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THE α-CONCENTRATION OF PROCACCIA OF INFINITE WORDS IN FINITELY GENERATED FUCHSIAN GROUPS.

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ABSTRACT. In order to study the spectral decomposition $(\alpha, f(\alpha))$ of Procaccia of the limit set L(G) of a finitely generated Fuchsian group G of rigid movements in the hyperbolic half plane $\mathbb{H}$, it is necessary to calculate the α of each element of L(G). Each such element is an allowed infinite word, each letter a generator of G. In this paper we calculate first the α of the periodic infinite words, and use this result in order to calculate the α of the non-periodic irrational words.

SECTION 1. INTRODUCTION.

In 1993, a method [1] was proposed to generate fractals Ω such that their multifractal decomposition $(\alpha, f(\alpha))$ of Procaccia modelled all $(\alpha, f(\alpha))$ curves in the Tel classification [2].

The importance of the curves $(\alpha, f(\alpha))$ in the Tel classification and their relevance to the study of a variety of physical phenomena is described in [1]. The fractal sets Ω generated in [1] are the limit sets Ω=L(G) of minimally generated groups G, all generators being rigid movements in $\mathbb{H}$ and having zero trace.

The importance of expressing the elements of Ω=L(G) by means of an infinite word code —each letter a generator of G— is reviewed in [3].

Let us deal then with the α – *concentration* of Procaccia of infinite words coding for elements in Ω=L(G), when G is minimally generated by zero-trace generators (three generators).

Generators A, B, and C have zero trace; then no two letters can be repeated in an allowed word, i.e. a word with correct spelling. Words $W_1 = ABABAB...$ and $W_2 = ABCABC...$ are allowed words denoting two different points in the fractal Ω=L(G), whereas word AABBAABBAABB... denotes no point in L(G), and does not have a correct spelling.

The transformations S=AB and T=ABC have $|trace| > 2$, i.e. they are hyperbolic transformations. Therefore words W_1 and W_2 can be written as infinite words $W_1 = SSSS...$ and $W_2 = TTTT...$ with hyperbolic letters.

This paper deals with the α of infinite words written with hyperbolic letters; specifically, we will calculate the α of infinite words in L(G), here G is a group generated by two hyperbolic operators: two rigid movements in $\mathbb{H}$.

The results can be easily extended to groups with any finite member of generators.

SECTION 2. CONSTRUCTION OF THE LIMIT SET $\mathbb{F}$ OF A FUCHSIAN SEMIGROUP GENERATED BY TWO HYPERBOLIC 2×2 MATRICES.

SECTION 2.1. GENERALITIES AND NOTATION.

Let $T(z) = \dfrac{az+b}{cz+d}$ be an element of the unimodular group U, i.e. $a, b, c,$ and d are integers, and $ad - bc = 1$. The transformation T(z) operates on the values

$$z \in \mathbb{H} = \{x + iy/y > o\}, \quad T : \mathbb{H} \to \mathbb{H}$$

Let us recall that the set $\{z \in \mathbb{H}/|cz + d| \leq 1\} = \{z/|T'(z)| \geq 1\}$ is the isometric circle of $T = T(z)$. With C_T, g_T, and r_T we will denote the isometric circle of T, its centre, and its radius, respectively.

We have $g = \dfrac{-d}{c}$ and $r = \dfrac{1}{|c|}$.

Let us also recall that every hyperbolic $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (i.e. $|traceT| = |a+d| > 2$) has two real fixed numbers, one an attractor, the other a repeller. The repeller belongs to C_T, and the attractor, hereafter denoted as ξ_T, is always inside $C_{T^{-1}}$. Let us recall that if A is hyperbolic then $\mathbb{H} - C_A$ is mapped, by A, onto $Int.C_{A^{-1}}$, and that ∂C_A is mapped onto $\partial C_{A^{-1}}$.

From now on, A and B will be hyperbolic elements of U such that $C_A, C_{A^{-1}}, C_B$ and $C_{B^{-1}}$, are disjoint (see Fig.1)

Let $S(A, B)$ denote the semigroup generated by A and B. Let $x \in \mathbb{H} - (C_A \cup C_B)$. Let $\mathbb{F}(x)$ denote the limit set of $\{T(x)/T \in S(A, B)\}$. It is not hard to prove that, if $y \in \mathbb{H} - (C_A \cup C_B)$, $y \neq x$, we have $\mathbb{F}(x) = \mathbb{F}(y)$. Hence, with $\mathbb{F}$ we will denote $\mathbb{F}(x)$ (for any x in $\mathbb{H}$), and we will call it the limit set of $S(A, B)$.

SECTION 2.2.

Let us now construct a fractal F associated with $S(A, B)$. We will construct it in stages, following an iterative process similar to the one that yields the Cantor ternary. Let us write

$$R = \{x \in \mathbb{R}/x \notin C_A\} \text{ and } S = \{x \in \mathbb{R}/x \notin C_B\}$$

STEP 1. We have, then, $cl.A(R) = C_{A^{-1}} \cap \mathbb{R}$ and $cl.B(S) = C_{B^{-1}} \cap \mathbb{R}$. These two segments, disjoint by our assumption on the transformations A and B, will be the analogue of the two segments $[0,1/3]$ and $[2/3,1]$ which constitute the first step in the construction of the Cantor ternary. *Par abus de langage*, and only when there is no danger of confusion, we will denote with the letters A and B (the same letters that denote the hyperbolic generators), these two sets A(R) and B(S), which are the two segments of the first step; see Fig.2.

STEP 2. In strict analogy to the construction of the Cantor set, we continue with the second step of our iterative process, as shown in Fig.3.

STEP 3. The third step is shown in Fig 4.

...and so on ad infinitum. The fractal F is obtained like the Cantor ternary, i.e. it is the intersection of all these steps.

Note. Hereafter, with a word of two letters A and B, of length N, we will refer indistinctly to the corresponding transformation in $S(A,B)$, and to the corresponding segment in step N in the construction of F just described. **Notice that F is well constructed**: all segments in step N are disjoint and contained in some segment in step N-1:

They are disjoint, since $C_{A^{-1}} \cap C_{B^{-1}} = \emptyset$ by the hypothesis, and since both A and B are one-to-one.

They are contained in some segment in step N-1: let us prove, e.g., that segment ABA is contained in segment AB:

$$ABA = AB[A(R)] = AB(C_{A^{-1}} \cap \mathbb{R}) \subset AB(S) = A(C_{B^{-1}} \cap \mathbb{R}) = AB$$

The same reasoning holds for every case, as we only use $C_{B^{-1}} \cap \mathbb{R} \subset R$ and $C_{A^{-1}} \cap \mathbb{R} \subset S$.

Thus, the 2^N disjoint segments in step N are a covering of F.

SECTION 2.3.

We will prove now that $\mathbb{F} = \mathrm{F}$.

1) $\mathbb{F} \subset \mathrm{F}$. The proof is quite easy: Let us first notice that we can associate a semicircle to each segment in any step N of the construction of F, as shown in Fig.5.

Par un tres grand abus de langage indeed, we will denote, with a word of N letters $A\&B$, **three** things now: the corresponding transformation, the corresponding segment in the step N of the construction of F, and the corresponding associated semicircle, and we will make sure that there will be no danger of confusion.

Let us now consider $\xi \in \mathbb{F}$. ξ is, then, a point in $\mathbb{R}$, approximated by elements of a convergent sequence $\{T_N(x)\}_{N\in\mathbb{N}}$, where T_N is a transformation of N letters A and B, and x is, as before, in $\mathbb{H} - (\mathrm{C_A} \cup \mathrm{C_B})$. The reader can infer that ξ is in F by pondering on the following facts:

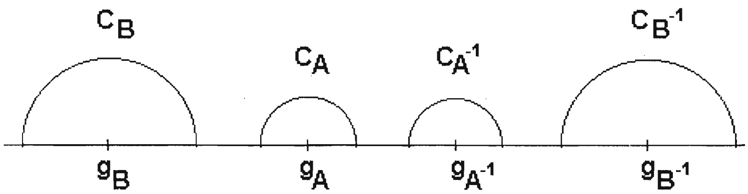

Fig.1

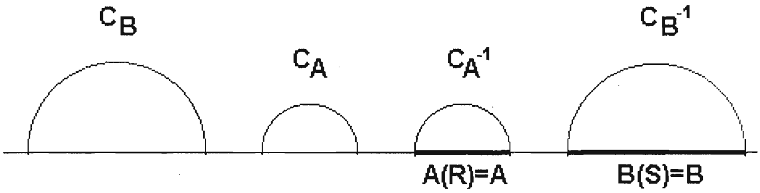

Fig.2

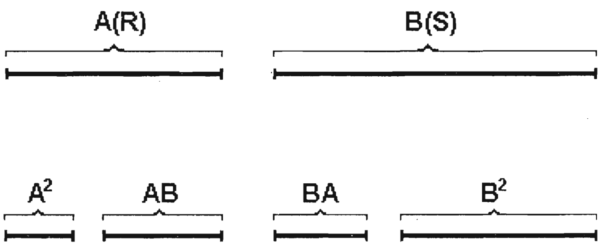

Fig.3

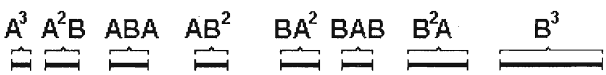

Fig.4

a) $\xi \in \mathbb{R}$,

b) $T_N(x) \longrightarrow \xi$ as $N \longrightarrow \infty$,

c) $T_N(x)$ belong to smaller and smaller semicircles T_N, like the ones in Fig.5, which have to be —for big values of N— one inside the other, due to the convergence of $\{T_N(x)\}$, $N \in \mathbb{N}$.

d) The closeness of the segments in step N of the construction of F, and the inclusion of the boundary of the semicircles referred to in c) completes what we need to prove that $\xi \in F$.

2) $F \subset \mathbb{F}$ is an easy excercise, left to the reader.

SECTION 3. THE INFINITE WORDS IN F AND THEIR α-CONCENTRATION OF PROCACCIA.

SECTION 3.1. INFINITE WORDS.

Let us recall that the finite words of length N made up of two letters A and B are a covering of F by disjoint closed segments; with C_N we will denote this covering. Each $\xi \in F$ will belong to just one such segment $I_N(\xi)$ in C_N. For growing values of N, there is a unique sequence of such intervals of decreasing size, one inside the other, associated with a growing-in-length word in letters A and B. Therefore, ξ is represented by a unique infinite word.

Such an infinite word in two letters can have a structure analogous to that of a rational number written in a binary way, that is, it can have a period, indefinitely repeated, preceded by a finite number of letters which do not necessarily show a periodic arrangement. When such is the case, we will say that ξ is represented by a "rational word".

Observation: if the finite word T is the period of a rational infinite word ξ, then ξ, as a point, is the fixed point atractor ξ_T of the corresponding transformation T.

Lemma 1: *The set of rational word points in F is dense in F with the usual topology of $\mathbb{R}$. This density is also valid if the topology of $\mathbb{R}$ is replaced by the one associated with the Hausdorff measure corresponding to the Hausdorff dimension of the fractal set F.*

The proof is left to the interested reader.

SECTION 3.2. THE α-CONCENTRATION OF PROCACCIA $\alpha(\xi)$ ASSOCIATED WITH A POINT $\xi \in F$.

Following Procaccia, Hensen and others [4], we consider the set F endowed with a probability measure P, and let us recall that the concentration of Procaccia relates lengths of intervals I_N —in the covering by intervals C_N— to the corresponding probabilities $P(I_N \cap F)$ associated with each $F \cap I_N$, in the following way:

$$P(I_N \cap F) = [\mu(I_N)]^{\alpha(I_N)},$$

where μ is the usual measure in $\mathbb{R}^1$.

Hereafter, we will consider all such intervals I_N in C_N as equiprobable, so that $P(I_N \cap F) = \frac{1}{2^N}$ for any of the 2^N intervals in the N^{th} step of the construction of F.

If $\xi \in F$, then there is a unique $I_N = I_N(\xi)$ to which ξ belongs. We will define the "N-aproximated α -- *concentration* of ξ" —abbreviated as $\alpha^N(\xi)$— by the quotient

$$\alpha^N(\xi) = \frac{ln(1/2^N)}{ln(\mu(I_N(\xi)))}$$

We know that [4]

$$\alpha(\xi) = \lim_{N\to\infty} \alpha^N(\xi)$$

when the limit exists.

SECTION 3.3. THE CONCENTRATION $\alpha(\xi)$ OF POINTS ξ ASSOCIATED WITH AN INFINITE RATIONAL WORD.

We will prove

Theorem 1: *Let ξ be a point associated with an infinite rational word, in letters A and B. Let $m \in \mathbb{N}$ be the number of letters in the period of this rational word. Let T be the period itself, a finite word of m letters. Then*

$$\alpha(\xi) = \frac{m\ ln2}{2\ ln|autT|},$$

where autT indicates the largest eigenvalue of T, in absolute value.

Proof: The author has proved this lemma in [3].

SECTION 3.4. THE CONCENTRATION $\alpha(\xi)$ OF POINTS ξ ASSOCIATED WITH ANY INFINITE WORD, RATIONAL OR NOT.

The following theorem expresses the concentration $\alpha(\xi)$, ξ an irrational word, in terms of the $\alpha -$ *concentration* of different rational words.

Theorem 2: *Let $\xi \in F$ and $N \in \mathbb{N}$. Let $I_N(\xi)$ be the only interval in C_N to which ξ belongs. Let T_N be the word of N letters A and B associated with the interval $I_N(\xi)$. Let us consider the corresponding transformation T_N, and let us denote by ξ_N its fixed point (attractor).*

Then we have:

$$\lim_{N\to\infty} \alpha(\xi_N) = \alpha(\xi)$$

Proof: We need a

Lemma: *Under all hypothesis of theorem 2 we have*

$$|\alpha^N(\xi_N) - \alpha(\xi_N)| \longrightarrow 0 \text{ as } N \longrightarrow \infty .$$

Let us suppose the lemma already proved. Let us consider the infinite rational word of period T_N. Since we saw that the corresponding associated point is precisely ξ_N (see the observation in section 3.1), we will think of ξ_N also as an infinite rational word, with a period of N letters.

We will show that

$$\alpha(\xi_N) \longrightarrow \alpha(\xi) \text{ when } N \longrightarrow \infty,$$

that is, the concentration of ξ will be approximated by concentrations of rational words.

Now:

$$|\alpha(\xi_N) - \alpha(\xi)| \leq |\alpha(\xi_N) - \alpha^N(\xi_N)| + |\alpha^N(\xi_N) - \alpha^N(\xi)| + |\alpha^N(\xi) - \alpha(\xi)|.$$

Let $\epsilon > 0$ be arbitrary and fixed. By our lemma, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$ implies

$$|\alpha(\xi_N) - \alpha^N(\xi_N)| < \frac{\epsilon}{2}.$$

Next, we observe that $I^N(\xi) = I^N(\xi_N)$ for every $N \in \mathbb{N}$. Therefore, $\alpha^N(\xi_N) - \alpha^N(\xi) = 0$.

Since, by definition,

$$\alpha(\xi) = \lim_{N\to\infty} \alpha^N(\xi),$$

there exists $N_1 \in \mathbb{N}$ such that $N \geq N_1$ implies

$$|\alpha^N(\xi) - \alpha(\xi)| < \epsilon/2.$$

The theorem is proved.

SECTION 3.5. PROOF OF THE LEMMA IN SECTION 3.4.

We know that

$$|\alpha^N(\xi_N) - \alpha(\xi_N)| = |\frac{ln[1/2^N]}{ln[\mu(I^N(\xi_N))]} - \frac{N ln2}{2 ln|autT_N|}| =$$

$$= N ln2 |\frac{2ln|autT_N| + ln[\mu(I^N(\xi_N))]}{2ln[\mu(I^N(\xi_N))] ln|autT_N|}| = N\frac{ln2}{2}|\frac{ln[|autT_N|^2 \mu(I^N(\xi_N))]}{ln[\mu(I^N(\xi_N))] ln|autT_N|}| \qquad (1)$$

Let us follow the three steps shown below:

Claim I

$$|ln[\mu(I^N(\xi_N))]| \geq ln|\frac{c|p-q|}{L^{2N}\delta^2}|,$$

where c, p, q, δ, and L are constants depending only on A and B, and $L > 1$.

Claim II

$$|ln|autT_N|| \geq ln|L^N \frac{\delta}{c}|,$$

where L, δ, and c are the same constants in Claim I.

Claim III

$$|ln[|autT_N|^2 \mu(I^N(\xi_N))]| \leq K,$$

where K is a constant not depending on N.

Proof of Claim I.

Let us first work with $\mu(I^N(\xi_N))$.

Let p and q be the extremes of the interval $C_{A^{-1}}$ if T_N ends in A; otherwise they are the extremes of the segment $C_{B^{-1}}$.

Let $T_N(z) = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$, where $a_N d_N - b_N c_N = 1$.

Then we have that

$$\mu(I^N(\xi_N)) = |T_N(p) - T_N(q)| = |\frac{a_N p + b_N}{c_N p + d_N} - \frac{a_N q + b_N}{c_N q + d_N}| =$$

$$= \frac{|a_N d_N (p-q) - b_N c_N (p-q)|}{|c_N p + d_N||c_N q + d_N|} = \frac{|p-q|}{|c_N^2||p + \frac{d_N}{c_N}||q + \frac{d_N}{c_N}|} \tag{2}$$

a) Let us deal next with $|p + \frac{d_N}{c_N}|$ and $|q + \frac{d_N}{c_N}|$. In order to fix ideas let us suppose that T_N ends in B. We can write

$$|p + \frac{d_N}{c_N}| = |p - (-\frac{d_N}{c_N})|,$$

and let us recall that $-\frac{d_N}{c_N} = g_{T_N}$ is the centre of isometric circle of T_N.

We have that $C_{T_N} \subset C_A$ since T_N ends in B, and therefore $g_{T_N} \in C_A$ —see Fig 6.

Let us define

$$\delta = min\{\text{distances between all extremes of the segments } C_A,\ C_B,\ C_{A^{-1}},\ C_{B^{-1}}\}$$

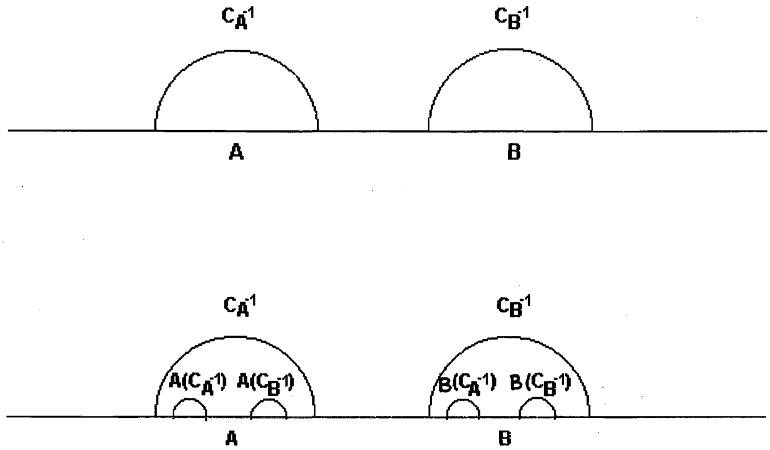
C_A^{-1}
C_B^{-1}
A
B
C_A^{-1}
C_B^{-1}
$A(C_A^{-1})$ $A(C_B^{-1})$
$B(C_A^{-1})$ $B(C_B^{-1})$
A
B

Fig.5

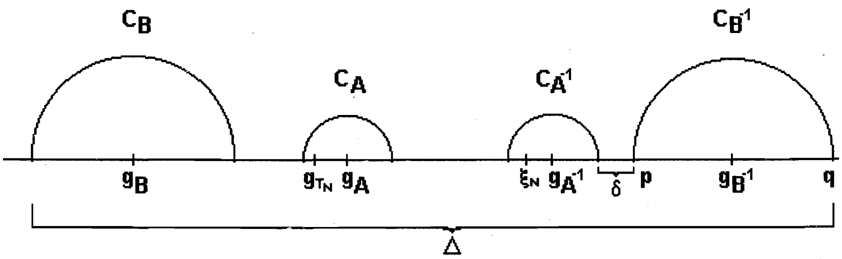
C_B
C_A
C_A^{-1}
C_B^{-1}
g_B
g_{T_N} g_A
ξ_N g_A^{-1}
δ
p
g_B^{-1}
q
Δ

Fig.6