Revista de la
Unión Matemática Argentina
Volumen 40, 1996.

ON THE MEASURE OF SELF-SIMILAR SETS II

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ABSTRACT. In §1 we show a condition for $\mathcal{H}^s(K_b) > 0$ for almost all $b = (b_1, \ldots, b_\ell) \in R^{n\ell}$ where $K_b = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K_b)$ and ψ_i are similitudes, $\psi_i(x) : R^n \to R^n$ defined by $\psi_i(x) = k_i A_i x + b_i$, A_i an orthogonal matrix, $0 < k_i < 1/3$, b_i a vector of R^n.

In §2 we give a (geometrical) criterion for a set $K = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K)$ to be $\mathcal{H}^s(K) = 0$ if the Hausdorff dimension is equal to its similarity dimension.

In §3 we develop a method for calculating the measure of $K = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K)$ when K meets certain conditions, generalizing a method shown in [7]. We also calculate dimensions of sets K such that their dimensions do not coincide with their similarity dimensions.

Finally we give some examples (Sierpinski sets with overlapping).

§1

Let $\psi_i(x)$, $i = 1, \ldots, \ell$, be similitudes in R^n *i.e.* $\psi_i(x) : R^n \to R^n$, $\psi_i(x) = k_i A_i x + b_i$ with $0 < k_i < 1$, A_i an orthogonal matrix and b_i a vector. Let $b = (b_1, \ldots, b_\ell) \in R^{n\ell}$ and let K_b be the (unique) compact set such that

$$K_b = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K_b) \tag{0}$$

The following theorem is due to Falconer

Theorem 1 [1]. *If $max\ k_i < 1/3$, then the Hausdorff dimension of K_b is $inf\ (n, s)$, where $\sum_{i=1}^{\ell} k_i^s = 1$, for almost all $b \in R^{n\ell}$ in the sense of the Lebesgue measure $\mathcal{L}^{n\ell}$.*

The number s, $\sum_{i=1}^{\ell} k_i^s = 1$, is usually known as the similarity dimension of K_b. In this paragraph we **assume that** $\{max\ k_i\} < 1/3$ **and** $b \in \mathcal{A} :=\{ a \in R^{n\ell} : K_a$ has Hausdorff dimension s with $\sum_{i=1}^{\ell} k_i^s = 1\}$.

It is easy to show that $\mathcal{H}^s(K_b) < \infty$ for $b \in \mathcal{A}$ (see [3] pg.122).

One natural question is to ask whether $\mathcal{H}^s(K_b) > 0$ for $b \in \mathcal{A}$. (It should be noted that if ψ_i are affine contractions instead of similitudes then $\mathcal{H}^s(K_b)$ may be infinite, cf. [4].).

We prove

Theorem 2. *Let $g := \{min\ k_i\}$ and $G := \{max\ k_i\}$. Suppose*

(1) $$\ell^{\left(\frac{2log\ g}{log\ G}\right)} \cdot g^n < 1 \equiv \ell^2.G^n < 1$$

Then $\mathcal{H}^s(K_b) > 0$ for almost all $b \in \mathcal{R}^{n\ell}$.

Condition (1) right implies that the similarity dimension is less than n and therefore by theorem 1, $\mathcal{A}$ is almost all $\mathcal{R}^{n\ell}$. That both formulas in (1) are equivalent follows from taking logarithm to the left hand formula i.e. $log\left[\ell^{\left(\frac{2log\ g}{log\ G}\right)} g^n\right] < 0 \equiv \frac{log\ g}{log\ G} log\ \ell^2 + n\ log\ g < 0$. Multiply this last expression by $\frac{log\,G}{log\,g}$, getting the right hand formula.

We recall a theorem of McLaughlin [5], generalized by Falconer [2], which we shall use. It should be noted that theorem 3, lemma 1 and corollary 1 are true **without** assuming $\{max\ k_i\} < 1/3$ or $b \in \mathcal{A}$ (or both).

Theorem 3 [2]. *Suppose that K_b has the following property: there exist a natural number m and α , $r_o > 0$ such that for any set $N \subset K_b$ with $|N| < r_o$ there are sets N_j with $N \subset \bigcup_{j=1}^{m} N_j$ and mappings $\varphi_j :\ N_j \to K_b\,(1 \leqslant j \leqslant m)$ such that*

($d(.,.)$ is the euclidean distance)

$$\alpha\ d(x, y) \leqslant |N|\, d(\varphi_j(x), \varphi_j(y))$$

for $x, y \in N_j$. Then $\mathcal{H}^s(K_b) > 0$.

To prove Theorem 2 we need

Lemma 1 [6]. *Fix b. If $\psi_i(K_b) \cap \psi_j(K_b) = \emptyset$ for $i \neq j$ then $\mathcal{H}^s(K_b) > 0$*

Proof.

One can use theorem 3 or one can notice that K_b satisfies an open set condition. See [3] ■

Let $\mathcal{C}(K)$ denote the convex hull of a subset K of R^n. Let i_j be natural numbers such that $1 \leqslant i_j \leqslant \ell$. I stands for a finite tuple of such i_j i.e. $I = i_1 \ldots i_m$, and $|I|$ denotes the length of such a tuple. We write for short $\psi_I(\cdot) = \psi_{i_1}(\ldots(\psi_{i_m}(\cdot))\ldots)$.

Given I,J two tuples, we say that I is a curtailment of J (we write $I \leqslant J$) iff $I = i_1 \ldots i_m$; $J = i_1 \ldots i_m$, $j_{m+1} \ldots j_s$, $s \geqslant m$. It is not difficult to see that $\leqslant$ defines a partial order in the set of all finite tuples.

Corollary 1. *Suppose K_b has the following property: there exists a finite family of tuples $\mathcal{F}$ (not necesarily of equal length) such that*

(i) *For any I' , $|I'| \geqslant \max_{J\in\mathcal{F}}|J|$, there exists $I \in \mathcal{F}$ such that $I \leqslant I'$ ($\mathcal{F}$ is secure).*

(ii) *$\psi_I(K_b) \cap \psi_J(K_b) = \emptyset$ for any pair $I, J \in \mathcal{F}$ with $i_1 \neq j_1$*

Then we have $\mathcal{H}^s(K_b) > 0$.

Proof.

Suppose $p \in \psi_i(K_b)$ $1 \leqslant i \leqslant \ell$, say $p \in \psi_1(K_b)$. Choose an index $I' = 1i'_2 \dots i'_m$ such that $|I'| = \max_{J\in\mathcal{F}} |J|$ and $p \in \psi_{I'}(K_b)$. Therefore by (i) we have an index $I \in \mathcal{F}$ such that $I \leqslant I'$ and $p \in \psi_I(K_b) \supset \psi_{I'}(K_b)$ with $I = 1i'_2 \dots i'_{j(j\leqslant m)}$. A similar argument and (ii) shows that $p \notin \psi_i(K_b)$, $1 \leqslant i \neq 1 \leqslant \ell$ *i.e.* $\psi_1(K_b) \cap \psi_i(K_b) = \emptyset$ $\forall i \neq 1$. By lemma 1, $\mathcal{H}^s(K_b) > 0$.■

Proof of theorem 2.

Let $b_o \in R^{n\ell}$. There is no loss of generality if we assume $0 \in K_{b_o}$. Let Q_{b_o} be a cube in $R^{n\ell}$ centered at b_o, $\mathcal{L}^{n\ell}(Q_{b_o}) \leqslant 1$. Therefore $|K_b \cup \{0\}| < c_o$, for some constant c_o if $b \in Q_{b_o}$. This is possible since $K_b = \{b_{i_1} + k_{i_1} A_{i_1} b_{i_2} + k_{i_1} k_{i_2} A_{i_1} A_{i_2} b_{i_3} + \dots : i_j \in \{1, \dots, \ell\}\}$. We will show that $\mathcal{H}^s(K_b) > 0$ for almost all $b_o \in Q_{b_o}$. This will prove our theorem.

Let $Q^j_{b_o} = \{(\ell - 1) - tuples\ (b_1, \dots, \hat{b}_j, \dots, b_\ell) : b \in Q_{b_o}\}$. Fix $\mathcal{O}$ a large natural number and $\mathcal{F}_\mathcal{O}$ be the set of all tuples $I = i_1 \dots i_m$ such that $g^\mathcal{O} < k_{i_1} \dots k_{i_{m-1}}$ and $k_{i_1} \dots k_{i_m} \leqslant g^\mathcal{O}$. Then $\mathcal{F}_\mathcal{O}$ has property i of corollary 1 and $K_b = \bigcup_{I\in\mathcal{F}_\mathcal{O}} \psi_I(K_b)$ for all b. Let

$$\sum_I (b) := b_{i_1} + k_{i_1} A_{i_1} b_{i_2} + \dots + k_{i_1} \dots k_{i_{m-1}} A_{i_1} \dots A_{i_{m-1}} b_{i_m}$$

$$= \psi_I(0) \tag{2}$$

and

$$T_I(b) := \mathcal{C}(\psi_I(K_b)) = \psi_I(\mathcal{C}K_b)$$

The number of elements of $\mathcal{F}_\mathcal{O}$ is not greater than $c_1 \cdot \ell^{\left(\mathcal{O}\frac{\log g}{\log G}\right)}$ where $c_1 \leqslant \ell$. Let $I, J \in \mathcal{F}_\mathcal{O}$: $i_1 \neq j_1$. We want to measure $A_{IJ} := \{b : b \in Q_{b_o}$ and $T_I(b) \cap T_J(b) \neq \emptyset\}$. Let $b \in A_{IJ}$. Then

$$\Big|\sum_I (b) - \sum_J (b)\Big| = |\psi_I(0) - \psi_J(0)| \leqslant |\psi_I(K_b \cup \{0\}) \cup \psi_J(K_b \cup \{0\})|$$

$$\leqslant |\psi_I(\mathcal{C}(K_b \cup \{0\})) \cup \psi_J(\mathcal{C}(K_b \cup \{0\}))| \leqslant 2c_o g^\mathcal{O} \tag{3}$$

Therefore if $b, b' \in A_{IJ}$ and

$$b = (b_1, \dots, b_{i_1 - 1}, b_{i_1}, b_{i_1+1}, \dots, b_\ell)$$
$$b' = (b_1, \dots, b_{i1-1}, b'_{i_1}, b_{i_1+1}, \dots, b_\ell)$$

Then from (3) and (2) and $i_1 \neq j_1$ we get

$$\left|\left[\sum_I (b) - \sum_J (b)\right] - \left[\sum_I (b') - \sum_J (b')\right]\right| = |(b_{i_1} - b'_{i_1}) + \triangle| \leqslant 4c_o g^\mathcal{O}$$

with

$$|\triangle| \leqslant \frac{2G}{1-G}|b_{i_1} - b'_{i_1}|$$

Combining this last two inequalities we get $|b_{i_1} - b'_{i_1}| \leqslant c_2 \cdot g^{\mathcal{O}}$ with c_2 depending on k_i and c_o. Therefore projecting along the axis i_1 we have $\mathcal{L}^{n\ell}(A_{IJ}) \leqslant c_3 \mathcal{L}^{n(\ell-1)}(Q^{i_1}_{b_o}) \cdot g^{\mathcal{O}\cdot n} \leqslant c_3 \cdot g^{\mathcal{O}\cdot n}$ with c_3 depending on k_i , c_o and n .

If $b \notin \{\bigcup_{I,J\in\mathcal{F}_{\mathcal{O}}\,;\,i_1\neq j_1} A_{IJ}\}$ then K_b has property ii) stated in Corollary 1 and then $\mathcal{H}^s(K_b) > 0$. But the number of pairs (IJ) $I, J \in \mathcal{F}_{\mathcal{O}}$, $i_1 \neq j_1$, is not greater than $c_1^2 \cdot \ell^{2\mathcal{O}\frac{log\ g}{log\ G}}$. Therefore the set $\{\bigcup_{I,J\in\mathcal{F}_{\mathcal{O}}\,;\,i_1\neq j_1} A_{IJ}\}$ has (outer) measure at most $c_1^2 \cdot c_3 \cdot \left(g^n \cdot \ell^{\left(\frac{2log\ g}{log\ G}\right)}\right)^{\mathcal{O}}$ and this tends to zero by hypothesis if $\mathcal{O} \to \infty$. The theorem follows.■

§2

Let K be a self-similar set i.e. $K = \bigcup_{i=1}^{\ell} \psi_i(K)$ where ψ_i are similitudes of ratio $0 < k_i < 1$, K compact. Recall that $\mathcal{H}^s(K) < \infty$ if s is the similarity dimension (cf [3] pg.122). Assume that the Hausdorff dimension of K is equal to its similarity dimension. Under such hypothesis we want to give some geometrical criterion for $\mathcal{H}^s(K) = 0$. This is proposition 1 below. To prove it we need some tools. The following function $f(\delta)$ has been defined in [7] for K, s as above , with the extra condition $0 < \mathcal{H}^s(K)$: let, for $\delta > 0$,

$$f(\delta) := sup\{\mathcal{H}^s(K \cap C_\delta)/\delta^s \ :\ C_\delta \, is\ a\ convex\ compact\ set\ of\ diameter\ \delta\}$$

In [7] it was proved that $f(\delta) \leqslant 1$ for all $\delta > 0$ (see also §3 of this paper).

We follow the notation of the proof of theorem 2, $\mathcal{F}_{\mathcal{O}}$ being the set of all tuples $I = i_1 \dots i_m$ such that $g^{\mathcal{O}} < k_{i_1} \dots k_{i_{m-1}}$ and

$$(4) \qquad g^{\mathcal{O}+1} < k_{i_1} \dots k_{i_m} \leqslant g^{\mathcal{O}}$$

Recall $K = \bigcup_{I\in\mathcal{F}_{\mathcal{O}}} \psi_I(K)$. Write for short $T_I := \mathcal{C}(\psi_I(K))$.

Let $h(\mathcal{O})$:=the maximum number of elements $I_1 \dots I_q \in \mathcal{F}_{\mathcal{O}}$ such that

$$(5) \qquad d(T_{I_i}, T_{I_j}) \leqslant g^{\mathcal{O}}|K|$$

The function $h(\mathcal{O})$, roughly speaking, measures the overlapping of the sets of approximately equal diameter $\psi_I(K), I \in \mathcal{F}_{\mathcal{O}}$.

Proposition 1. *If the Hausdorff and similarity dimension of K are equal to s then:* $\mathcal{H}^s(K) = 0 \Longleftrightarrow \overline{\lim_{\mathcal{O}\to\infty}}\ h(\mathcal{O}) = \infty$

Proof.

$\Rightarrow$) Suppose $\overline{\lim_{\mathcal{O}\to\infty}} h(\mathcal{O}) \leqslant \beta$. Therefore if $\mathcal{O}$ is any large natural number then by definition of $h(\mathcal{O})$ any set N such that $N \subset K$, $g^{\mathcal{O}+1} < |N| \cdot |K| \leqslant g^{\mathcal{O}}$ can be decomposed in at most β sets $N_j = \psi_{I_j}(K) \cap N$ with $I_j \in \mathcal{F}_{\mathcal{O}}$ *i.e.* $N \subset \bigcup_1^\beta N_j$. Apply theorem 3 with $\varphi_j = \psi_{I_j}^{-1}$.

$\Leftarrow$) Suppose $\overline{\lim_{\mathcal{O}\to\infty}}\ h(\mathcal{O}) = \infty$ and $\mathcal{H}^s(K) > 0$. Then we are in condition to define the function $f(\delta)$ as above. Also observe that $\mathcal{H}^s(\psi_i(K) \cap \psi_j(K)) = 0$ for $1 \leqslant i \neq j \leqslant \ell$ and therefore $\mathcal{H}^s(\psi_I(K) \cap \psi_J(K)) = 0$ *if* $I \neq J \in \mathcal{F}_{\mathcal{O}}$.

By definition of $h(\mathcal{O})$ there exits $I_1 \ldots I_{h(\mathcal{O})}$ such that $d(T_{I_i}, T_{I_j}) \leqslant g^{\mathcal{O}}|K|$.

Let $C_{\delta_o} = \{x : d(x, T_{I_1}) \leqslant 2g^{\mathcal{O}}|K|\}$.

Therefore (by (4)) $C_{\delta o}$ contains $T_{I_1}, \ldots, T_{I_{h(\mathcal{O})}}$ and has diameter $\delta_o \leqslant 5 \cdot g^{\mathcal{O}} \cdot |K|$. Therefore, since $\mathcal{H}^s(\psi_{I_j}(K) \cap \psi_{I_i}(K)) = 0$ we have

$$1 \geqslant f(\delta_o) \geqslant \frac{\mathcal{H}^s(K \cap C_{\delta_o})}{\delta_o^s} \geqslant \frac{h(\mathcal{O})\{\min\limits_{i=1\ldots h(\mathcal{O})} \mathcal{H}^s(\psi_{I_i}(K))\}}{5^s|K|^s g^{\mathcal{O}s}} \geqslant (by 4) \geqslant$$
$$\geqslant \frac{h(\mathcal{O})\mathcal{H}^s(K) g^s}{5^s|K|^s}$$

This is absurd taking $\mathcal{O} \to \infty$ ■

§3

In this section we assume K to be a *self − similar set*, i.e. $K = \bigcup_{i=1}^{\ell} \psi_i(K)$, ψ_i a similitude of ratio $0 < k_i < 1$. In [7] a method was given which permits to approximate the Hausdorff measure of $\mathcal{H}^s(K)$ assuming that:

(i) $0 < \mathcal{H}^s(K) < \infty$

(ii) K has property A (see below)

(iii) $\sum_{i=1}^{\ell} k_i^s = 1$ i.e. the Hausdorff dimension of K is equal to its similarity dimension.

In this section we want to generalize this result by dropping condition iii). Recall that for a self similar set K satisfying conditions i) and iii) one must have $\mathcal{H}^s(\psi_i(K) \cap \psi_j(K)) = 0 \ \ \forall i \neq j$; $1 \leqslant i, j \leqslant \ell$.

Theorem 4. *Assume K to be a self similar set and* $0 < \mathcal{H}^s(K) < \infty$. *Define*

$$f(\delta) := \sup_{\substack{C_\delta \text{ convex} \\ \text{compact} \\ \text{of diameter } \delta > 0}} \mathcal{H}^s(K \cap C_\delta)/\delta^s$$

Then $f(\delta) \leqslant 1 \qquad \forall \delta > 0$.

Proof.

If the Hausdorff dimension of K is zero then K has to be a point (if K had two points at least then being K a self similar set defined by similitudes then it should have infinite points. This would contradict $\mathcal{H}^o(K) < \infty$). The theorem is true in this case.

Therefore we assume $s > 0$. Suppose $\frac{\mathcal{H}^s(K\cap C_\delta)}{\delta^s} = \frac{\mathcal{H}^s(K\cap C_\delta)}{|C_\delta|^s} \geqslant \beta > 1$ for some C_δ convex and compact. Moreover one can assume $|K \cap \partial C_\delta| = |C_\delta| = \delta$. Let $A_n = \bigcup_{|I|\geqslant n} \psi_I(K \cap C_\delta)$. Therefore $A_{n+1} \subset A_n$. Let $A = \bigcap_{i=1}^{\infty} A_i$. For A we have $\mathcal{H}^s(A) > 0$ or $\mathcal{H}^s(A) = 0$.

Assume $\mathcal{H}^s(A) > 0$. Let n_o be such that $\mathcal{H}^s(A_{n_o}/A) < \varepsilon$ and observe that $\{\psi_I(K \cap C_\delta)\}$, $|I| \geqslant n_o$ is a Vitali family for A. Since for any countable disjoint subfamily we have

$$\begin{aligned} \sum|\psi_I(K\cap C_\delta)|^s &\leqslant \frac{1}{\beta}\sum\mathcal{H}^s(\psi_I(K\cap C_\delta)) \\ &\leqslant \frac{\mathcal{H}^s(K)}{\beta} < \infty, \end{aligned} \tag{6}$$

there exists a disjoint countable subfamily indexed by $\mathcal{F}$ such that $\mathcal{H}^s(A/\bigcup_{I\in\mathcal{F}}\psi_I(K\cap C_\delta)) = 0$ (cf.[3], pg.11). Besides, we can assume

$$\beta\mathcal{H}^s(A) - \varepsilon \leqslant \beta\sum_{I\in\mathcal{F}}|\psi_I(K\cap C_\delta)|^s \leqslant \sum_{I\in\mathcal{F}}\mathcal{H}^s(\psi_I(K\cap C_\delta)) \leqslant \mathcal{H}^s(A_{n_o}) \leqslant \varepsilon + \mathcal{H}^s(A)$$

This is absurd if ε is sufficiently small.

If $\mathcal{H}^s(A) = 0$, let c be a fixed positive number such that $0 < \delta < c$ and

$$K \subset [K \cap C_\delta]_c = \{x : d(x, K \cap C_\delta) \leqslant c\} \tag{7}$$

For any $\varepsilon > 0$, let $n_o = n_o(\varepsilon)$ be a natural number such that $\mathcal{H}^s(A_{n_o}) \leqslant \varepsilon$. Then

$$\mathcal{H}^s\left(\bigcup_{I\in\mathcal{F}'}\psi_I(K\cap C_\delta)\right) \leqslant \varepsilon \tag{8}$$

where $\mathcal{F}'$ denotes a maximal family of indices I of lenght n_o chosen in the following way: first choose I_o ($|I_o| = n_o$) such that $|\psi_{I_o}(K\cap C_\delta)| = \max_{|I|=n_o}|\psi_I(K\cap C_\delta)|$. From all the indices I ($|I| = n_o$) such that $\psi_I(K\cap C_\delta)\cap\psi_{I_o}(K\cap C_\delta) = \emptyset$ choose one such that its diameter is maximum, call this index I_1 and so on. Using (7) we see that

$$K \subset \bigcup_{|I|=n_o} \psi_I([K\cap C_\delta]_c) \tag{9}$$

and by the way $\mathcal{F}'$ is chosen (recall $0 < \delta < c$)

$$K \subset \bigcup_{I \in \mathcal{F}'} \psi_I([K \cap C_\delta]_c) \tag{10}$$

Moreover

$$\frac{\mathcal{H}^s(\psi_I(K \cap C_\delta))}{|\psi_I(K \cap C_\delta)|^s} \geqslant \beta > 1 \qquad \forall I \in \mathcal{F}' \tag{11}$$

Using this last formula, (8) and (10), we get

$$\begin{aligned} \frac{\varepsilon}{\beta} \geqslant \sum_{I \in \mathcal{F}'} |\psi_I(K \cap C_\delta)|^s \geqslant \left(\frac{\delta}{3c}\right)^s \left(\sum_{I \in \mathcal{F}'} |\psi_I([K \cap C_\delta]_c)|^s\right) \geqslant \\ \left(\frac{\delta}{3c}\right)^s \mathcal{H}^s_{3c\{max\, k_i\}^{n_o}}(K) \end{aligned} \tag{12}$$

It follows that $\mathcal{H}^s(K) = 0$, an absurd.■

Corollary 2. *Assume the hypothesis of theorem 4. Then:*
(i) $f(\delta) \leqslant 1 \qquad \forall \delta > 0$
(ii) $\overline{lim}_{\delta \to 0} f(\delta) = 1$
(iii) $f(\delta)$ *is continuous from the right*

Proof.

(ii) follows from elementary density bounds (see [3], p. 24).

(iii) From Blaschke selection theorem follows that for any δ there is a compact, convex set of diameter δ, C_δ, such that $f(\delta) = \frac{\mathcal{H}^s(K \cap C_\delta)}{\delta^s}$. Notice that $f(\delta)\delta^s$ is non decreasing. Using these last observations and the continuity of $\mathcal{H}^s(\cdot)$ we get (iii) (cf. [7] §1). ■

Property A. *We say K has property A if there exists $a > 0$ such that for any sphere $B_{r_1}(x)$ with $r_1 < a$ there exists an expanding similitude $\psi(z)$ with contraction ratio $\xi \geqslant 1$ and an index i_o, $1 \leqslant i_o \leqslant \ell$, such that*

$$\psi(B_{r_1}(x) \cap K) \subset \psi_{i_o}(K)$$

Property A is indeed quite strong.

Proposition 2. *If K is a self-similar set with property A and Hausdorff dimension s then K is an s-set i.e. $0 < \mathcal{H}^s(K) < \infty$.*

Proof.

As ψ_i are similitudes and $K = \bigcup_{i=1}^{\ell} \psi_i(K)$, by [2] page 550 we get $\mathcal{H}^s(K) < \infty$.

That $0 < \mathcal{H}^s(K)$ follows from the fact that property A implies the hypothesis of theorem 3. This assertion is proved as follows. Let N be any subset of K of diameter less than a of property A. Then by this property there exists an index i_o, $1 \leqslant i_o \leqslant \ell$ and ψ an expansive similitude such that $\psi(N) \subset \psi_{i_o}(K)$. Taking $\psi_{i_o}^{-1}$ in this inclusion one gets $\psi_{i_o}^{-1}\psi(N) \subset K$. If $|\psi_{i_o}^{-1}\psi(N)| < a$ then proceed as before with $\psi_{i_o}^{-1}\psi(N)$ as N. After a finite number of steps one gets

$$\psi_{i_n}^{-1}\psi' \dots \psi_{i_o}^{-1}\psi(N) \subset K$$

$$a \leqslant |\psi_{i_n}^{-1}\psi' \dots \psi_{i_o}^{-1}\psi(N)|$$

where $1 \leqslant i_j \leqslant \ell$ and $\psi', \dots, \psi$ are expansive similitudes. Therefore one can define $\varphi(x) = \psi_{i_n}^{-1}\psi' \dots \psi_{i_o}^{-1}\psi(x) : N \to K$. It is easily chequed that $a\, d(x,y) \leqslant |N|\, d(\varphi(x), \varphi(y)$ for all x, y in N.■

The following is a corollary of theorem 4.

Corollary 3. *If K has property A and $0 < \mathcal{H}^s(K) < \infty$ then $f(\delta) = 1$ for some δ_o such that $a \leqslant \delta_o \leqslant |K|$*

Proof.

Let δ be such that $0 < \delta < a$. We want to show that $f(\delta) \leqslant f(\delta\xi\{min\ k_i\}^{-1})$. For this let C_δ be a convex compact set such that $f(\delta) = \dfrac{\mathcal{H}^s(K \cap C_\delta)}{\delta^s}$. Obviously $C_\delta \subset B_{r_1}(x)$ for some sphere with radius $r_1 < a$. From this and property A we get

$$\psi(C_\delta \cap K) \subset \psi(B_{r_1}(x) \cap K) \subset \psi_{i_o}(K)$$

and therefore $\psi_{i_o}^{-1}(\psi(C_\delta \cap K)) = \psi_{i_o}^{-1}(\psi(C_\delta)) \cap \psi_{io}^{-1}(\psi(K)) \subset K$. Intersecting this last expression with $\psi_{i_o}^{-1}(\psi(C_\delta))$ we get $\psi_{i_o}^{-1}(\psi(C_\delta \cap K)) \subset K \cap \psi_{i_o}^{-1}(\psi(C_\delta))$ and therefore

$$\mathcal{H}^s(\psi_{i_o}^{-1}(\psi(K \cap C_\delta))) = \xi^s k_{io}^{-s} \mathcal{H}^s(K \cap C_\delta) \leqslant \mathcal{H}^s(K \cap \psi_{i_o}^{-1}(\psi(C_\delta)))$$

i.e.

$$f(\delta) = \frac{\mathcal{H}^s(K \cap C_\delta)}{\delta^s} \leqslant \frac{\mathcal{H}^s(K \cap \psi_{i_o}^{-1}(\psi(C_\delta)))}{\delta^s \xi^s k_{i_o}^{-s}} \leqslant f(\delta\xi k_{i_o}^{-1})$$

(notice that $\psi_{i_o}^{-1}(\psi(C_\delta))$ is convex, compact, of diameter $\delta\xi k_{io}^{-1}$). This proves the assertion. From this, i) and ii) of corollary 2 we get that $\sup_{[a,\infty)} f(\delta) = 1$ and therefore $\sup_{[a,|K|]} f(\delta) = 1$ because $f(\delta) = \dfrac{\mathcal{H}^s(K)}{\delta^s}$ for $\delta > |K|$. Since $f(\delta)$ is continuous from the right and $f(\delta) \cdot \delta^s$ is non decreasing we get $f(\delta_o) = 1$ for some $a \leqslant \delta_o \leqslant |K|$.■

Definition. *K is $\varepsilon-$ discrete if $0<\mathcal{H}^s(K)<\infty$ and there exist (non void) sets $K_1,\ldots,K_q$, $q=q(\varepsilon)$, such that:*

(i) $K=\bigcup_{i=1}^{q} K_i$ *and* $\mathcal{H}^s(K_i\cap K_j)=0$ *for* $i\neq j$

(ii)$|K_i|\leqslant\varepsilon\ \forall i$

(iii) *the numbers* $\alpha_i:=\frac{\mathcal{H}^s(K_i)}{\mathcal{H}^s(K)}$ *,* $i=1\ldots q$*, can be calculated.*

The reader should observe that if a self similar set K is such that $0<\mathcal{H}^s(K)<\infty$ and its Hausdorff dimension s is equal to its similarity dimension then K is ε -discrete. To see this, just take as K_i in the above definition the sets $\psi_I(K)$, $|I|=|i_1\ldots i_{no}|=n_o$ i.e. $q=\ell^{n_o}$. Properties i), ii) are easily verified and $\dfrac{\mathcal{H}^s(K_i)}{\mathcal{H}^s(K)}=$ $\dfrac{\mathcal{H}^s(\psi_I(K))}{\mathcal{H}^s(K)}=k_{i_1}^s\ldots k_{i_{no}}^s$.

Therefore the Sierpinski set and the Cantor set are ε-discrete. See §4 this paper for other examples.

If K is $\varepsilon-$ *discrete* with $\varepsilon<a_o$ then we can obtain approximations of $f(\delta)$ on $[a_o,|K|]$. This is theorem 5 below. Later we shall use $a_o=a$, with a of property A. For this theorem we need some definitions that **only** assume that K is $\varepsilon-$ *discrete* with $\varepsilon<a_o$.

Let $\mathcal{P}$ be the family of all non void sets $\{i_1,\ldots,i_t\}$ with $1\leqslant i_1<\cdots<i_t\leqslant q$. If $p\in\mathcal{P}$ **define** $G(p)=|\bigcup_{i\in p} K_i|$. It is clear that $G(\mathcal{P})$ is a finite set of non negative numbers and there exists some $d\in G(\mathcal{P})$ such that $d\leqslant\varepsilon<a_o$ (by ii) of the above definition). **Define** $\widetilde{\mathcal{U}}$ on $G(\mathcal{P})$ in the following way: $\widetilde{\mathcal{U}}(d):=\max_{\substack{p \text{ such that}\\ G(p)=d}}(\sum_{i\in p}\alpha_i)$.

Next we define $\mathcal{U}(\delta)$, for $\delta\geqslant a_o$, ($\mathcal{U}(\delta)$. $\mathcal{H}^s(K)$ will be an approximation of $f(\delta)\cdot\delta^s$). **Define** $\mathcal{U}(\delta):=\max_{\substack{d\leqslant\delta\\ d\in G(\mathcal{P})}}\widetilde{\mathcal{U}}(d)$. Easy consequences of the definition of $\mathcal{U}(\delta)$ are that it is a non decreasing function and that $\mathcal{U}(\delta)$ is constant on the intervals $[a_o,a_1),[a_1,a_2),\ldots,[a_w,\infty)$ with $a_w\leqslant|K|$ where $a_1<\cdots<a_w$ are points of $G(\mathcal{P})$ and $\mathcal{U}(\delta)=1$ for $\delta\in[a_w,\infty)$ by i) of the above definition. We also recall that $f(\delta)\cdot\delta^s$ is non decreasing and continuous from the right and that s does not coincide necesarily with the similarity dimension.

Theorem 5. *Assume that K is $\varepsilon-$ discrete with $\varepsilon<a_o$. Then*

(i) $\mathcal{U}(\delta)\leqslant\frac{f(\delta)\delta^s}{\mathcal{H}^s(K)}\leqslant\mathcal{U}(\delta+2\varepsilon)$ *for* $\delta\geqslant a_o$

(ii) *Suppose that K is $\varepsilon-$ discrete for each $\varepsilon>0$. Then*

$$\left(\sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s}-\sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}\right)\to 0\ for\ \varepsilon\to 0.$$

Proof.

We assume $s>0$. If $s=0$ then K must be a point and the theorem is trivial.

(i) Fix δ. By Blaschke selection theorem $f(\delta)\cdot\delta^s = \mathcal{H}^s(K\cap C_\delta)$ for some C_δ convex compact. Let $d\in G(\mathcal{P}), d\leqslant\delta$. Then $\tilde{\mathcal{U}}(d)\cdot\mathcal{H}^s(K) = \max_{G(p)=d}(\sum_{i\in p}\alpha_i)\mathcal{H}^s(K) = \max_{G(p)=d}(\sum_{i\in p}\mathcal{H}^s(K_i)) \leqslant f(\delta)\cdot\delta^s$ This last inequality because $|\bigcup_{\substack{i\in p\\ G(p)=d}} K_i| = d\leqslant\delta$ and $f(\delta)\cdot\delta^s$ is non decreasing. From this the left hand inequality of i) follows. To prove the right hand inequality, suppose that C_δ is such that $f(\delta)\cdot\delta^s = \mathcal{H}^s(K\cap C_\delta)$ and let $p_o\in\mathcal{P}$ be the set of indices i such that K_i intersects C_δ. As $|K_j|\leqslant\varepsilon\ \forall j$ we get $G(p_o) = |\bigcup_{i\in p_o} K_i| \leqslant \delta+2\varepsilon$ and therefore by the definition of $\mathcal{U}(\delta)$ we have

$$
\begin{aligned}
f(\delta)\cdot\delta^s = \mathcal{H}^s(K\cap C_\delta) &\leqslant \mathcal{H}^s(K\cap(\bigcup_{i\in p_o} K_i)) \\
&= \left(\sum_{i\in p_o}\alpha_i\right)\mathcal{H}^s(K) \leqslant \mathcal{U}(G(p_o))\mathcal{H}^s(K) \\
&\leqslant \mathcal{U}(\delta+2\varepsilon)\mathcal{H}^s(K)
\end{aligned}
$$

proving the other inequality of i).

(ii) From i) we get $\sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s} \leqslant \sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s} \leqslant$
$\leqslant \sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{(\delta+2\varepsilon)^s}\sup_{\delta\in[a_o,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s} \leqslant \sup_{\delta\in[a_o+2\varepsilon,|K|+2\varepsilon]}\frac{\mathcal{U}(\delta)}{\delta^s}\sup_{\delta\in[a_o,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s}$. But $\mathcal{U}(\delta)= 1$ if $\delta\geqslant|K|$. Then $\sup_{\delta\in[a_o+2\varepsilon,|K|+2\varepsilon]}\frac{\mathcal{U}(\delta)}{\delta^s} \leqslant \sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}$. From this and the above inequality we get $0\leqslant \sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s} - \sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}$
$\leqslant \sup_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}(\sup_{\delta\in[a_o,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s}-1) \leqslant (by i) \leqslant$
$\leqslant \sup_{\delta\in[a_o,|K|]}\frac{f(\delta)}{\mathcal{H}^s(K)}(\sup_{\delta\in[a_o,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s}-1)$ which proves ii).■

Finally, we show how to obtain bounds for $\mathcal{H}^s(K)$. Assume

(i) K has property A

(ii) K is $\varepsilon-$ *discrete* with $\varepsilon < a$, a of property A

Then, from theorem 5 and corollaries 2 i) and 3 we get

$$
\sup_{\delta\in[a,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s} \leqslant \frac{1}{\mathcal{H}^s(K)} \leqslant \sup_{\delta\in[a,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s} \tag{13}
$$

If K is $\varepsilon-$ *discrete* for any $\varepsilon>0$ we get from theorem 5 ii) that in (13) the two suprema tend to $1/\mathcal{H}^s(K)$, yielding the algorithm. For an example see §4.

We finally point out that in practice, knowing α_i in the definition of the $\varepsilon-$ *discreteness* requires the knowledge of s, the dimension of K. We prove a lemma showing how this dimension can be obtained in some cases. We **assume** that $0<\mathcal{H}^s(K)<\infty$, which in practice will follow from theorem 3 and [2].

Lemma 2. *Let* $K = \bigcup_{i=1}^{\ell} \psi_i(K)$. *Assume* $\psi_i(K) \cap \psi_j(K)$, $i \neq j$, *is a disjoint union of sets* $K_t^{i,j}$ $t = 1, \ldots, n(i,j)$ *where* $K_t^{i,j} = \psi_t^{ij}(K)$ *and* ψ_t^{ij} *is a similitude with contraction ratio* ξ_t^{ij}. *Assume also* $\mathcal{H}^s(\psi_i(K) \cap \psi_j(K) \cap \psi_k(K)) = 0$ *for any triple* (i,j,k), $i \neq j \neq k \neq i$. *Then the Hausdorff dimension* s *verifies the following relation*

$$1 = \sum_{i=1}^{\ell} \xi_i^s - \left(\sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(i,j)} (\xi_t^{ij})^s \right)$$

(here ξ_i *is the contraction ratio of* ψ_i*)*

Proof.

From the hipothesis we know that

$$\mathcal{H}^s(\psi_i(K)) = \mathcal{H}^s \left(\psi_i(K)/(\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \psi_j(K)) \right) + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sum_{t=1}^{n(i,j)} \mathcal{H}^s(K_t^{i,j}) \tag{14}$$

and therefore

$$\xi_i^s \mathcal{H}^s(K) = \mathcal{H}^s \left(\psi_i(K)/(\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \psi_j(K)) \right) + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sum_{t=1}^{n(i,j)} (\xi_t^{ij})^s \mathcal{H}^s(K) \tag{15}$$

Also from the hypothesis we get

$$\mathcal{H}^s(K) = \sum_{i=1}^{\ell} \mathcal{H}^s \left(\psi_i(K)/(\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \psi_j(K)) \right) + \sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(i,j)} \mathcal{H}^s(K_t^{i,j}) \tag{16}$$

Putting (15) in (16) we get

$$\mathcal{H}^s(K) = \left(\sum_{i=1}^{\ell} \xi_i^s \right) \cdot \mathcal{H}^s(K) - \left(\sum_{i=1}^{\ell} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sum_{t=1}^{n(ij)} (\xi_t^{ij})^s \right) \mathcal{H}^s(K) + \left(\sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(ij)} (\xi_t^{ij})^s \right) \mathcal{H}^s(K)$$

As $0 < \mathcal{H}^s(K) < \infty$ and $\xi_t^{ij} = \xi_t^{ji}$,

$$1 = \sum_{i=1}^{\ell} \xi_i^s - \left(\sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(ij)} (\xi_t^{ij})^s \right) \blacksquare$$

§4 *Example* (*Sierpinski set with overlapping*) : Let $\triangle$ be an equilateral triangle of side 1 and let p_1, p_2, p_3 be its vertices (fig.1). Let $\triangle_1$, $\triangle_2$, $\triangle_3$, be three smaller equilateral triangles inside $\triangle$ and touching p_1, p_2, p_3, respectively. We define $\psi_i(x)$; $i = 1,2,3$, as the similitudes that transform $\triangle$ onto $\triangle_i$ (without rotation). We assume that all contracting ratios are equal to $\xi\,, 0 < \xi < 1$. Notice that $\mathcal{C}(K) = \triangle$ ($\mathcal{C}$ = convex hull). We write for short: $\triangle_I = \psi_I(\triangle) = \psi_I(\mathcal{C}(K))$ and assume that ξ satisfies the following equation:

$$2\xi - \xi^n = 1 \; ; \; n \geqslant 3 \; , \; n \text{ an integer.} \tag{17}$$

We will see that we can apply the previous theory to K i.e. we will prove that K has property A and therefore by proposition 2, K is an $s - set$. We will also show that K is $\varepsilon - discrete$. To prove all these assertions one needs lemmas 3 and 4 and the following discussion.

It is easy to prove that there is a unique ξ , $0 < \xi < 1$ satisfying (17), for the polinomial $2x - x^n - 1$ has only two roots in $[1/2, 1]$, being 1 one of them.

Let $0 < \xi_3 < 1$ where ξ_3 satisfies (17) with $n = 3$. Then $\xi_3 = \frac{\sqrt{5}-1}{2}$. It is easy to prove that if $0 < \xi < 1$ and satisfies (17) then $1/2 < \xi \leqslant \xi_3$. Therefore ξ must also satisfy

$$1 \geqslant \xi + \xi^{n-1} \tag{18}$$

with equality only if n=3 and

$$1/2 < \xi \leqslant \xi_3 = \frac{\sqrt{5}-1}{2} < 2/3 \tag{19}$$

From (19) it is seen that $\triangle_1 \cap \triangle_2 \cap \triangle_3 = \emptyset$ and $\triangle_i \cap \triangle_j \neq \emptyset$ for $i \neq j$. Notice that by (17), $\triangle_i \cap \triangle_j$ is an equilateral triangle of side ξ^n. From the construction we get

$$\triangle_i \cap \triangle_j = \triangle_{\underbrace{ij\ldots j}_{n}} = \triangle_{\underbrace{ji\ldots i}_{n}} \tag{20}$$

and from this it follows that

$$\psi_{\underbrace{ij\ldots j}_{n}}(x) = \psi_{\underbrace{ji\ldots i}_{n}}(x) \tag{21}$$

Fig. 4 a) shows K for $\xi = \xi_3$. Our aim is lemma 4 below which will be used to prove the mentioned properties of K. For the following lemma it is useful to look at figures 3, 2 a), b), c).

Lemma 3. *For any index I beginning with 1 of length $q = m(n-1)+1$, $m \geqslant 1$, such that $\triangle_I \cap \triangle_{\underbrace{12\ldots 2}_{n}} \neq \emptyset$, we have a) or b) or c):*

a) if $\triangle_I \subset \triangle_{\underbrace{12\ldots2}_{n}}$ then there exists $J = \underbrace{12\ldots2}_{n} j_{n+1}\cdots j_q$ such that $\psi_I(x) = \psi_J(x)$.

b) if $\triangle_I \not\subseteq \triangle_{\underbrace{12\ldots2}_{n}}$ and $\triangle_I \cap \triangle_{\underbrace{12\ldots2}_{n}}$ is not a point then there exists $J = \underbrace{12\ldots2}_{n} j_{n+1}\cdots j_q$ such that one side of $\triangle_I$ and one side of $\triangle_J$ are on the same line and $\triangle_I \cap \triangle_J = \triangle_I \cap \triangle_{\underbrace{12\ldots2}_{n}}$ is an equilateral triangle of side of length $\xi^{(m+1)(n-1)+1}$ (Observe that there are only two possibilities, see fig.2 a, b)

c) $\triangle_I \cap \triangle_{\underbrace{12\ldots2}_{n}}$ is a point, fig. 2 c).

The same holds interchanging 1 with 2.

Only a) will be used but b) and c) are needed in the proof.

Proof.

The proposition is true if $m = 1$ because the only sets $\triangle_I$ with $|I| = n$, I beginning with 1 that can touch $\triangle_{\underbrace{12\ldots2}_{n}}$ are, by (18), $\triangle_{12\ldots22}$, $\triangle_{12\ldots21}$, $\triangle_{12\ldots23}$ and in case n=3 also $\triangle_{112}$, $\triangle_{132}$ (fig. 3). The last two satisfy c). The first one satisfies a) and the others satisfy b) with $J = \underbrace{12\ldots2}_{n}$.

Assume that the lemma is true for m. Take an index I beginning with 1 and of lenght $(m+1)(n-1)+1$ with $\triangle_I$ touching $\triangle_{\underbrace{12\ldots2}_{n}}$. Then $\triangle_I \subset \triangle_{I'}$ where $I'I'' = I$, I' of lenght m(n-1)+1 (I' is a curtailment of I). For m the lemma was assumed, therefore if $\triangle_{I'} \subset \triangle_{\underbrace{12\ldots2}_{n}}$ then there exists $J' = \underbrace{12\ldots2}_{n} j_{n+1}\cdots j_{m(n-1)+1}$ such that $\psi_{J'}(x) = \psi_{I'}(x)$ and therefore $\psi_{J'I''}(x) = \psi_{I'I''}(x) = \psi_I(x)$.

Now, assume $\triangle_{I'} \not\subseteq \triangle_{\underbrace{12\ldots2}_{n}}$. If c) is true for I' then $\triangle_I \cap \triangle_{\underbrace{12\ldots2}_{n}}$ is at most a point. If b) is true for I' then there exists J' with the mentioned properties. We assume that $\triangle_{I'}$ and $\triangle_{J'}$ are located as in fig. 2 a. As $\triangle_{I'} \cap \triangle_{J'}$ is an equilateral triangle of side $\xi^{(m+1)(n-1)+1}$ we get (by 17) that $\triangle_{I'} \cap \triangle_{J'} = \triangle_{I'} \cap \triangle_{\underbrace{12\ldots2}_{n}} = \triangle_{I'\underbrace{2\ldots2}_{n-1}} = \triangle_{J'\underbrace{3\ldots3}_{n-1}}$.

Therefore $\triangle_{I'I''} = \triangle_I$ must touch $\triangle_{I'\underbrace{2\ldots2}_{n-1}}$. But the only sets $\triangle_{I'L}$, $|I'L| = (m+1)(n-1)+1$ that could touch $\triangle_{I'\underbrace{2\ldots2}_{n-1}}$ are (by 18) $\triangle_{I'2\ldots22}$, $\triangle_{I'2\ldots23}$, $\triangle_{I'2\ldots21}$, $\triangle_{I'2\ldots212}$, $\triangle_{I'2\ldots232}$. Therefore I must be one of the above. If I is the first then as $\triangle_{I'\underbrace{2\ldots2}_{n-1}} = \triangle_{J'\underbrace{3\ldots3}_{n-1}}$ we get $\psi_{I'2\ldots2}(x) = \psi_{J'3\ldots3}(x)$ and a) follows. If I is one of the last two then c) is true and if (for example) $I'\underbrace{2\ldots 23}_{n-1} = I$ then it is seen that b) is true with $J = J'\underbrace{3\ldots 3}_{n-1}$. ■

Lemma 4. (a) $\triangle_1 \cap \triangle_2 \cap K = \psi_{\underbrace{12\ldots 2}_{n}}(K) = \psi_{\underbrace{21\ldots 1}_{n}}(K)$

(b) $\triangle_1 \cap \triangle_3 \cap K = \psi_{\underbrace{13\ldots 3}_{n}}(K) = \psi_{\underbrace{31\ldots 1}_{n}}(K)$

(c) $\triangle_2 \cap \triangle_3 \cap K = \psi_{\underbrace{23\ldots 3}_{n}}(K) = \psi_{\underbrace{32\ldots 2}_{n}}(K)$

Proof.

We prove the first proposition, the others follow from symmetry. Obviously $\psi_{\underbrace{12\ldots 2}_{n}}(K) \subset \triangle_1 \cap \triangle_2 \cap K$. Let $p \in (\mathrm{int}(\triangle_1 \cap \triangle_2)) \cap K$, then there exists an index I beginning with 1 (or 2) of lenght m(n-1)+1 with m great enough such that $p \in \psi_I(K)$ and $p \in \triangle_I \subset \triangle_1 \cap \triangle_2$. Recalling (20), by Lemma 3 a) we get $p \in \psi_{\underbrace{12\ldots 2}_{n}}(K)$ (or $p \in \psi_{\underbrace{21\ldots 1}_{n}}(K)$ respectively). Therefore using (21) we get a).

If $p \in \partial(\triangle_1 \cap \triangle_2)$ then obvioulsy $p \in \psi_{\underbrace{12\ldots 2}_{n}}(K)$. ■

Lemma 5. *K has property A.*

Proof.

Let $q_1 = \psi_{\underbrace{12\ldots 2}_{n}}(p_3)$ and $q_2 = \psi_{\underbrace{12\ldots 2}_{n}\underbrace{1\ldots 1}_{n-1}}(p_3)$ (see fig.3). We want to show that the 'shape' of K near q_1 is the same as near q_2. Notice that from (18) the only sets $\triangle_I$, $|I| = n$ that touch q_1 are $\triangle_{12\ldots 2} = \triangle_{21\ldots 1}$,$\triangle_{12\ldots 23}$,$\triangle_{21\ldots 13}$ and eventually $\triangle_{132}$ and $\triangle_{231}$ if n=3. We assume for simplicity that $n > 3$ and let the reader fill the gaps if n=3. Therefore, if ε is small enough, from $K = \cup\{\psi_I(K) : |I| = n\}$ we get,

(22)
$$K \cap B_\varepsilon(q_1) = (\psi_{\underbrace{12\ldots 2}_{n}}(K) \cup \psi_{\underbrace{12\ldots 23}_{n}}(K) \cup \psi_{\underbrace{21\ldots 13}_{n}}(K)) \cap B_\varepsilon(q_1)$$

By lemma 4 c) if ε is small enough we get $\psi_2(K) \cap B_{\varepsilon\xi^{1-n}}(\psi_2(p_3)) \subset \psi_{\underbrace{32\ldots 2}_{n}}(K)$ and applying $\psi_{\underbrace{12\ldots 2}_{n-1}}(x)$ to this last relation we get $\psi_{\underbrace{12\ldots 22}_{n}}(K) \cap B_\varepsilon(q_1) \subset \psi_{\underbrace{12\ldots 23}_{n}\underbrace{2\ldots 2}_{n-1}}(K) \subset \psi_{\underbrace{12\ldots 23}_{n}}(K)$. Applying this to (22) we get

$$K \cap B_\varepsilon(q_1) = (\psi_{\underbrace{12\ldots 23}_{n}}(K) \cup \psi_{\underbrace{21\ldots 13}_{n}}(K)) \cap B_\varepsilon(q_1) \tag{23}$$

Also q_2 only touches $\triangle_{12\ldots 2} = \triangle_{21\ldots 1}$ and $\triangle_{12\ldots 21}$ of sets $\triangle_I$, $|I| = n$. Therefore $K \cap B_\varepsilon(q_2) = (\psi_{\underbrace{12\ldots 22}_n}(K) \cup \psi_{\underbrace{12\ldots 21}_n}(K)) \cap B_\varepsilon(q_2)$. Using this last formula, (23) and the fact that $\psi_{\underbrace{12\ldots 23}_n}(x) + (q_2 - q_1) = \psi_{\underbrace{12\ldots 21}_n}(x)$; $\psi_{\underbrace{21\ldots 13}_n}(x) + (q_2 - q_1) = \psi_{\underbrace{12\ldots 22}_n}(x)$ we get that there exists $\varepsilon > 0$ such that

(24) $$K \cap B_\varepsilon(q_2) = (K \cap B_\varepsilon(q_1)) + (q_2 - q_1) \text{ and } B_\varepsilon(q_2) \subset \triangle_1$$

Using this last assertion one can prove that K has property A as follows.

Let $a << \varepsilon$, a will be the parameter of property A. We assume that $B_a(x)$ is a ball touching $\triangle_1 \cap \triangle_2$ (if $B_a(x)$ touches only $\triangle_1$ but neither $\triangle_2$ nor $\triangle_3$ then take $i_0 = 1$, $\psi =$ identity). Assume $B_a(x) \subset B_\varepsilon(q_1)$ then use (24) , lemma 4 a) and take $\psi(x) = x + (q_2 - q_1)$, $i_0 = 1$ in property A to get $\psi(B_a(x) \cap K) \subset \psi_1(K)$.

If $B_a(x) \not\subset B_\varepsilon(q_1)$ then use lemma 4 a) $\psi = identity$ and $i_0 = 1$ or 2, depending where x is placed.■

Because of proposition 2, K is an s-set, s the Hausdorff dimension of K.

Next we calculate s. Apply lemma 2 to get that s must satisfy

(25) $$\xi^s - \xi^{ns} = 1/3$$

Then, $z = \xi^s$ is a root of $1/3 = z - z^n$, $0 < z < 1$, and $s = \frac{\log z}{\log \xi}$. This last polynomial has two roots r_1 , r_2, in (0,1): $0 < r_1 < \xi_3 < r_2 < 1$. Since K contains a segment, we have $s \geqslant 1$. From (17) and (25) we get for $s = 1$, $\xi = 2/3$, which is in contradiction with (19). Then $s > 1$. From $z = \xi^s$ and (19) it follows that $z = r_1$ and $s < 2$.

K is indeed ε-discrete for any $\varepsilon > 0$. We prove this fact for $n = 3$ and a similar argument works for the other cases.

Fig.4 a) shows a decomposition of K in closed sets K_i, $i = 1, \ldots, 4$; $\mathcal{H}^s(K_i \cap K_j) = \emptyset$ for $i \neq j$; $K = K_1 \cup K_2 \cup K_3 \cup K_4$. From lemma 4 it follows that $K_1 = \psi_2(K)$, $K_3 = \psi_{11}(K)$, $K_2 = \psi_{33}(K)$ and therefore $\mathcal{H}^s(K_1) = \xi_3^s \mathcal{H}^s(K)$; $\mathcal{H}^s(K_2) = \mathcal{H}^s(K_3) = \xi_3^{2s}\mathcal{H}^s(K)$ and

(26) $$\mathcal{H}^s(K_4) = (1 - \xi_3^s - 2\xi_3^{2s})\mathcal{H}^s(K)$$

Fig. 4 b) shows a decomposition of the set K_4 in closed sets K_5, K_6, K_7; $K_4 = K_5 \cup K_6 \cup K_7$; $\mathcal{H}^s(K_i \cap K_j) = 0$, $i \neq j$ with K_5, K_6 similar to K_4 and K_7 similar to K ($K_7 = \psi_{311}(K)$). Moreover, using lemma 4 and symmetry, $\mathcal{H}^s(K_5) = \mathcal{H}^s(K_6) = \xi_3^s \mathcal{H}^s(K_4) = by\ (26) = (\xi_3^s - \xi_3^{2s} - 2\xi_3^{3s})\mathcal{H}^s(K)$; $\mathcal{H}^s(K_7) = \xi_3^{3s}\mathcal{H}^s(K)$. From this follows that K is ε-discrete because one can apply the decomposition of figures 4 a) and 4 b) again and again to the smaller pieces which are similar to K and K_4.

Fig. 5 shows a variant of our example where the ratios of the contractions are not equal: $\xi_1 = {\xi_2}^2, \xi_2$ a root of $x + x^2 - x^5 - 1 = 0$ (this forces $\psi_{1222}(x) = \psi_{211}(x)$) and ξ_3 such that Δ_3 does not intersect Δ_1 nor Δ_2. In the case of the figure $\xi_3 = 1/4, \xi_2 \simeq 0,682\ldots, \xi_1 \simeq 0,465\ldots$.

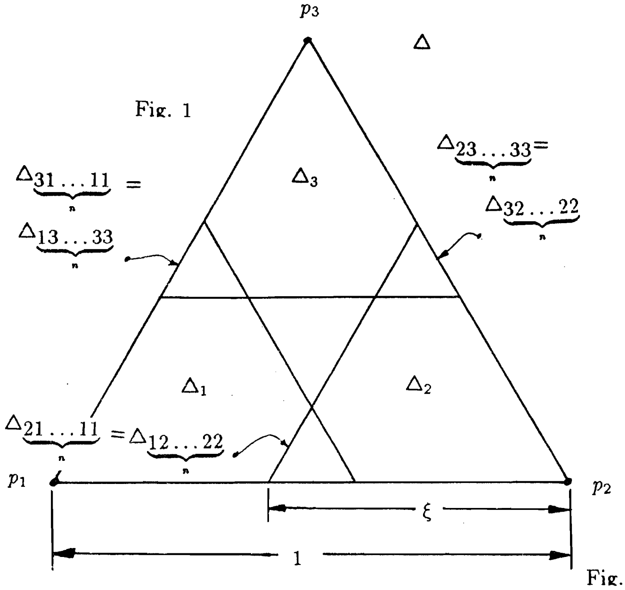


Fig. 1

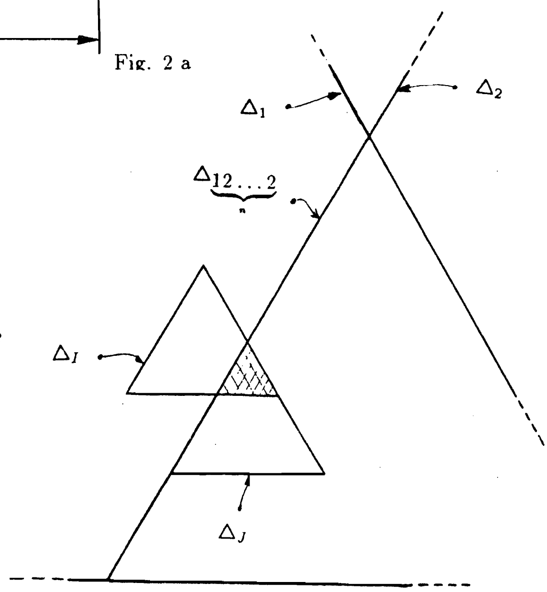


Fig. 2 a

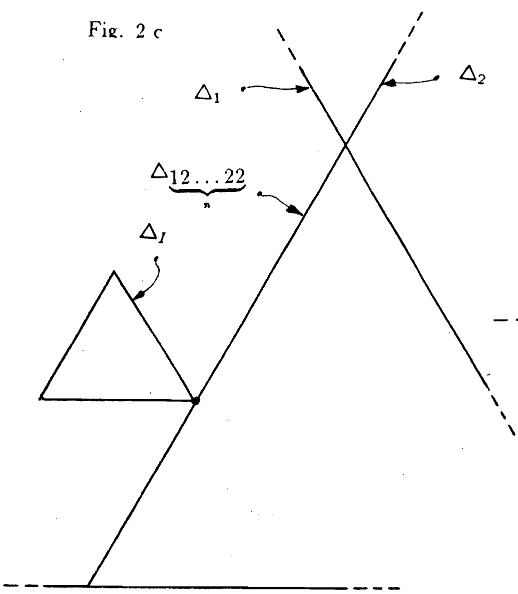


Fig. 2 c

Fig. 2 b

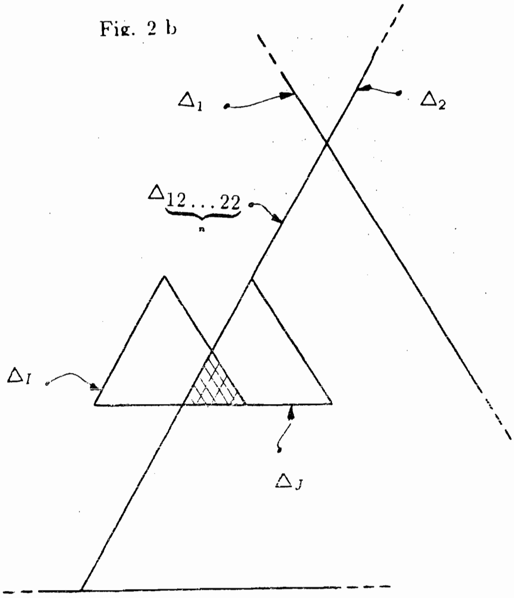
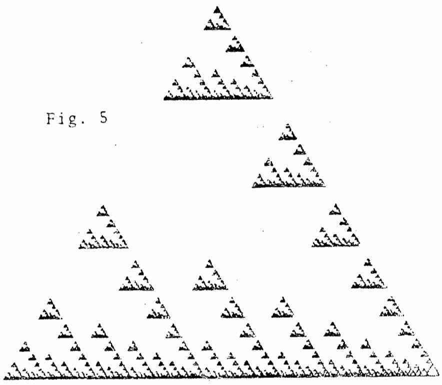


Fig. 5



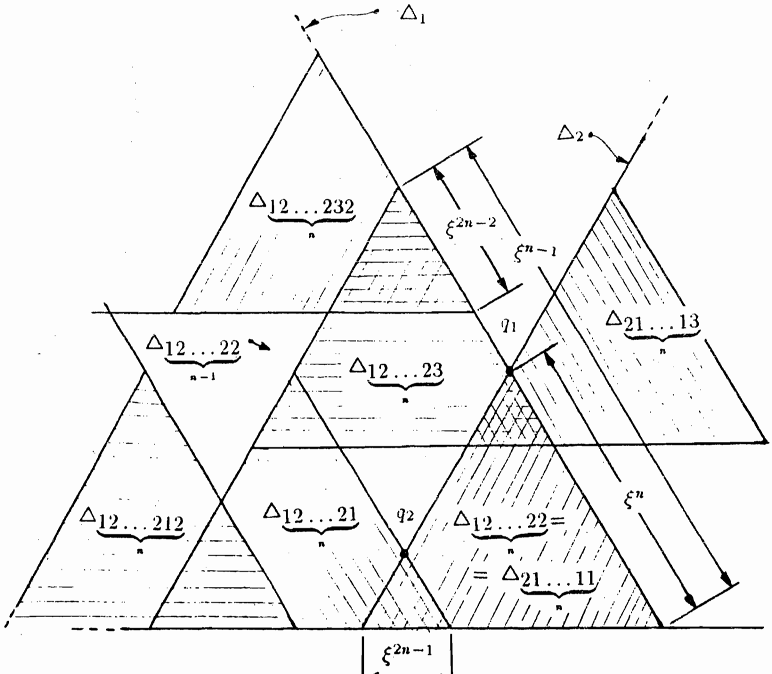


Fig. 3

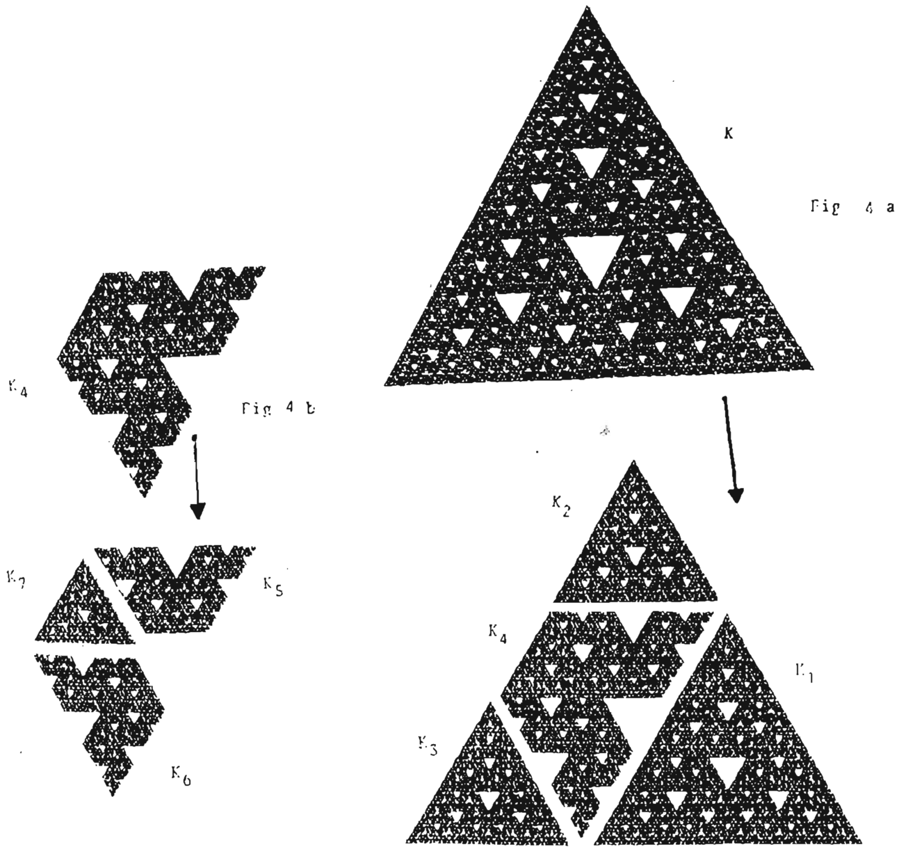


Fig 4 a

Fig 4 b

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Recibido en Diciembre 1995