Revista de la
Unión Matemática Argentina
Volumen 40, 1996.

A MODIFICATION OF THE ERA AND A DETERMINANTAL APPROACH TO THE STABILITY OF COMPLEX SYSTEMS OF DIFFERENTIAL EQUATIONS

ZIAD ZAHREDDINE

Department of Mathematics, Faculty of Sciences,
U. A. E. University, Al-Ain, P. O. Box: 17551

Abstract

Recently, a new stability criterion for systems of differential equations with complex coefficients has been advanced. It is based on a sequence of polynomials associated with the system. This criterion known as the Extended Routh Array (ERA) suffers the defect that it gets cumbersome and highly complicated as the dimension of the system gets large. In this paper, we propose a modification of the ERA which reduces significantly the burden of computations. The modified array requires only computations of a set of second order determinants. The new algorithm is then applied to produce a determinantal criterion for the stability of the above systems.

A M S Subject Classification: 34E05.

Key Words and Phrases: Stability Criteria, Extended Routh Array, Systems of Differential Equations.

1. Introduction

Tests of stability of systems of differential equations are crucial in many areas of mathematical analysis. In the case of real coefficients, the classical Routh-Hurwitz criterion gives a quite complete solution, among many others see [1,2,3,5]. The case of complex coefficients has recently become an active area of research. Different approaches to this interesting problem are recorded in the literature, see for example [4,6,7,8]. The Extended Routh Array (ERA) introduced in [7] settles the stability of systems with complex coefficients.

However, a large amount of computations will be involved to produce the ERA when the dimension of the system becomes high. Therefore, there is a need to work towards more simplified versions of the ERA. The establishment of simpler and more easily realizable criteria in practice will also further the theoretical development of the subject.

In this paper we address this problem and we propose a modification of the ERA which we call the Modified Extended Routh Array (MERA), where much simpler arithmetic is performed. At each step of the MERA only the calculation of a second order determinant is required. Furthermore, we exploit the MERA towards a new determinantal criterion for the asymptotic stability of a system of differential equations with complex coefficients.

In section 2 we give a quick reminder of the ERA and the way it is constructed. In section 3, we introduce the MERA and we prove that it is in fact another algorithm for testing the stability of complex systems, from which the equivalence of the MERA and the ERA follows. A determinantal approach to the stability problem is introduced in section 4. We end up in section 5 with some concluding remarks.

2. The Extended Routh Array

All the terminology of this section is taken from [7]. Suppose

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-2} z^2 + a_{n-1} z + a_n \tag{1}$$

is the characteristic polynomial of a system of differential equations with complex coefficients and of arbitrary dimension. Consider the rational function:

$$h(z) = \frac{z^n + i\,\mathrm{Im}a_1 z^{n-1} + \mathrm{Re}a_2 z^{n-2} + i\,\mathrm{Im}a_3 z^{n-3} + \mathrm{Re}a_4 z^{n-4} + \ldots..}{\mathrm{Re}a_1 z^{n-1} + i\,\mathrm{Im}a_2 z^{n-2} + \mathrm{Re}a_3 z^{n-3} + i\,\mathrm{Im}a_4 z^{n-4} + \ldots..}$$

The function h(z) is sometimes referred to as the test fraction associated with the system [8].

Let f_1 be the numerator and f_2 the denominator of h. Suppose $\mathrm{Re}a_1 \neq 0$, and call f_3 the remainder of the division of f_1 by f_2. By induction, define the polynomial f_j to be the remainder of the division of f_{j-2} by f_{j-1} for j = 3, ..., n+1. Lemma 4.1 of [7] expresses explicitly the coefficients of f_j in terms of those of f_{j-1} and f_{j-2}. The ERA is the following array in which the j-th row represents the coefficients of f_j for j = 1, 2, 3, ..., n+1 and where each row is completed by zeros to the size of the first row. We assume no zeros exist in the first column:

$$\begin{array}{cccccccccc}
1 & i\,\mathrm{Im}a_1 & \mathrm{Re}a_2 & i\,\mathrm{Im}a_3 & \mathrm{Re}a_4 & i\,\mathrm{Im}a_5 & . & . & . \\
\mathrm{Re}a_1 & i\,\mathrm{Im}a_2 & \mathrm{Re}a_3 & i\,\mathrm{Im}a_4 & \mathrm{Re}a_5 & & . & . & . \\
b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & . & & . & . & . \\
b_{4,1} & b_{4,2} & b_{4,3} & . & . & & . & . & . \\
. & . & . & . & . & & . & . & . \\
. & . & . & . & . & & . & . & . \\
. & . & . & . & . & & . & . & . \\
b_{n,1} & b_{n,2} & 0 & . & . & & . & . & . \\
b_{n+1,1} & 0 & . & . & . & & . & . & .
\end{array}$$

where

$$b_{3,1} = \frac{1}{\mathrm{Re}a_1}(\mathrm{Re}a_1.\mathrm{Re}a_2 - \mathrm{Re}a_3) - \frac{i\,\mathrm{Im}a_2}{(\mathrm{Re}a_1)^2}(i\,\mathrm{Re}a_1.\mathrm{Im}a_1 - i\,\mathrm{Im}a_2)\ ,$$

$$b_{3,2} = \frac{1}{\mathrm{Re}a_1}(i\,\mathrm{Re}a_1.\mathrm{Im}a_3 - i\,\mathrm{Im}a_4) - \frac{\mathrm{Re}a_3}{(\mathrm{Re}a_1)^2}(i\,\mathrm{Re}a_1.\mathrm{Im}a_1 - i\,\mathrm{Im}a_2)\ ,$$

$$b_{4,1} = \frac{1}{b_{3,1}}(b_{3,1}.\mathrm{Re}a_3 - \mathrm{Re}a_1.b_{3,3}) - \frac{b_{3,2}}{b_{3,1}^2}(ib_{3,1}.\mathrm{Im}a_2 - \mathrm{Re}a_1.b_{3,2})\ ,$$

$$b_{4,2} = \frac{1}{b_{3,1}}(ib_{3,1}.\mathrm{Im}a_4 - \mathrm{Re}a_1.b_{3,4}) - \frac{b_{3,3}}{b_{3,1}^2}(ib_{3,1}.\mathrm{Im}a_2 - \mathrm{Re}a_1.b_{3,2})\ ,$$

and so on.

Theorem 4.1 of [7] states the following:

Theorem 1. The system with characteristic polynomial (1) is asymptotically stable if and only if each term of the first column of the ERA is positive, where asymptotic stability is as defined in [7].

3. The Modified Extended Routh Array

Consider the following array in which the first and second row are the same as in the ERA. We call the new array the Modified Extended Routh Array (MERA) for reasons to become clear later. The c's and the d's along with their respective subscripts have been so chosen for technical purposes.

$$\begin{array}{cccccccc}
d_{01} & d_{02} & d_{03} & d_{04} & d_{05} & . & . \\
d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & . & . \\
c_{11} & c_{12} & c_{13} & c_{14} & . & . & . \\
d_{21} & d_{22} & d_{23} & d_{24} & . & . & . \\
c_{21} & c_{22} & c_{23} & . & . & . & . \\
d_{31} & d_{32} & d_{33} & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & .
\end{array}$$

where

$$d_{01} = 1, \quad d_{0k} = \begin{cases} \mathrm{Re}\, a_{k-1} & , \ k \geq 3 \ \ \text{and} \ \ \text{odd} \\ i\,\mathrm{Im}\, a_{k-1} & , \ k \ \ \text{even} \end{cases}, \quad d_{1k} = \begin{cases} \mathrm{Re}\, a_k & , \ k \ \ \text{odd} \\ i\,\mathrm{Im}\, a_k & , \ k \ \ \text{even} \end{cases}$$

and

$$c_{11} = \frac{d_{11}.d_{02} - d_{01}.d_{12}}{d_{11}}, \quad c_{12} = \frac{d_{11}.d_{03} - d_{01}.d_{13}}{d_{11}}, \quad c_{13} = \frac{d_{11}.d_{04} - d_{01}.d_{14}}{d_{11}}, \ldots$$

$$d_{21} = \frac{d_{11}.c_{12} - c_{11}.d_{12}}{d_{11}}, \quad d_{22} = \frac{d_{11}.c_{13} - c_{11}.d_{13}}{d_{11}}, \quad d_{23} = \frac{d_{11}.c_{14} - c_{11}.d_{14}}{d_{11}}, \ldots$$

$$c_{21} = \frac{d_{21}.d_{12} - d_{11}.d_{22}}{d_{21}}, \quad c_{22} = \frac{d_{21}.d_{13} - d_{11}.d_{23}}{d_{21}}, \quad c_{23} = \frac{d_{21}.d_{14} - d_{11}.d_{24}}{d_{21}}, \ldots$$

$$d_{31} = \frac{d_{21}.c_{22} - c_{21}.d_{22}}{d_{21}}, \quad d_{32} = \frac{d_{21}.c_{23} - c_{21}.d_{23}}{d_{21}}, \quad d_{33} = \frac{d_{21}.c_{24} - c_{21}.d_{24}}{d_{21}}, \ldots$$

The following theorem implies the equivalence between the ERA and the MERA.

Theorem 2. The system with characteristic polynomial (1) is asymptotically stable if and only if $d_{k1} > 0$ for all k = 1, ..., n.

Proof. Suppose the system is asymptotically stable, then by [7, theorem 3.2] the test fraction h(z) can be expanded in the following continued fraction expansion:

$$h(z) = a_0 + b_0 z + \cfrac{1}{a_1 + b_1 z + \cfrac{1}{a_2 + b_2 z + \ldots + \cfrac{1}{a_{n-2} + b_{n-2} z + \cfrac{1}{a_{n-1} + b_{n-1} z}}}}$$

where $\mathrm{Re}\, a_k = 0$ and $b_k > 0$ for k = 0, ..., n-1.

The proof of theorem 4.1 of [7] makes it clear how the coefficients b_k in the above expansion relate to the first column of the ERA, namely

$$b_k = \frac{b_{k+1,1}}{b_{k+2,1}},$$

for k = 0, ..., n-1, where we suppose that $b_{1,1} = 1$ and $b_{2,1} = \mathrm{Re}\, a_1$.

The polynomials $f_1, f_2, \ldots, f_{n+1}$ forming the ERA are related by the recurrence relations:

$f_{k+1} = (a_k + b_k z) f_{k+2} + f_{k+3}$, k = 0, ..., n-1 ,

$f_{n+2} = 0$.

These recurrence relations are simply another version of lemma 4.1 of [7]. Upon checking these relations, we see that the terms that arise are exactly those contained in the MERA. Therefore, the following continued fraction expansion arises out of the terms of the MERA,

$$h(z) = c_0 + d_0 z + \cfrac{1}{c_1 + d_1 z + \cfrac{1}{c_2 + d_2 z + \ldots + \cfrac{1}{c_{n-1} + d_{n-1} z}}}$$

where $d_k = \dfrac{d_{k1}}{d_{(k+1)1}}$ for k = 0, 1, ..., n-1.

From the uniqueness of the continued fraction expansion of h(z) [7, section 3], we conclude that $b_k = d_k$ for k = 0, 1, ..., n-1.

We claim that

$$d_{k1} = b_{k+1,1}$$

for all k = 0, 1, ...,n.

It is clear that $b_0 = d_0 = \dfrac{1}{\mathrm{Re}\, a_1}$, and the relation $b_1 = d_1$ leads to $\dfrac{b_{2,1}}{b_{3,1}} = \dfrac{d_{11}}{d_{21}}$.

Since $b_{2,1} = d_{11} = \mathrm{Re}\, a_1$, we conclude that $d_{21} = b_{3,1}$.

By induction suppose that $d_{(k-1)1} = b_{k,1}$ for some k, $3 \le k \le n$, then

$$d_{k-1} = \frac{d_{(k-1)1}}{d_{k1}} \text{ and } b_{k-1} = \frac{b_{k,1}}{b_{k+1,1}}.$$

By combining the relations $d_{k-1} = b_{k-1}$ and $d_{(k-1)1} = b_{k,1}$ we get $d_{k1} = b_{k+1,1}$ which proves our claim.

Since $b_{k+1,1} > 0$ for all k = 1, ..., n we conclude that $d_{k1} > 0$ for k = 1, ..., n.

4. Determinantal approach

In this section we exploit the results of section 3 to advance a new determinant-type algorithm for the stability of the systems described above.

Theorem 3. The system with characteristic polynomial (1) is asymptotically stable if and only if

$$(-1)^{k(k-1)/2} \Delta_k > 0$$

for k = 1, ..., n and where $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the first n principal minors indicated in the arrangement

$$\begin{array}{c|cc|cc|cc}
\mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \mathrm{Re}\,a_5 & i\,\mathrm{Im}\,a_6 & \cdot\ \cdot \\
\cline{1-1}
1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \mathrm{Re}\,a_4 & i\,\mathrm{Im}\,a_5 & \cdot\ \cdot \\
0 & \mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \mathrm{Re}\,a_5 & \cdot\ \cdot \\
\cline{1-3}
0 & 1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \mathrm{Re}\,a_4 & \cdot\ \cdot \\
0 & 0 & \mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \cdot\ \cdot \\
\cline{1-5}
0 & 0 & 1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot
\end{array}$$

where each row is completed by zeros to the size of the first row.

Proof. Suppose the system is asymptotically stable, then by theorem 2 $d_{k1} > 0$ for k = 0, 1, ..., n.

Consider the determinant Δ_k of order 2k-1 for $2 \le k \le n$. Subtract $1/\mathrm{Re}\,a_1$ times the (2j-1)-st row from the 2j-th row, for j = 1, 2, ..., k-1, and with the use of the MERA, we find that

$$\Delta_k = d_{11}.\det\begin{bmatrix} c_{11} & c_{12} & c_{13} & . & . \\ d_{11} & d_{12} & d_{13} & . & . \\ 0 & c_{11} & c_{12} & . & . \\ 0 & d_{11} & d_{12} & . & . \\ 0 & 0 & c_{11} & . & . \\ . & . & . & . & . \end{bmatrix}$$

where obviously $d_{11} = \text{Re}a_1$, $d_{12} = i\,\text{Im}a_2$, $d_{13} = \text{Re}a_3$ and so on. Clearly the new determinant is of order 2k-2.

Now subtract c_{11}/d_{11} times the 2j-th row from the (2j-1)-st row for j = 1, 2,..., k-1 and again with the help of the MERA, we obtain

$$\Delta_k = (-1)^{k-1} d_{11}^2 \Delta_{k-1}^{(1)}$$

for k = 2, 3, ..., n, where $\Delta_r^{(j)}$ denotes the determinant Δ_r with both the subscripts of all its elements increased by j. From the last relation we find immediately that

$$\Delta_k = (-1)^{k(k-1)/2} d_{11}^2 d_{21}^2 \ldots d_{(k-1)1}^2 d_{k1}$$

or

$$(-1)^{k(k-1)/2} \Delta_k = d_{11}^2 d_{21}^2 \ldots d_{(k-1)1}^2 d_{k1}$$

for k = 2, 3, ..., n. From theorem 2 it follows that $d_{k1} > 0$ for k = 0, 1, ..., n, therefore

$$(-1)^{k(k-1)/2} \Delta_k > 0$$

for k = 1, ..., n.

Conversely, suppose that $(-1)^{k(k-1)/2} \Delta_k > 0$ for k = 1, ..., n. If k = 1, then $\Delta_1 = d_{11} > 0$. In the relation $(-1)^{k(k-1)/2} \Delta_k = d_{11}^2 d_{21}^2 \ldots d_{(k-1)1}^2 d_{k1}$ for k = 2, 3, ..., n, put k = 2, then $d_{21} > 0$.

Continuing by induction we get $d_{k1} > 0$ for k = 1, ..., n, and by theorem 2 the system is asymptotically stable and that completes the proof.

5. Concluding Remarks

The complexity of computation in the ERA stability test has been reduced by exploiting special features of the continued fraction expansion of the test fraction associated with the system. With the introduction of the MERA, this paper contributes to ongoing research to finding algorithms which are computationally attractive, numerically simple and accurate for assessing the stability of a system of differential equations. However, the search for tests with reduced computational efforts is still continuing. We have also applied the new MERA to obtain a determinantal criterion for the stability of the system.

REFERENCES

1. **Gantmacher F. R.**, The Theory of Matrices, Chelsea Pub. Co., New York, 1960.

2. **Hurwitz A.**, On The Conditions Under Which an Equation Has Only Roots With Negative Real Parts, Selected Papers on Mathematical Trends in Control Theory, Edited by R. Bellman and R. Kalaba, Dover, New York, 1964.

3. **Krein M. G.** and **Naimark M. A.**, The Method of Symmetric and Hermitian Forms in The Separation of Roots of Algebraic Equations, Linear and Multilinear Alg., vol.10, 265-308, 1981.

4. **Levinson N.** and **Redheffer R. M.**, Complex Variables, McGraw-Hill Publishing Company Limited, 1980.

5. **Marden M.** Geometry of Polynomials, Amer. Math. Soc., 2nd Edition, 1966.

6. **Zahreddine Z.**, Explicit Relationships Between Routh-Hurwitz and Schur-Cohn Types of Stability, Irish Math. Soc. Bull., #29, 49-54, 1992.

7. **Zahreddine Z.**, An Extension of The Routh Array For The Asymptotic Stability of a System of Differential Equations With Complex Coefficients, Applicable Analysis, vol.49, 61-72, 1993.

8. **Zahreddine Z.**, Wide Sense Stability of Complex Systems of Differential Equations of Arbitrary Dimension, Chin. Ann. of Math., 15B:2, 193-204, 1994.

Recibido en Diciembre 1995