement of finite dimension is dropped, and the condition of compactness of S is substituted by that of lnc S. As we remark here, these improvements yield easily better results about the dimension of the kernel and new Krasnoselsky-type theorems.

Let p and q be points of S. The point p *has higher visibility via* S *than* q if $st(q,S) \subset st(p,S)$. The *visibility cell of* p *in* S is the set vis(p,S) of all the points of S having higher visibility via S than p. Of course, $p \in vis(p,S)$ always.

THEOREM 3.5 (Toranzos, [12]) *Let* S *be a closed connected set such that* lnc S *be compact. The family of visibility cells of all points of local nonconvexity of* S *is a convex crown of* S.

A *simple smooth Jordan domain* is a compact set $S \subset \mathbf{R}^2$ whose boundary is a simple closed smooth Jordan curve having a finite number of inflection points.

THEOREM 3.6 (Forte Cunto, [2]) *Let* S *be a simple smooth Jordan domain. The family of stars of the inflection points of* bdry S *is a finite crown of* S.

Let $y \in$ bdry S and $x \in st(y,S)$. We say that $R(x \rightarrow y)$ is an *inward ray through* y if there exists $t \in R(x\, y \rightarrow)$ such that $(y\ \ t) \subset$ int S. Otherwise, we say that $R(x \rightarrow y)$ is an *outward ray through* y. The *inner stem of* y *in* S is the set ins(y,S) formed by y and all the points of st(y,S) that issue outward rays through y . A *regular domain* is a set S having connected interior and such that S = cl int S .

THEOREM 3.7 (Toranzos, [13]) *Let* S *be a nonconvex regular domain. Then the family* $\mathfrak{I} = \{\text{ins}(x,S) \mid x \in \text{Inc } S\}$ *is a crown of* S .

EXAMPLE 3.8 *Example of a crown without minimal subcrowns.*

Let S be a planar set consisting of three quarters of a circular disk, that is, using polar cordinates :

$$S = \left\{(r,w) \in \mathbb{R}^2 \,\middle|\, 0 \le r \le 1; \tfrac{p}{2} \le w \le 2p\right\}.$$

Let O be the origin, $p = \left(1, \frac{p}{2}\right)$ and $q = (1,p)$. The convex components of S are the closed semidisks obtained by intersection of S with a halfplane limited by a line through O . Each of these convex components is characterized by the point of the arc $[\overline{p\quad q}]$ where its limiting line intersects this arc. If x is a point of this arc, let K_x be the corresponding convex component. It is easy to verify that if L is a subset of the mentioned circular arc such that the points p and q are accumulation points of L, then the family $\Re_L = \{K_x \mid x \in L\}$ is a crown of S . Consider now the family $\mathscr{L}$ of all the convex components of S, with the exception of K_p and K_q . Then $\mathscr{L}$ is a crown of S that has no minimal subcrown.❑

4.- REPRESENTATION AND DIMENSION OF THE CONVEX KERNEL.

The natural way to begin a study on starshapedness is to prove a theorem of representation (or construction) of the convex kernel of a starshaped set. The format of such a theorem is :

THEOREM 4.1 *Let* S *be a starshaped set with property* $\mathcal{P}$ *included in the space* E *with structure* $\mathcal{Q}$ *. Then the family* $\mathfrak{R}$ *of subsets of* S *is a crown of* S.

Unless we determine explicitly the property (or properties) $\mathcal{P}$, the structure $\mathcal{Q}$ and the family $\mathfrak{R}$, this statement is not a real theorem but a *theorem-format* i.e. a logical template that can be filled with real mathematical contents. All of the theorems quoted in the previous paragraph fit into this format. The proof of a theorem having this format is a particular solution of the Problem (P_2) stated in the first paragraph. Once solved the *Representation Problem* of the convex kernel, the *Dimension Problem,* stated above as Problem (P_1) , can be approached in the same way by means of another *theorem-format* .

THEOREM 4.2 *Let* S *be a set with property* $\mathcal{P}$ *included in the space* E *that has structure* $\mathcal{Q}$ *, and let* $\mathfrak{R}$ *be a crown of* S . *Then* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* $x \in \text{relint}(F \cap S)$, *and a neighborhood* $\mathcal{U}_x$ *of* x *such that for each* $M \in \mathfrak{R}$ *holds* $(\mathcal{U}_x \cap F \cap S) \subset M$.

<u>Proof</u> : The *'if'* part is simple since the definition of crown implies $(\mathcal{U}_x \cap F \cap S) \subset M$ where the set between brackets has dimension α . For the converse implication it is enough to take $F = \text{aff} \ker S$ and $x \in \text{relint} \ker S$.❑

Let us now apply this *theorem-format* to the examples of crowns that were introduced in the previous paragraph.

THEOREM 4.3 *Let* E *be a locally convex linear topological space, and* S *a starshaped subset of* E . *Then* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in \text{relint}(F \cap S)$, *and a neighborhood* $\mathcal{U}_x$ *of* x *such that* $\forall t \in S$, $(\mathcal{U}_x \cap F \cap S) \subset \text{st}(t,S)$.

<u>Proof</u> : This is just the conjunction of Theorem 3.1 and Theorem 4.2 .❑

THEOREM 4.4 *Let* E *be a locally convex linear topological space,* S *a starshaped subset of* E, *and* $\mathfrak{R}$ *a covering family of convex components of* S. *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of x such that* $\forall K \in \mathfrak{R}$ *holds* $(\mathcal{U}_x \cap F \cap S) \subset K$.

Proof : Conjunction of Theorem 3.2 and Theorem 4.2.❑

THEOREM 4.5 *Let* $E = \mathbf{R}^d$, *and* S *be a compact starshaped subset of* E . *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of x such that* $\forall t \in ext_{d-1} S$ *holds* $(\mathcal{U}_x \cap F \cap S) \subset st(t,S)$.

Proof : Conjunction of Theorem 3.3 and Theorem 4.2 .❑

THEOREM 4.6 *Let* $E = \mathbf{R}^d$ *and* S *be a compact connected subset of* E. *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that* $\forall t \in$ Inc S *holds* $(\mathcal{U}_x \cap F \cap S) \subset nova(t,S)$.

Proof : Conjunction of Theorem 3.4 and Theorem 4.2 . It is important to recall that precisely the present result was proved in [9] , where Stavrakas introduced the notion of clear visibility. ❑

THEOREM 4.7 *Let* E *be a locally convex linear topological space and* S *a closed connected subset of* E *such that* Inc S *be compact. Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that every point of* $(\mathcal{U}_x \cap F \cap S)$ *has higher visibility via* S *than each of the points of local nonconvexity of* S .

Proof : This is the conjunction of Theorem 3.5 and Theorem 4.2 .❑

THEOREM 4.8 *Let* E = $\mathbf{R}^2$ *and* S *be a simple smooth Jordan domain. Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* x $\in$ relint (F $\cap$ S) *and a neighborhood* $\mathcal{U}_x$ *of* x *such that every point of* ($\mathcal{U}_x \cap$ F $\cap$ S) *see via* S *every inflection point of* bdry S .

Proof : Conjunction of Theorem 3.6 and Theorem 4.2 .❑

THEOREM 4.9 *Let* E *be a locally convex linear topological space and* S *a nonconvex regular domain included in* E . *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* x $\in$ relint (F $\cap$ S) *and a neighborhood* $\mathcal{U}_x$ *of* x *such that every point of* ($\mathcal{U}_x \cap$ F $\cap$ S) *issues outward rays through each of the points of local nonconvexity of* S .

Proof : This is the conjunction of Theorem 3.7 and Theorem 4.2 .❑

We have shown, by means of these seven examples, that any solution to the Problem (P_2) of representation of the convex kernel by a crown yields almost inmediately, via the theorem-format 4.2, a solution to the problem (P_1) of the dimension of the convex kernel.

5.- KRASNOSELSKY-TYPE THEOREMS.

Every theorem that fits into the theorem-format 4.1 of representation of the convex kernel by means of a crown is essentially a result about the intersection of a certain family of sets. The literature on Convexity has, in the finite-dimensional case, a large *corpus* of theory usually labelled as **Helly-type Theorems** , that deals with the intersection of families of sets and has a strong combinatorial flavor. The conjunction of this type of result with the theorems exhibited in the previous paragraph is highly desirable, but a technical problem arises : Helly-type theorems usually refer to families of **convex** sets, while the members of a crown are not necessarily convex. The difficulty is solved by means of an auxiliary lemma whose proof is usually far from simple.

LEMMA 5.1 (K-lemma) *Let* S *be a set with property* $\mathfrak{P}$ *included in the space* E *that has structure* $\mathfrak{Q}$ *and* $\mathfrak{R}$ *be a nonconvex crown of* S . *Let* $x \in S$ *but* $x \notin \ker S$. *Then,* $\exists\, M \in \mathfrak{R}$ *such that* $x \notin \operatorname{conv} M$ $[x \notin \operatorname{cl} \operatorname{conv} M]$.

This lemma implies inmediately that $\ker S$ is the intersection of the convex hulls [the closed convex hulls] of the members of the crown $\mathfrak{R}$ **.** The use of the alternative enclosed in square brackets depends on the topological conditions of the crown considered. It is clear that this lemma is superfluous if the crown is convex, as in examples 3.2 and 3.5 above. We quote here for later reference the three most commonly used Helly-type theorems.

THEOREM 5.2 (Helly,[4]) *Let* $E = \mathbf{R}^d$ *and* $\mathfrak{R}$ *be a finite family of convex subsets of* E *such that each subfamily of* k *members of* $\mathfrak{R}$, *with* $k \le d+1$, *has nonempty intersection. Then, the intersection of all the members of* $\mathfrak{R}$ *is nonempty. The condition of finiteness of* $\mathfrak{R}$ *can be dropped if it is required the compactness of all its members.*

THEOREM 5.3 (Grünbaum, [3]) *Let* $E = \mathbf{R}^d$ *and* $\mathfrak{R}$ *be a finite family of convex subsets of* E . If N *denotes the set of positive integers, we define a function* $g : N \times N \to N$ *by* $g(n,1) = 2n$, $g(n,n) = n+1$, *and* if $n > k > 1$ *then* $g(n,k) = 2n-k$. *Any other value of* $g(n,k)$ *is irrelevant. The dimension of the* intersection of all *the members of* $\mathfrak{R}$ *is greater than or equal to* α *if and only if the dimension of the intersection of every subfamily of* $\mathfrak{R}$ *that has at most* $g(d,\alpha)$ *members is at least* α .

THEOREM 5.4 (Klee, [7]) *Let* $E = \mathbf{R}^d$ **,** $\mathfrak{R}$ *be a finite family of convex subsets of* E *and* $\delta > 0$. *The intersection of all the members of* $\mathfrak{R}$ *contains a ball of radius* δ *if and only if for every subfamily of* d+1 *members of* $\mathfrak{R}$, *its intersection contains such a ball. As in Theorem 5.2, the finiteness of* $\mathfrak{R}$ *can be dropped provided the compactness of all its menbers is required.*

The knowledge of a crown for a certain class of starshaped sets, plus the previous theorems, produce three different Krasnoselsky-type theorem-formats. As we have observed at the beginning of thiis paragraph, either the crown considered is convex or it must verify a K-Lemma that follows the format of Lemma 5.1.

THEOREM 5.6 (Krasnoselsky-type 1) *Let* $E = \mathbf{R}^d$, S *be a compact subset of* E , *and* $\mathfrak{R}$ *be a crown of* S *that either is convex or verifies Lemma 5.1. Then* S *is starshaped if and only if the intersection of every subfamily of* d+1 *members of* $\mathfrak{R}$ *is nonempty.*

Proof : Theorem 5.2 and , if needed, Lemma 5.1. The compactness of S can be substituted by the finiteness of the crown $\mathfrak{R}$.❑

THEOREM 5.6 (Krasnoselsky-type 2) *Let* $E = \mathbf{R}^d$, S *be a subset of* E , *and* $\mathfrak{R}$ *be a finite crown of* S *that either is convex or verifies Lemma 5.1. If* N *denotes the set of positive integers, define a function* $g : N \times N \to N$ *by* $g(n,1) = 2n$, $g(n,n) = n+1$, *and for* $n > k > 1$ $g(n,k) = 2n-k$. *Any other value of* g(n,k) *is irrelevant. Then,* S *is starshaped and* $\dim \ker S \geq \alpha$ *if and only if the dimension of the intersection of each subfamily of* $g(d,\alpha)$ *members of the crown is at least* α .

Proof : Theorem 5.3 and, if the crown is not convex, Lemma 5.1. In this case the finiteness of the crown is essential and admits no substitution by any compactness condition.❑

THEOREM 5.7 (Krasnoselsky-type 3) *Let* $E = \mathbf{R}^d$, S *be a compact subset of* E , *and* $\mathfrak{R}$ *be a crown of* S *that either is convex or verifies Lemma 5.1. Then* S *is starshaped and* ker S *contains a ball of radius* $\delta > 0$ *if and only if the intersection of each subfamily of* $\mathfrak{R}$ *having at most* d+1 *members contains a ball of radius* δ .

Proof : Theorem 5.4 and, if needed, Lemma 5.1. Once more, the compactness of S can be substituted by the finiteness of the crown.❑

These three theorem-formats combined with the seven types of crowns described in Paragraph 3 can give rise to twenty one Krasnoselsky-type theorems. Some of those results are already known. M. Breen (in [1] and other papers) has derived several Krasnoselsky-type theorems from the Stavrakas' crown described in Theorem 3.4. The theorem that can be obtained by the conjunction of Theorem 3.1 and Theorem 5.5 is the original 1946 Krasnoselsky's Theorem [9]. The nine Krasnoselsky-type theorems that can be derived from Theorems 3.5, 3.6 and 3.7 have already been proved in the papers ([12], [2] and [13]) where the respective crowns were described. As an example we state the theorems that can be derived from the crown described in Theorem 3.2.

THEOREM 5.8 *Let* $E = \mathbf{R}^d$, S *be a subset of* E, *and* $\mathfrak{R}$ *be a covering family of convex components of* S. *Then* S *is starshaped if and only if every* $d+1$ *members of* $\mathfrak{R}$ *have nonempty intersection.*

THEOREM 5.9 *Let* $E = \mathbf{R}^d$, S *be a subset of* E, *and* $\mathfrak{R}$ *be a finite covering family of convex components of* S. *If* N *denotes the set of positive integers, define a function* $g : \mathrm{N}\times\mathrm{N} \to \mathrm{N}$ *by* $g(n,1) = 2n$, $g(n,n) = n+1$, *and for* $n > k > 1$ $g(n,k) = 2n-k$. *Any other value of* $g(n,k)$ *is irrelevant. Then,* S *is starshaped and* $\dim \ker S \geq \alpha$ *if and only if the dimension of the intersection of each subfamily of* $g(d,\alpha)$ *members of* $\mathfrak{R}$ *is at least* α.

THEOREM 5.10 *Let* $E = \mathbf{R}^d$, S *be a subset of* E, *and* $\mathfrak{R}$ *be a covering family of convex components of* S. *Then* S *is starshaped and* $\ker S$ *contains a ball of radius* δ *if and only if the intersection of each subfamily of* $\mathfrak{R}$ *that has at most* $d+1$ *members contains a ball of radius* δ.

6.- CONCLUDING REMARKS.

In the previous sections we have shown that once proved a theorem about the construction of the convex kernel that fits the format of Theorem 4.1 and, in the case that it would be necessary, a *K-Lemma* like 5.1, the whole Starshapedness Theory including theorems about construction and dimension of the kernel and Krasnoselsky-type theorems follows easily. The main tool in this development has been the idea of *crown of a starshaped set.* We claim that this notion is worthy of systematic study. The study of minimal crowns generated by some of the known types of crowns seems specially promising.

In the Krasnoselsky-type theorems that fit the theorem-format 5.6 , sometimes it is possible to obtain a slight improvement if the analogous theorem of Katchalsky [5] is substituted instead of Grünbaum's Theorem 5.3. The application of different Helly-type theorems and/or adimensional theorems regarding intersections of convex sets to the present approach remains to be studied in the future.

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Departamento de Matemática
Universidad de Buenos Aires
Pabellón 1 - Ciudad Universitaria
(1428) Buenos Aires - ARGENTINA

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