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ESTIMATES ON THE $(L^p(w), L^q(w))$ OPERATOR NORM OF THE FRACTIONAL MAXIMAL FUNCTION.

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Abstract: In $\mathbb{R}^n$, given $\gamma \in [0, n)$ and $p \in (1, n/\gamma)$, it is well known that $w^q \in A^r$, with $1/q = 1/p - \gamma/n$ and $r = 1 + q\frac{p-1}{p}$, is a necessary and sufficient condition for the boundedness of the Maximal Fractional Operator M_γ between $L^p(w^p)$ and $L^q(w^q)$ spaces. In this work we study the dependence of the operator norm on the constant of the A_r condition. The result extends the obtained by S. Buckley for the Hardy-Littlewood Maximal Function (i.e.: $\gamma = 0$).

§1.

Let μ be a positive Borel measure in $\mathbb{R}^n$. For each γ in $(0, n)$, the fractional maximal operator M_γ with respect to μ is defined by

$$M_\gamma f(x) = \sup_{x \in Q} \frac{1}{\mu(Q)^{1-\frac{\gamma}{n}}} \int |f| \, d\mu, \tag{1.1}$$

for $f \in L^1_{loc}(d\mu)$, where the sup is taken over all cubes in $\mathbb{R}^n$ containing x. It is well known that for each p in $(1, n/\gamma)$ there exists a constant C, independent of f, such that the inequality

$$\left(\int_{\mathbb{R}^n} (|M_\gamma f| \, w)^q \, d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} (|f| \, w)^p \, d\mu \right)^{\frac{1}{p}}, \tag{1.2}$$

holds with $1/q = 1/p - \gamma/n$ for every f in $L^p(w^p d\mu)$ if and only if w is a weight in the $A(p, q)$ class with respect to μ, that is, w is a non negative function satisfying

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(1.3) $$K_{w,p,q} = \sup_Q \left(\frac{1}{\mu(Q)}\int_Q w^q d\mu\right)^{\frac{1}{q}} \left(\frac{1}{\mu(Q)}\int_Q w^{-p'} d\mu\right)^{\frac{1}{p'}} < \infty,$$

where the sup is taken over all cubes in $I\!R^n$ and $p' = p/(p-1)$. From the classical proofs of the above result, it can be obtained that the constant C in (1.2) depends on $K_{w,q,p}$, but they do not show explicitly the dependence. In 1993, S. Buckley ([B]) solved the problem for the Hardy-Littlewood maximal function (i.e.: $\gamma = 0$ in (1.1)). The purpose of this work is to extend that result to the general case of the operator in (1.1). Actually, our main result is the following theorem.

(1.4) **Theorem:** *If* $0 \leq \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$ *and* w *is a nonnegative function on* $I\!R^n$ *such that, for every cube* Q, *(1.3) holds, then*

(1.5) $$\left(\int_{I\!R^n} (|M_\gamma f\,|\,w)^q \, d\mu\right)^{\frac{1}{q}} \leq C K_{w,p,q}^{p'(1-\frac{\gamma}{n})} \left(\int_{I\!R^n} |f\,w|^p \, d\mu\right)^{\frac{1}{p}}$$

The power $K_{w,p,q}^{p'(1-\frac{\gamma}{n})}$ *is the best possible.*

As it can be seen in §2, our techniques to prove the above theorem are extensions of those used by Buckley in the case $\gamma = 0$. An important point in order to obtain these extensions is to recall the obvious relation between the $A(p,q)$ classes, defined as in (1.3), and the Muckenhoupt's classes A_r with respect to μ. In fact, since a weight w is in A_r, $1 < r < \infty$, when

(1.6) $$B_{w,r} = \sup_Q \left(\frac{1}{|Q|}\int_Q w d\mu\right) \left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{r-1}} d\mu\right)^{r-1} < \infty,$$

where the sup is taken over all cubes in $I\!R^n$, it is clear that w belongs to $A(p,q)$ if and only if w^q belongs to $A_{1+q/p'}$, with $p' = p/(p-1)$. Moreover, we have $B_{w^q,1+q/p'} = K^q_{w,p,q}$.

§2

As in the case $\gamma = 0$, we are going to prove theorem (1.4) by using an argument of interpolation. For this reason, let us first to state the following version of the Marcinkiewicz's interpolation theorem with respect to a positive Borel measure μ.

(2.1) **Theorem:** *Suppose that a quasi-linear operator* T *is simultaneously of weak types* (p_1, q_1) *y* (p_2, q_2), $1 \leq p_i, q_i \leq \infty$, $q_1 \neq q_2$, *with norms* M_1 *y* M_2 *respectively (i.e.:* $\mu(\{x : Tf(x) > \alpha\}) \leq \left(\frac{M_i}{\alpha}\|f\|_{L^{p_i}(d\mu)}\right)^{q_i}$, $i = 1, 2$*). Then for any* (p,q) *with*

$$\frac{1}{p} = \frac{t}{p_1} + \frac{(1-t)}{p_2} \quad , \quad \frac{1}{q} = \frac{t}{q_1} + \frac{(1-t)}{q_2}, \quad 0 < t < 1,$$

the operator T is of strong type (p,q), and we have

$$\|Tf\|_{L^q(d\mu)} \leq H\, M_1^t M_2^{1-t}\, \|f\|_{L^p(d\mu)}$$

Proof: See [Z], p. 111, vol.2.∎

(2.2) **Remark:** From the proof of (2.1) in the case $p_1 \leq p_2$ y $q_1 < q_2$, it follows that

$$H^q = 2^q q \left(\frac{(p_1/p)^{\frac{q_1}{p_1}}}{q - q_1} + \frac{(p_2/p)^{\frac{q_2}{p_2}}}{q_2 - q} \right)$$

To apply the above theorem we need weak type inequalities for M_γ . They will be given by the next two results. The first one was proved by S. Buckley and provides an estimate concerning a known property of A_r classes. The proof of the second one is due to B. Muckenhoupt and R. Wheeden ([MW]). However, accordingly to our purpose, here we are going to examine carefully that proof in order to obtain a more precise conclusion.

(2.3) **Theorem:** *If w satisfies A_p then w satisfies $A_{p-\varepsilon}$ with $\varepsilon \sim B_{w,p}^{1-p'}$ and $B_{w,p-\varepsilon} \leq C B_{w,p}$, where $C = C(n,p)$.*

Proof: See [B], p. 255, lemma 2.1.∎

(2.4) **Theorem:** *If $0 \leq \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$, $\alpha > 0$, E_α is the set where $M_\gamma f > \alpha$, and w is a nonnegative function on $I\!R^n$ satisfying (1.3) then there is a C , independent of f , such that*

$$\left(\int_{E_\alpha} w^q \, d\mu \right)^{\frac{1}{q}} \leq C \frac{K_{w,p,q}}{\alpha} \left(\int_{I\!R^n} |fw|^p \, d\mu \right)^{\frac{1}{p}} \tag{2.5}$$

Proof: Fix $M > 0$ and let $E_{\alpha,M} = E_\alpha \cap B(0,M)$. It is clear that for each $x \in E_{\alpha,M}$ there exists a cube Q containing x such that

$$\frac{1}{\alpha \mu(Q)^{1-\frac{\gamma}{n}}} \int_Q |f| \, d\mu > 1$$

Using Besicovitch's theorem we can choose a sequence $\{Q_k\}$ of these cubes such that $E_{\alpha,M} \subset \cup Q_k$ and no point of $I\!R^n$ is on more that $C = C(n)$ of these cubes , i.e. $\sum \chi_{Q_k} \leq C$. Then, since $p/q \leq 1$ and w satisfies (1.3), we have

$$
\begin{aligned}
\left(\int_{E_{\alpha,M}} w^q\, d\mu\right)^{\frac{p}{q}} &\leq \sum_k \left(\int_{Q_k} w^q\, d\mu\right)^{\frac{p}{q}} \\
&\leq \sum_k \left(\int_{Q_k} w^q\, d\mu\right)^{\frac{p}{q}} \left(\frac{1}{\alpha\mu(Q_k)^{1-\frac{\gamma}{n}}} \int_{Q_k} |f|\, d\mu\right)^p \\
&\leq \sum_k \left(\int_{Q_k} w^q\, d\mu\right)^{\frac{p}{q}} \frac{1}{\alpha^p \mu(Q_k)^{(1-\frac{\gamma}{n})p}} \\
&\leq \left(\int_{Q_k} |f|^p\; w^p d\mu\right) \left(\int_{Q_k} w^{-p'}\, d\mu\right)^{\frac{p}{p'}} \\
&\leq \frac{K^p_{w,p,q}}{\alpha^p} \sum_k \left(\int_{Q_k} |f|^p\; w^p d\mu\right) \\
&\leq \frac{K^p_{w,p,q} C}{\alpha^p} \left(\int_{I\!R^n} |f|^p w^p d\mu\right)
\end{aligned}
$$

Finally, letting $M \to \infty$ we get (2.5).∎

Now, we are able to proceed with the proof of our main result.

Proof of Theorem (1.4): In the next, for the sake of simplicity, we are going to denote $K_{w,p,q}$ by K. As we said in §1, the fact that w satisfies (1.3) implies w^q belongs to A_r with $r = 1 + q/p'$ and $B_{w^q,r} = K^q$. Then, from (2.2), there exists $\varepsilon \sim K^{q(1-r')}$ such that w^q belongs to A_s with $s = r - \varepsilon > 1$ and $B_{w^q,s} \leq CK^q$, $C = C(n,p,q)$. Now, we choose numbers p_1 and q_1 such that $1 < p_1 < p$, $1/q_1 = 1/p_1 - \gamma/n$ and $s = 1 + q_1/p_1'$. So w^{q/q_1} satisfies $A(p_1,q_1)$ with $K_{w^{q/q_1},p_1,q_1} \leq CK^{q/q_1}$, $C = C(n,p,q)$. Then, by theorem (1.4),

$$
\int_{\{M_\gamma f>\alpha\}} w^q d\mu \leq C\frac{K^q}{\alpha^{q_1}} \left(\int_{I\!R^n} |f|^{p_1}\, w^{q\,p_1/q_1} d\mu\right)^{\frac{q_1}{p_1}} \quad . \tag{2.6}
$$

By defining $Tg\,(x) = M_\gamma\left(g\, v^{\frac{\gamma}{n}}\,(x)\right)$, with $v = w^q$, and taking $f = g\, v^{\frac{\gamma}{n}}\,(x)$, it is clear that (2.6) can be written in the form

$$
\int_{\{Tg(x)>\alpha\}} v\, d\mu \leq C\frac{K^q}{\alpha^{q_1}} \left(\int_{I\!R^n} |g|^{p_1}\, v d\mu\right)^{\frac{q_1}{p_1}} \tag{2.7}
$$

In the following step of the proof we shall asume $\varepsilon \leq \frac{q}{4}\frac{n}{n-\gamma}$. This hypothesis can be ensured by taking ε min $\left(1, \frac{q}{4}\frac{n}{n-\gamma}\right)$ instead of the original ε in the choice of p_1 and q_1 (note that this change preserves the relation between ε and K). Now, we can pick q_2 and p_2 such that $1/q - 1/q_2 = 1/q_1 - 1/q$ and $1/q_2 = 1/p_2 - \gamma/n$.

It is clear that $1+q_2/p_2' > 1+q/p'$, so $v \in A_{1+q_2/p_2'}$ with $B_{v,1+q_2/p_2'} \leq CK^q$, $C = C(n,q,p)$. Then, by reasoning as before, we get

$$\int_{\{Tg(x)>\alpha\}} v\,d\mu \leq C\frac{K^q}{\alpha^{q_2}}\left(\int_{I\!R^n} |g|^{p_2}\,v\,d\mu\right)^{\frac{q_2}{p_2}} \tag{2.8}$$

Since there exists $t \in (0,1)$ such that

$$\frac{1}{p} = \frac{t}{p_1} + \frac{(1-t)}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{t}{q_1} + \frac{(1-t)}{q_2},$$

theorem (2.1) allows us to obtain, from (2.7) and (2.8), the inequality

$$||Tg||^q_{L^q(v)} \leq CH^qK^q\,||g||^q_{L^p(v)}\,, \tag{2.9}$$

where $C = C(n,p,q)$ and H is as in (2.2). From our choice of q_1, p_2 and q_2 and the assumption on ε, we have

$$q_1 = q - \frac{\varepsilon n}{n-\gamma},$$

$$q_2 = \frac{qq_1}{2q_1 - q} = q\frac{q - \frac{\varepsilon n}{n-\gamma}}{q - \frac{2\varepsilon n}{n-\gamma}} \leq \frac{q^2}{q - 2q/4} = 2q,$$

$$p_2 = \frac{nq_2}{n+\gamma q_2} \leq 2q.$$

Then, H can be estimated as follows

$$H^q = 2^q q\left(\frac{(p_2/p)^{\frac{q_2}{p_2}}}{q_2 q} + \frac{(p_1/p)^{\frac{q_1}{p_1}}}{qq_1}\right)\frac{qq_1}{q-q_1} \leq 2^q q^3\left(\left(\frac{2q}{p}\right)^{\frac{2q}{p}} + 1\right)\frac{(n-\gamma)}{\varepsilon n}.$$

The above inequality, (2.9) and the fact that $\varepsilon \sim K^{q(1-r')}$ allow us to obtain

$$||Tg||^q_{L^q(v)} \leq CK^{qr'}\,||g||^q_{L^p(v)} = C\,K^{q\left(\frac{p'+q}{q}\right)}\,||g||^q_{L^p(v)}\,,$$

with $C = C(n,p,q)$. Finally, (1.5) follows from the definition of T by taking $g = f\,w^{-q\frac{\gamma}{n}}$ and $v = w^q$.

To see that the power of K in (1.5) is the best possible , we give an example in $I\!R$ (a similar one works in $I\!R^n$ for any n). Let $r = 1 + q/p'$, where p and q are as in the hypothesis of the theorem, and δ belonging to $(0,1)$. By a simple computation we can see that $w(x) = |x|^{\frac{(r-1)(1-\delta)}{q}}$ satisfies $A(p,q)$ with $K_{w,p,q} \simeq \delta^{\frac{1-r}{q}}$, when μ is the Lebesgue measure. Then, from (1.5) with this weight, we have

$$\|M_\gamma f\|^q_{L^q(w^q)} \leq C\delta^{-r} \ \|f\|^q_{L^p(w^p)}. \tag{2.10}$$

Now, we take $f(x) = |x|^{(\delta-1)} \chi_{[0,1)}(x)$. It is not difficult to prove that

$$M_\gamma f(x) \geq \frac{C}{\delta} |x|^\gamma f(x),$$

for every $x \in I\!R$, where C is independent of δ. Then, the above inequality and the fact that $\|f\|^q_{L^p,w^p} = \delta^{-q/p}$ allow us to get the estimate

$$||M_\gamma f||^q_{L^q(w^q)} \geq C\,\delta^{-1-q} = C\delta^{-r} \ \|f\|^q_{L^p(w^p)},$$

where C is independent of δ. Finally, we complete the proof by combining the above inequality with (2.10).■

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