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SL(2,R)-MODULE STRUCTURE OF THE EIGENSPACES OF THE CASIMIR OPERATOR

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ABSTRACT. In this paper, on the space of smooth sections of a $SL(2,R)$–homogeneous vector bundle over the upper half plane we study the $SL(2,R)$ structure for the eigenspaces of the Casimir operator. That is, we determine its Jordan-Hölder sequence and the socle filtration. We compute a suitable generalized principal series having as a quotient a given eigenspace. We also give an integral equation which characterizes the elements of a given eigenspace. Finally, we study the eigenspaces of twisted Dirac operators.

§1. *Introduction*

Let $G = SL(2,\mathbf{R})$ and K be a fixed maximal compact subgroup K of G. Let (τ, V) be a representation of K, we denote

$$C^\infty(G/K,V) = \{ f : G \to V \ / f \text{ is } C^\infty \text{ and } f(gk) = \tau(k)^{-1} f(g) \quad \text{for all } k \in K \}$$
$$L^2(G/K,V) = \{ f : G \to V \ / f(gk) = \tau(k)^{-1} f(g) \quad \text{for all } k \in K,\ \|f\|_2^2 < \infty \}$$

where $\| \ \|_2$ is computed with respect to Haar measure. On $L^2(G/K,V)$ we fix the obvious topology. On $C^\infty(G/K,V)$ we fix the topology of uniform convergence on compacts of the functions and their derivatives. Both spaces are representations of G under the left regular action $L_g f(x) = f(g^{-1}x)$ for all $g, x \in G$.

Let Ω the Casimir element of the universal algebra $\mathcal{U}(g_o)$ of the Lie algebra g_o of G, Ω define a G-left invariant operator on $C^\infty(G/K,V)$. Here, we obtain the G-module structure of each eigenspace of the Casimir operator

$$\Omega : C^\infty(G/K,V) \quad \to \quad C^\infty(G/K,V)$$

whenever V is an irreducible representation of K. Actually, we prove that whenever an eigenspace is irreducible, then it is infinitesimally equivalent to a principal series representation, and when an eigenspace is reducible then we have an exact sequence $0 \to W \to A_\lambda^n \to M \to 0$, where A_λ^n is the λ–eigenspace of Ω in $C^\infty(G/K,V)$, W is a Verma module and M an irreducible Verma module.

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As a corollary we obtain the eigenvalues and eigenspaces of

$$\tilde{\Omega}\colon L^2(G/K,V) \to L^2(G/K,V)$$

From this, it results that if λ is an eigenvalue of $\tilde{\Omega}$ the corresponding eigenspace is a proper subset of the respective one of Ω. We also compute the L^2-eigenspaces of the Dirac operator $\mathbf{D}$.

Knapp-Wallach [K-W] obtained an integral operator which sends an adjusted principal series onto the $K-$finite vector of the L^2-kernel of the Dirac operator $\mathbf{D}$. In this work we obtain a similar result for each L^2-eigenspace of $\mathbf{D}$ (c.f §4).

Let $\phi_{\lambda,n}$ be the Eisenstein function (cf. ***) in $C^\infty(G/K,V)$ that belongs to the $\lambda-$eigenspace of Ω, we prove:

(i) a continuous function that satisfies the integral equation

$$\int_K f(gkx)dk = f(g)\phi_{\lambda,n} \text{ for all } g,x \in G$$

is smooth and is an eigenfunction of Ω corresponding to the eigenvalue λ.

(ii) Any $\lambda-$eigenfunction of Ω satisfies the integral equation in (i).

Now, we stablish some notations,

$$
\begin{aligned}
K &= \left\{k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad : \quad \theta \in \mathbf{R}\right\} \\
A &= \left\{a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad : \quad t \in \mathbf{R}^+\right\} \\
M &= \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \qquad (1.2) \\
N &= \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad : \quad x \in \mathbf{R}\right\} \\
A^+ &= \{a_t \in A \quad : \quad 1 < t\} \\
A^- &= \{a_t \in A \quad : \quad 0 < t < 1\}
\end{aligned}
$$

We will use the decompositions $G = KAN$ and $G = KAK = K\overline{A^+}K = K\overline{A^-}K$ [K]. If we denote by

$$(1.3) \qquad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Iwasawa decomposition of the Lie algebra g_o of G is $g_o = k_o \oplus a_o \oplus n_o$ where $k_o = \mathbf{R}X$, $a_o = \mathbf{R}H$, $n_o = \mathbf{R}Y$. We denote by g, k, a, n their complexifications.

The Casimir operator Ω is an element of the universal algebra $\mathcal{U}(g)$ of g, moreover, the center of $\mathcal{U}(g)$ is $\mathbf{C}[\Omega]$ [L]. It is defined by

$$(1.4) \qquad \Omega = \tfrac{1}{2}\left(H^2 - H - YX\right)$$

If

$$W = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad E_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad E_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \tag{1.5}$$

another expression of Casimir operator is

$$\Omega = \tfrac{1}{8}\left(W^2 + 2W + 4E_-E_+\right) \tag{1.6}$$

W, E_+ and E_- satisfy the relations

$$\overline{W} = -W \qquad \overline{E_\pm} = E_\mp \qquad [E_+, E_-] = W \qquad [W, E_\pm] = \pm 2E_\pm$$

Let θ be the usual Cartan involution on g_o. Therefore, k_o is the subspace of fix points of θ. Let p_o be the (-1)–eigenspace of θ.

The Killing form in g_o is

$$B(X,Y) = 4\mathrm{Trace}(XY).$$

Thus $\{\frac{1}{2}E_+, \frac{1}{2}E_-\}$ is an orthonormal base of p with respect to the hermitian form

$$-B(X, \theta\bar{Y})$$

The Iwasawa decomposition for E_+ and E_- is

$$\begin{aligned} \tfrac{1}{2}E_+ &= \tfrac{1}{4}W + \tfrac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \\ \tfrac{1}{2}E_- &= -\tfrac{1}{4}W + \tfrac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{1.7}$$

§2.*Eigenspaces of* Ω

Since K is abelian, the irreducible representations of K are onedimensional. They are (τ_n, V_n) with $n \in \mathbf{Z}$, where

$$\dim V_n = 1 \; and \; \tau_n(k_\theta)v = e^{in\theta}v \qquad \text{for all } v \in V_n$$

Given $n \in \mathbf{Z}$, the elements of the center of the universal enveloping algebra of g will be considered acting on $C^\infty(G/K, V_n)$ as left invariant operators.

For all $\lambda \in \mathbf{C}$ define

$$A^n_\lambda = \left\{ f \in C^\infty(G/K, V_n) \;\Big/\; \Omega f = \frac{\lambda^2 - 1}{8} f \right\} \tag{2.1}$$

Since Ω is a continuous linear operator on $C^\infty(G/K, V_n)$, it follows that A^n_λ is a closed subspace of $C^\infty(G/K, V_n)$. Thus, A^n_λ is a subrepresentation of $C^\infty(G/K, V_n)$ with infinitesimal character χ_{λ_δ} , where δ is the linear functional of a_o such that $\delta(H) = \frac{1}{2}$ and $\chi_{\lambda\delta}$ is the character of $\mathbf{C}$ multiplication by $\frac{\lambda^2-1}{8}$.

We denote by $A^n_\lambda[m]$ the K-type τ_m of A^n_λ.

PROPOSITION 2.1.

Given $n \in \mathbf{Z}$, $\lambda \in \mathbf{C}$, the representation A^n_λ of G is admissible and finitely generated. Moreover,

(i) $\dim A^n_\lambda[m] \leq 1$ *for all* $m \in \mathbf{Z}$

(ii) If $A^n_\lambda[m] \neq \{0\}$, *then* n *and* m *have the same parity.*

Remark: The converse of (*ii*) is also true. It follows from proposition 2.4.

We need some results to prove the proposition 2.1

Let $f \in A^n_\lambda[m]$, f is a spherical function of type (m, n) because

$$f(k_\theta g k_\psi) = e^{-im\theta} f(g) e^{-in\psi} \quad \text{for all } g \in G,\ k_\theta,\ k_\psi \in K$$

Since $G = KAK$, the values of f are determined by its values on A. Besides, if $m \neq n$ then $f|_K \equiv 0$. In fact, the equallity $f(k_\theta) = f(k_\theta.1) = e^{-im\theta} f(1)$, implies that $f|_K \neq 0 \Leftrightarrow f(1) \neq 0$, now since f is spherical of type (m, n) we have that $f(k_\theta) = f(1.k_\theta) = f(1)e^{-in\theta} = f(1)e^{-im\theta}$, therefore if $f|_K$ were nonzero we would have that $m = n$.

The subgroup A is Lie isomorphic to $\mathbf{R}^+$ (positive real numbers with the usual product) by the isomorphism $\alpha(a_t) = t^2$.

Lemma 2.2.

If $f \in A^n_\lambda[m]$, *the function* $F : \mathbf{R}^+ \to \mathbf{C}$ *associated to* f *given by* $F(\alpha(a)) = f(a)$ *for all* $a \in A$ *satisfy the differential equation*

$$(2.2) \quad z^2 \frac{d^2}{dz^2} - \frac{2z^3}{1 - z^2} \frac{d}{dz} - \frac{z^2}{(1 - z^2)^2}(m^2 + n^2) + \frac{z(1 + z^2)}{(1 - z^2)^2} nm - \frac{\lambda^2 - 1}{4} = 0$$

The equation has regular singularities at the points $0, \pm 1, \infty$.

A proof of this lemma is in [Ca-M].

Proof of the Proposition 2.1. Since Ω is an elliptic operator in $C^\infty(G/K, V_n)$, if $f \in A^n_\lambda$, $f|_A$ is real analytic. Therefore, the function $F : \mathbf{R}^+ \to$ defined in (2.2) is a real analytic function. Hence there is a holomorphic extension of F to a neighborhood of $\mathbf{R}^+$ in the right half plane.

On the other hand by the Frobenius theory for differential equations with regular singular points [C-page 132] the equation (2.2) has an analytic solution on a neighborhood of 1 if and only if m and n have the same parity. Moreover, any holomorphic solution of (2.2) is a multiple of the power series

$$(2.3) \qquad (z-1)^{\frac{1}{2}|m-n|} \sum_{j=0}^{\infty} c_j (z-1)^j \qquad c_0 = 1$$

In fact, the indicial equation of (2.2) is

$$s(s-1)+s-\frac{1}{4}(m-n)^2=0$$

and its roots are $\pm\frac{1}{2}(m-n)$. Thus, as the roots differ by an integer, the exponent of the first term of (2.3) is $\frac{1}{2}|m-n|$, if this number were not an integer the function (2.3) would not be analytic on a neighborhood of 1, this forces the same parity for n and m.

As the other singularities of (2.2) are $0,-1,\infty$, there is an extension of the analytic solution on a neighborhood of 1 to an analytic solution on a neighborhood of $\mathbf{R}^+$. So (*i*) and (*ii*) holds. □

Remark. Since A^n_λ has infinitesimal character $\chi_{\lambda\delta}$ and A^n_λ is admissible by Proposition 2.1, A^n_λ has finite length by a known rwsult of Harish-Chandra [V,Corollary 5.4.16].

Corollary 2.3.

Given $n\in\mathbf{Z}$, $\lambda\in\mathbf{C}$, the K-type τ_n occurs in any subrepresentation of A^n_λ. Moreover, A^n_λ has a unique irreducible G-submodule.

Proof. Let W be a nontrivial closed submodule of A^n_λ and denote by W_K the set of K-finite elements in W, we consider the map

$$(*)\qquad \begin{array}{ccc} \mathrm{Hom}_G(W,A^n_\lambda) & \longrightarrow & \mathrm{Hom}_K(W_K,V_n) \\ T & \longrightarrow & \left(v\rightarrow \tilde{T}v=Tv(1)\right) \end{array}$$

This map is well defined. In fact, if $v\in W_K$,

$$\tilde{T}(kv)=T(kv)(1)=\left(L_k.Tv\right)(1)=Tv(k^{-1})=\tau_n(k)Tv(1)$$

Moreover, it is inyective. In fact, suppose that $\tilde{T}\equiv 0$, so $Tv(1)=0$ for all $v\in W_K$. As T is a continuous linear transformation, W_K is a dense subset of W [L-page 24], and there exists a sequence $\{v_m\}$ in W_K such that $v_m\rightarrow w$ for each $w\in W$, then

$$Tv_m\rightarrow Tw \quad\Longrightarrow\quad 0=Tv_m(1)\rightarrow Tw(1)$$

that is, $Tw(1)=0$ for all w. Now, for $w\in W$,

$$Tw(g)=\left(L_{g^{-1}}.Tw\right)(1)=T(g^{-1}w)(1)=0 \quad \text{for all } g\in G,$$

so $T\equiv 0$. If W is a closed submodule of A^n_λ, by (*) $W[n]\neq 0$, and by (*i*) $W[n]=A^n_\lambda[n]$. This concludes the first statement of the corollary. The second follows from the equality $W[n]=A^n_\lambda[n]$. □

Fix $n \in \mathbf{Z}$, $\lambda \in \mathbf{C}$, let δ be the linear functional on a_o such that $\delta(H) = \frac{1}{2}$, $\log a_t = t\ H$, and denote by $(-1)^n$ the character of M such that $\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \to (-1)^n$. As usual, define

$$(2.4)\quad \mathrm{I}^G_{MAN}((-1)^n \otimes e^{\lambda\delta} \otimes 1) = \\ = \{f : G \to \mathbf{C} \quad C^\infty \text{ such that} \\ f(xman) = e^{-(\lambda+1)\delta(\log a)}(-1)^n(m^{-1})f(x) \ \text{ for all } x \in G,\ man \in MAN\}$$

the representation of G induced by the representation $(-1)^n \otimes e^{\lambda\delta} \otimes 1$ of MAN. G acts by left translation. Recall that $\mathrm{I}^G_{MAN}((-1)^n \otimes e^{\lambda\delta} \otimes 1)$ has infinitesimal character $\chi_{\lambda\delta}$ and $\mathrm{I}^G_{MAN}((-1)^n \otimes e^{\lambda\delta} \otimes 1)$ is irreducible if and only if $\lambda \not\equiv (n+1)\,mod(2)$ [B].

Define linear transformations

$$(2.5)\quad \begin{array}{ccc} \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\pm\lambda\delta} \otimes 1\right) & \xrightarrow{T} & A^n_\lambda \\ f & \longrightarrow & \left(x \to Tf(x) = \int_K f(xk)\tau_n(k)dk\right) \end{array}$$

Whenever it becomes necessary to sea which is the domain of the operators, we will write $T_\pm$, otherwise we will write T.

The linear transformation T is well defined because

$$Tf(xk') = \int_K f(xk'k)\tau_n(k)\,dk = \tau(k')^{-1}\int_K f(xk)\tau_n(k)\,dk.$$

Besides, since $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\pm\lambda\delta} \otimes 1\right)$ has infinitesimal character $\chi_{\lambda\delta}$, T is a left G-morphism and left infinitesimal translation by Ω agrees with right infinitesimal translation, ($L_\Omega.f = R_\Omega.f$ for all $f \in C^\infty(G/K, V_n)$). Hence the image of T is contained in A^n_λ.

T is not zero because

$$T\tau_{-n}(1) = \int_K \tau_{-n}(k)\tau_n(k)dk = \int_K dk \neq 0$$

Note that A^n_λ and $A^n_{\lambda'}$ is the same eigenspace of Ω if $\lambda^2 = (\lambda')^2$. So, if $\lambda \in \mathbf{Z}$ we will always assume that $\lambda \geq 0$.

PROPOSITION 2.4.

Given $n \in \mathbf{Z}$,

(i) *If* $\lambda \in \mathbf{C} \setminus \mathbf{Z}$, *or* $\lambda \in \mathbf{Z}$ *and* $\lambda \not\equiv (n+1)\,\mathrm{mod}(2)$, A^n_λ *is infinitesimally equivalent to* $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$.

(ii) If $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n\,\mathrm{mod}(2)$ *and* $\lambda > |n|$, A^n_λ *is infinitesimally equivalent to* $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$.

(iii) *If* $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n\,\mathrm{mod}(2)$ *and* $\lambda < n$, *the* (g, K)*-module structure of* A^n_λ *is the following*

$$\begin{array}{ll} E_+A^n_\lambda[m] \neq 0 & \textit{for all } m \textit{ such that } A^n_\lambda[m] \neq 0 \\ E_-A^n_\lambda[m] \neq 0 & \textit{for all } m \neq \pm\lambda \textit{ such that } A^n_\lambda[m] \neq 0 \\ E_-A^n_\lambda[\pm\lambda + 1] = 0 & \end{array}$$

(iv) If $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n \bmod(2)$, $n < 0$ and $\lambda < -n$, the (g, K)-module structure of A^n_λ is the following

$$E_- A^n_\lambda[n] \neq 0 \quad \textit{for all } m \textit{ such that } A^n_\lambda[m] \neq 0$$
$$E_+ A^n_\lambda[m] \neq 0 \quad \textit{for all } m \neq \pm\lambda + 1 \textit{ such that } A^n_\lambda[m] \neq 0$$
$$E_+ A^n_\lambda[\pm\lambda + 1] = 0.$$

Remark 1: Under the hypothesis *(iii)* or *(iv)* we have that A^n_λ is not a quotient of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\pm\lambda\delta} \otimes 1\right)$.
Remark 2: A^n_λ is irreducible if and only if $\lambda \not\equiv (n+1) \bmod(2)$.

We need the following lemma to prove *(iii)* of proposition 2.4.

Lemma 2.5.
Given $n \in \mathbf{Z}$, let $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n \bmod 2$ and $\lambda < n$, there exist $m \in \mathbf{Z}$, $m < -\lambda$ such that $A^n_\lambda[m]$ is not zero.

Proof of Lemma 2.5. Let m be an integer such that

$$m \equiv n \bmod 2 \qquad m < -\lambda \qquad \tfrac{1}{2}(n-m) \quad \text{is even} \tag{2.6}$$

The conditions on m and n ensure the existence of a smooth solution F of (2.2) on the interval $(0, \infty)$. In fact, using the Frobenius method for differential equations with regular singularities, that (2.2) has a analytic solution in a neighbordhood of 1 if and only if m and n have the same parity. Besides, the singularities of (2.2) are $0, \pm 1$, ∞. Therefore, this solution extends to a solution on the interval $(0, \infty)$. Moreover, any smooth solution of (2.2) in the interval $(0, \infty)$ is a multiple of the power series

$$(z-1)^{\frac{1}{2}|m-n|} \sum_{j=0}^{\infty} c_j (z-1)^j \qquad c_0 = 1$$

Therefore, F has a zero of order $\frac{1}{2}|m-n|$ at 1.
We have to prove that F extends to an element of $A^n_\lambda[m]$. This will take some work.
Let $N_K(A)$ be the normalizer of A on K.
Consider $C^\infty_{\tau_{n-m}}(A)$ to be the set of smooth funtions on A such that

(j) $\phi(kak^{-1}) = \tau_{n-m}(k)\,\phi(a) \qquad$ for all $a \in A$, $k \in N_K(A)$
(jj) $\dfrac{\phi(a)}{\delta(\log a)^{\frac{1}{2}(n-m)}}$ is a smooth function and even on A.

Let $f \colon A \to \mathbf{C}$ given by $f(a) = F(\alpha(a))$, with α the isomorphism between A and $\mathbf{R}^+$ defined in (2.2). Let's prove that the function f is in $C^\infty_{\tau_{n-m}}(A)$. In fact, the normalizer of A on K, is exactly

$$N_K(A) = \{\pm I\} = \left\{k_{\frac{\pi}{2}}, k_{-\frac{\pi}{2}}\right\}$$

As $n-m$ and $\frac{1}{2}(n-m)$ are even numbers,

$$\tau_{n-m}(\pm I) = \tau_{n-m}(k_{\pm\frac{\pi}{2}}) = e^{\pm i(n-m)\frac{\pi}{2}} = 1$$

So, f satisfy (j) if and only if $f(a) = f(a^{-1})$ for all $a \in A$, or equivalently $F(x) = F(x^{-1})$ for all $x \in \mathbf{R}^+$. Let's prove that $F(x) = F(x^{-1})$. Let h be the function given by $h(z) = F(z^{-1})$, we want to prove that $h = F$. We claim that h satisfies the same differential equation that F does. In fact, let $w = z^{-1}$, then

$$\begin{aligned}\frac{dh}{dz}(z) &= \frac{dF}{dw}(w)\,w' \\ &= -w^2\frac{dF}{dw}(w)\end{aligned}$$

$$\begin{aligned}\frac{d^2F}{dz^2}(z) &= -2ww'\frac{dF}{dw}(w) + w^4\frac{d^2F}{dw^2}(w) \\ &= 2w^3\frac{dF}{dw}(w) + w^4\frac{d^2F}{dw^2}(w)\end{aligned}$$

and

$$-\frac{2z^3}{1-z^2} = -\frac{2w^{-3}}{1-w^{-2}} = \frac{2w^{-1}}{1-w^2}$$

$$-\frac{z^2}{(1-z^2)^2} = -\frac{w^{-2}}{(1-w^{-2})^2} = -\frac{w^2}{(1-w^2)^2}$$

$$\frac{z(1+z)}{(1-z^2)^2} = \frac{w^{-1}(1+w^{-2})}{(1-w^{-2})^2} = \frac{w(w^2+1)}{(1-w^2)^2}$$

So,

$$z^2\frac{d^2h}{dz^2}(z) - \frac{2z^3}{1-z^2}\frac{dh}{dz}(z) + \left(-\frac{z^2}{(1-z^2)^2}(m^2+n^2) + \frac{z(1+z^2)}{(1-z^2)^2}nm - \frac{\lambda^2-1}{4}\right)h(z) =$$

$$= w^2\frac{d^2F}{dw^2}(w) + \left(2w - \frac{2w^{-1}}{1-w^2}w^2\right)\frac{dF}{dw}(w) + \left(-\frac{w^2}{(1-w^2)^2}(m^2+n^2) + \frac{w(1+w^2)}{(1-w^2)^2}nm - \frac{\lambda^2-1}{4}\right)F(w)$$

The right hand side is exactly the equation(2.2) on F, so it is zero. Both h and F are smooth functions on $(0,\infty)$ and solutions of the differential equation (2.2). So, by (2.6) they are multiple of each other in a neighborhood of 1. Hence, we write,

$$\begin{aligned}h(z) &= (z-1)^{\frac{1}{2}|n-m|}\psi_h(z) \\ F(z) &= (z-1)^{\frac{1}{2}|n-m|}\psi_F(z)\end{aligned}$$

with ψ_h and ψ_F power series, such that $c\psi_h(z) = \psi_F(z)$ for a suitable nonzero complex number. Therefore,

$$h(z) = F(z^{-1}) = (z^{-1} - 1)^{\frac{1}{2}|n-m|}\psi_F(z^{-1}) = (z-1)^{\frac{1}{2}(n-m)}z^{-\frac{1}{2}|n-m|}\psi_F(z^{-1})$$

Thus, $\psi_h(z) = (z-1)^{-\frac{1}{2}(n-m)}\psi_F(z^{-1})$. This imply that

$$c\psi_h(z) = (z-1)^{-\frac{1}{2}(n-m)}\psi_F(z^{-1})$$

Hence, $F(z) = F(z^{-1})$ in a neighborhood of 1. As F is real analytic in $(0,\infty)$, $F(z) = F(z^{-1})$ for all $z \in \mathbf{R}^+$. Equivalently, $f(a) = f(a^{-1})$ for all $a \in A$. Thus, f satisfies (j).

We want to prove that f satisfies (jj). The function $\delta(\log a)^{-\frac{1}{2}(n-m)}$ is even on A because

$$\begin{aligned} \delta(loga_t)^{-\frac{1}{2}(n-m)} &= (t\ \delta(H))^{-\frac{1}{2}(n-m)} \\ &= (-t\ \delta(H))^{-\frac{1}{2}(n-m)} \qquad \text{by (2.6)} \\ &= \delta(\log a_t^{-1})^{-\frac{1}{2}(n-m)} \end{aligned}$$

Thus, the function $f(a)\delta(\log a)^{-\frac{1}{2}(n-m)}$ is even. The function $f(a)\delta(\log a)^{-\frac{1}{2}(n-m)}$ is smooth because f is real analytic and has a zero of order $\frac{1}{2}(n-m)$ at 1. Therefore, we have proved that $f \in C^{\infty}_{\tau_{n-m}}(A)$. We want to extend f to an element of $A^n_\lambda[m]$

Let $C^\infty(G/K)[\tau_{n-m}]$ be the space of smooth complex valued functions on G/K such that $f(kx) = \tau_{n-m}(k)f(x)$ for all $k \in K$, $x \in G$.

We need to prove:

Sublemma 2.6.

The restriction map from $C^\infty(G/K)[\tau_{n-m}]$ to $C^\infty_{\tau_{n-m}}(A)$ is biyective.

Proof of sublemma 2.6. : The equallity $G = KAK$ implies that the restriction map is inyective. To prove that is suryective we appeal to a theorem of Helgason. Let $\mathcal{H}$ be the set of harmonic polynomial functions on p_o. We consider the usual action of K on $\mathcal{H}$. That is, the one determinated by the isotropy representation of K in p_o. We now set ourselves in §10 of [H-1], with $\delta = \tau_{n-m}$. Since $n \equiv m mod(2)$, we have that $\tau_{n-m} \in \hat{K}_o$. Let $degQ^\delta(\lambda) = p(\delta)$. A formula due to Kostant and cited on pag 203 of [H-1] says that $p(\delta) = d(\delta)$ =degree of the harmonic homogeneous polynomials in the $\delta-$isotypic component of $\mathcal{H}$. To compute $d(\delta)$ we proceed as follow: If e_1, e_2 is an orthonormal basis for p_o, we know that $k(\theta)\dot{e}_1 = \cos(2\theta)e_1 - \sin(2\theta)e_2$, $k(\theta)\dot{e}_2 = \sin(2\theta)e_1 + \cos(2\theta)e_2$. Since $(n-m)/2$ is a whole number the polynomial function on p_o, $(e_1 + ie_2)^{(n-m)/2}$ is harmonic and has degree $(n-m)/2$, moreover $k(\theta)(e_1 + ie_2)^{(n-m)/2} = e^{i(n-m)\theta}(e_1 + ie_2)^{(n-m)/2}$. Thus, we have that $p(\delta) = (n-m)/2$. Therefore, our space $C^\infty_{\tau_{n-m}}(A)$ contains the space $\mathcal{D}^{\tau_{n-m}}(A)$ of page 211 in [H-1]. Hence, lemma 10.1 of [H-1] implies that the restricction map from $\mathcal{D}^{\tau_{n-m}}(G/K)$ into $\mathcal{D}^{\tau_{n-m}}(A)$ is a linear homeomorphism. We remark that $\mathcal{D}^{\tau_{n-m}}(G/K) \subset C^\infty(G/K)[\tau_{n-m}]$. A density argument together with the fact that K is compact imply sublemma 2.6. □

We proceed with the proof of lemma 2.5. For this end, we now have that the function f admits a smooth extension $\tilde{f}\colon \exp p_o \to \mathbf{C}$ which satisfies

$$\tilde{f}(kak^{-1}) = \tau_{n-m}(k)\,\tilde{f}(a) \tag{2.7}$$
$$= \tau_m(k)^{-1}\tilde{f}(a)\tau_n(k)$$

The diffeomorphism between G and $\exp p_o K$ ensures that the function $\hat{f}\colon G \to \mathbf{C}$ given by

$$\hat{f}(pk) = \tilde{f}(p)\tau_n(k)^{-1} \qquad \text{for all } p \in \exp p_o\,, k \in K$$

is well defined and it is smooth. Also, $\hat{f}$ is in the K-type τ_m of $C^\infty(G/K, V_n)$. In fact, for $x \in G$ we write $x = k_2 a k_2^{-1} k_1$ with $k_1, k_2 \in K$, and $a \in A$, hence

$$\begin{aligned}(L_k\hat{f})(x) = \hat{f}(k^{-1}k_2ak_2^{-1}k_1) &= \tilde{f}(k^{-1}k_2ak_2^{-1}k)\tau_n(k^{-1}k_1)^{-1}\\ &= \tau_{n-m}(k^{-1}k_2)f(a)\tau_n(k^{-1}k_1)^{-1}\\ &= \tau_{n-m}(k^{-1})\tau_{n-m}(k_2)f(a)\tau_n(k^{-1}k_1)^{-1}\\ &= \tau_{n-m}(k^{-1})\tilde{f}(k_2ak_2^{-1})\tau_n(k^{-1})^{-1}\tau_n(k_1)^{-1}\\ &= \tau_n(k^{-1})\tau_m(k)\tilde{f}(p)\tau_n(k^{-1})^{-1}\tau_n(k_1)^{-1}\\ &= \tau_m(k)\tilde{f}(p)\tau_n(k_1)^{-1}\\ &= \tau_m(k)\hat{f}(x) \qquad \square\end{aligned}$$

A comutation like the one in [Wa] page 280, implies that

$$(\Omega\hat{f})(x) = \tau_m(k_2^{-1})\tau_n(k_2^{-1}k_1)(z^2\frac{d^2F}{d^2z} + \dots) = 0$$

because F satisfies the equation 2.2.

This concludes the proof of lemma 2.5

Proof of the Proposition 2.4. (*i*) As T is not the zero function and since $\lambda \not\equiv n+1 \bmod(2)$ the module $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ is irreducible. Thus T is inyective. The K-types τ_m which occur in $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ are indexed by all the m with the same parity as n. Since T is one-to-one they must occur in A^n_λ. By proposition 2.1 (*i*), (*ii*), they are exactly the K-types of A^n_λ. Thus, T is suryective at the level of (g, K)-modules.

(*ii*) Since $\lambda \geq 0$, $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$ has only one irreducible submodule F which is finite dimensional and whose K-types are parametrized by $\{m : -(\lambda-1) \leq m \leq \lambda - 1,\ m \equiv n\,(2)\}$. The structure of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$ is

$$\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right) \quad \begin{matrix}\supset W_+\\ \supset W_-\end{matrix} \quad \supset F \quad \supset \quad 0$$

where W_+ is the G-submodule spanned by the K-types $\{-(\lambda-1), -(\lambda-3), \dots, \lambda-1, \lambda+1, \dots\}$ and W_- is the one spanned by the K-types $\{\dots, \lambda-3, \lambda-1\}$. As

$\lambda > |n|$ the K-type τ_n occur in F. On the other hand, we have verified that T maps non trivially the $K-$type τ_n, so F is not a submodule of Ker$T$. Since F is contained in every nonzero submodule of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$. T is 1:1; by a similar argument to the one used on (i) we get that T is suryective.

(iii) Suppose that n, $\lambda > 0$ $\lambda < n$, $\lambda \not\equiv n + 1(2)$. Then the image of T_- is the discrete serie $H_{\lambda\delta}$ of infinitesimal character $\chi_{\lambda\delta}$. We recall that the K-types of $H_{\lambda\delta}$ are parametrized by $\{\lambda+1, \lambda+3, \dots\}$. In fact, the nonzero quotients of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$ are $H_{\lambda\delta}$, $H_{-\lambda\delta}$, $H_{\lambda\delta} \oplus H_{-\lambda\delta}$ or itself. Now, the irreducible finite-dimensional submodule occurs in KerT_-, otherwise $T_-(F)$ would be an irreducible submodule of A^n_λ and do not have the K-type τ_n ($\lambda < |n|!$), that contradicts corollary 2.3. This contradiction ensures that T_- is not inyective. By corollary 2.3, A^n_λ has only one irreducible submodule, Im$T_- \neq H_{\lambda\delta} \oplus H_{-\lambda\delta}$. Furthermore, since the irreducible submodule contains the K-type τ_n ,so Im$T_- = H_{\lambda\delta}$. Therefore $H_{\lambda\delta}$ is the irreducible submodule of A^n_λ.

The structure of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ is the following

$$\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right) \supset H_{\lambda\delta} \oplus H_{-\lambda\delta} \quad \begin{matrix} \supset H_{\lambda\delta} \\ \supset H_{-\lambda\delta} \end{matrix} \quad \supset 0$$

T_+ is not inyective; otherwise $T_+(H_{-\lambda\delta})$ is an irreducible submodule of A^n_λ and does not have the K-type τ_n. Also Ker $T_+ \neq H_{\lambda\delta} \oplus H_{-\lambda\delta}$; otherwise, the finite dimensional representation F is a subrepresentation of A^n_λ, contradicting corollary 2.3. Thus,

$$\mathrm{Im}T_+ \cong \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)/H_{-\lambda\delta}$$

This implies that

$$(\mathrm{Im}T_+)_K = \bigcup_{\substack{m \geq -(\lambda-1) \\ m \equiv n(2)}} A^n_\lambda[m]$$

which is the Verma module of lowest weight $-(\lambda-1)$. Thus,

$$\begin{aligned} &E_+ A^n_\lambda[m] \neq 0 \qquad \text{for all } m \geq -(\lambda-1) \\ &E_- A^n_\lambda[m] \neq 0 \qquad \text{for all } m \geq -(\lambda-1) \text{ and } m \neq -\lambda+1 \end{aligned}$$

By lemma 2.5 there exists a K-type $A^n_\lambda[m] \neq 0$ for some $m < -\lambda$. This ensure that $A^n_\lambda[m] \neq 0$ for all $m < -\lambda$ and $m \equiv n\mathrm{mod}(2)$, on the other hand, A^n_λ would have a lowest weight submodule with lowest weight less than $-\lambda\delta$. The infinitesimal character of this lowest weight submodule would be different from $\chi_{\lambda\delta}$, giving a contradiction. Following the same argument, E_+ acts nontrivially on each $A^n_\lambda[m]$, $m < -\lambda$.

For the case $\lambda = 0$ and $\lambda + 1 \equiv n\mathrm{mod}(2)$ the proof is easier.

(iv) It has the same proof of (iii). This concludes the proof of proposition 2.4. □

Remark 1: Given $n \in \mathbf{Z}$ and $\lambda \in \mathbf{C}$, the K-types $A^n_\lambda[m]$ are not zero for all m with the same parity of n.

Remark 2: In view of [S] , in cases (i) and (ii) A_λ^n is equivalent to the maximal model of I_{MAN}^G which is the induced representation with hiperfunctions coefficients. In case (iii) A_λ^n is a quotient of the maximal model of a generalized principal series.

Remark 3: Given $n \in \mathbf{Z}_{\geq 0}$ and $\lambda \geq 0$ as in (iii) of proposition 2.4 , the G-module structure of A_λ^n is

$$\cdots \quad \bullet \quad -(\lambda+1) \bullet \underset{0}{\overset{\neq 0}{\rightleftarrows}} \bullet -(\lambda-1) \quad \cdots \quad \lambda-1 \bullet \underset{0}{\overset{\neq 0}{\rightleftarrows}} \bullet \lambda+1 \quad \bullet \quad \cdots$$

the right arrows represent the action of E_+ and the left ones the action of E_-. That is, we have proved

Corollary 2.6.
Let $\lambda \in \mathbf{Z}_{\geq 0}$ and $\lambda \equiv n+1 \mod(2)$. A composition series for A_λ^n is

$$0 \to V \to A_\lambda^n \to M \to 0$$

where V is the Verma module of lowest weight $-(\lambda-1)$ and M is the irreducible Verma module of highest weight $-(\lambda+1)$.

PROPOSITION 2.7.
Given $n \in \mathbf{Z}$ and λ as in (iii) of proposition 2.4 (i.e. $\lambda \equiv n+1 \mod(2)$ and $\lambda \geq 0$ an integer), then A_λ^n is quotient of a generalized principal series $\mathrm{I}_{MAN}^G(W_0)$ where $W_0 = \mathbf{R}^2$ and the representation of MAN is

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \to (-1)^n \exp t \begin{pmatrix} \lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

Proof. For $f = (f_1, f_2) \in \mathrm{I}_{MAN}^G(W_0)$ let

$$S : \mathrm{I}_{MAN}^G(W_0) \to C^\infty(G/K, V_n)$$

defined by

$$(Sf)(x) = \int_K f_1(xk)\tau_n(k)\, dk + \int_K f_2(xk)\tau_n(k)\, dk$$

Since $\mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ is contained in $\mathrm{I}_{MAN}^G(W_0)$ via the map $f \to F = (f, 0)$ and S restricted to $\mathrm{I}_{MAN}^G(W_0)$ is equal to T_+, hence $\mathrm{Im}(S)$ contains $\mathrm{Im}(T_+)$. An easy calculation shows that $\mathrm{Im}(S)$ contains properly $\mathrm{Im}(T_+)$. Now, corollary 2.6 implies that any K-finite vector in A_λ^n outside of $\mathrm{Im}(T_+)$ is cyclic in $A_\lambda^n/\mathrm{Im}(T_+)$. Therefore, S is onto. $\square$

Now, consider the Casimir operator acting on the subspace of compactly supported functions in $C^\infty(G/K, V_n)$. We denote by $\tilde{\Omega}$ the unique essentially selfadjoint extension of Ω to a dense subspace of

$$\mathrm{L}^2(G, V_n) = \left\{ f : G \to \mathbf{C} \quad \Big/ \quad \begin{array}{l} f(xk) = \tau_n(k)^{-1} f(x) \\ \int_G |f(x)|^2 \, dx < \infty \end{array} \right\}$$

(cf [A-S]).

PROPOSITION 2.8.
If $W_\lambda^n = \{f \in L^2(G/K, V_n) \;/\; \tilde{\Omega} f = \frac{\lambda^2 - 1}{8} f\}$, then W_λ^n is non zero if and only if $\lambda \in \mathbf{Z} - \{0\}$, $\lambda + 1 \equiv n \mathrm{mod}(2)$ and $|\lambda| < |n|$. Moreover, $W_\lambda^n = W_{-\lambda}^n$ is isomorphic to the discrete series of Harish-Chandra parameter $\lambda\delta$.

Proof. Suppose that $\lambda \in \mathbf{Z} - \{0\}$, $\lambda + 1 \equiv n \mathrm{mod}(2)$ and $|\lambda| < |n|$. As $\tilde{\Omega}$ is elliptic, a Connes-Moscovici result [C-M] ensure that W_λ^n is a sum of discrete series, actually , it is irreducible by the Frobenius Reciprocity. The K-finite elements of $L^2(G/K, V_n)$ are in the set of K-finite elements of $C^\infty(G/K, V_n)$, so $W_\lambda^n[m] \subset A_\lambda^n[m]$ for all $m \in \mathbf{Z}$. By proposition 2.4, A_λ^n has subspaces infinitesimally equivalent to a discrete series for λ such that

$$\lambda \in \mathbf{Z} \qquad \lambda \equiv n + 1 \bmod(2), \qquad 0 < |\lambda| < |n|$$

This "discrete series" subspaces are really contained in $L^2(G/K, V_n)$. In fact, if $f \in A_\lambda^n[m]$ and it belongs to a "discrete series", then f satisfies the differential equation (2.2) or the one which results from the identification of A^+ with $\mathbf{R}_{>0}$ via $a_t \leftrightarrow t$. Then the theory of leading exponents as in [K] says that $f(a_t)\; e^{-(\lambda-1)t}$ at $t = \infty$. Now, the integral formula for the Cartan decomposition together with $\lambda > 0$ imply that f is square integrable. For negative λ we have a similar proof.

For the converse we use the structure of the discrete series, Frobenius Reciprocity together with proposition 2.4. This concludes proposition 2.8. □

§3.L^2 *and* C^∞*-eigenspaces of the Dirac operator*

Let $g_o = k_o \oplus p_o$ be the Cartan decomposition of g_o, then p_o is the subspace of symmetric matrix of g_o.

If we fix a minimal left ideal S in the Clifford algebra of p_o, the resulting representation of $so(p_o)$ brakes down in two irreducible representations. Such representation composed with the adjoint representation of k_o restricted to p_o lift up at a representation of K called the spin representation of K. Let $\{X_1, X_2\}$ be an orthonormal base of p_o, let c be the Clifford multiplication and fix an integer n. The Dirac operator

$$\mathbf{D} \colon C^\infty(G/K, V_{n+1} \otimes S) \quad \to \quad C^\infty(G/K, V_{n+1} \otimes S)$$

is defined by

$$\mathbf{D} = \sum_{i=1}^{2} (1 \otimes c(X_i))\, X_i \tag{3.1}$$

where X_i act as left invariant operators for all i. The spin representation S decompose into a sum of two irreducible subrepresentations $S = S^+ \oplus S^-$ (c.f. 4.2 bellow). If $X \in p_o$, then $c(X)S^\pm = S^\mp$, so

$$\mathbf{D}^\pm : C^\infty\left(G/K, V_n \otimes S^\pm\right) \quad \rightarrow \quad C^\infty\left(G/K, V_n \otimes S^\mp\right) \tag{3.2}$$

are well defined.

We also consider

$$\tilde{\mathbf{D}}: L^2(G/K, V_{n+1} \otimes S) \quad \rightarrow \quad L^2(G/K, V_{n+1} \otimes S)$$

Some properties of the Dirac operators $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are: both are elliptic G-invariant differential operator. As the Rimannian metric of G/K is complete, $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{D}}^2$ are essentially selfadjoint in $L^2(G/K, V_{n+1} \otimes S)$ [W], that is, the minimal extension is the unique selfadjoint closed extension over the set of smooth compactly supported funtions. Thus, we consider $\tilde{\mathbf{D}}$ equal to this extension which coincides with the maximal one [A]. The eigenvalues of $\tilde{\mathbf{D}}$ are defined as the eigenvalues of the unique selfadjoint extension.

The following proposition is a corollary to proposition 2.8.

PROPOSITION 3.1.

If α is an eigenvalue of $\tilde{\mathbf{D}}$, then the α-eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is irreducible and it is a proper subspace of the α-eigenspace $\mathrm{W}_\alpha(\mathbf{D})$ of $\mathbf{D}$. The eigenvalues of $\tilde{\mathbf{D}}$ are $\alpha \in \mathbf{R}$ such that $\alpha^2 = \frac{1}{8}(n+2)^2 - \lambda^2$ with λ integer and $0 < |\lambda| \leq n+1$.

Proof. For $G = SL(2,R)$ The Parthasarathy equality [A-S] is

$$\begin{aligned} \mathbf{D}^2 &= -\Omega + \frac{(n+1)^2 - 1}{8}\, Id \\ \tilde{\mathbf{D}}^2 &= -\tilde{\Omega} + \frac{(n+1)^2 - 1}{8}\, Id \end{aligned} \tag{3.3}$$

If α is a non-zero eigenvalue of $\tilde{\mathbf{D}}$,

$$\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2) = \mathrm{W}_\alpha(\tilde{\mathbf{D}}) \oplus \mathrm{W}_{-\alpha}(\tilde{\mathbf{D}}) \tag{3.4}$$

(cf [G-V]). Because of (3.3), the left hand side of (3.4) is the $-\alpha^2 + (n+1)^2 - 1 = \frac{1}{8}(\lambda^2 - 1)$ eigenspace of the Casimir operator. Now, since $S = V_{-1} \oplus V_1$,

$$\mathrm{L}^2(G/K, V_{n+1} \otimes S) = \mathrm{L}^2(G/K, V_n) \oplus \mathrm{L}^2(G/K, V_{n+2})$$

Hence proposition 2.8 implies that $0 \leq \lambda \leq n+1$ and

$$\alpha^2 = \frac{(n+1)^2 - \lambda^2}{8}$$

Moreover,

$$\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2) = A_\lambda^n \cap L^2(G/K, V_n) \oplus A_\lambda^{n+1} \cap L^2(G/K, V_{n+2})$$

Thus, $\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2)$ is equal to the sum of two copies of the discrete series $H_{\lambda\delta}$. Since, $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is isomorphic to $H_{\lambda\delta}$ we get that $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is properly contained in $\mathrm{W}_\alpha(\mathbf{D})$. □

Corollary 3.2.
(τ_n, V_n) and (τ_{n+2}, V_{n+2}) are K-types of $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ for every non-zero eigenvalue α of $\tilde{\mathbf{D}}$. For the case $\alpha = 0$, (τ_{n+2}, V_{n+2}) is contained in $\mathrm{Ker}\tilde{\mathbf{D}}$ and (τ_n, V_n) is not.

§4. *Szegö kernels associated to the eigenspaces of* $\tilde{\mathbf{D}}$

In [K-W] Knapp and Wallach gave an integral operator to explicitly obtain a discrete serie as the image of a nonunitary principal serie when the discrete serie is realized as the kernel of Schmid operator. In §3 we have obtained that each eigenspace of the Dirac operator

$$\tilde{\mathbf{D}}\colon L^2(G/K, V_{n+1} \otimes S) \quad \rightarrow \quad L^2(G/K, V_{n+1} \otimes S)$$

is a discrete serie. The purpose of this section is to give an integral operator for each non zero eigenvalue α of $\tilde{\mathbf{D}}$ which will realize the eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ as a quotient of an appropiated principal serie. From §3 it is easy to deduce which will be the principal serie corresponding to each eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$, the problem is to obtain the G-invariant integral operator onto $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$. Let $G = SL(2, \mathbf{R})$ and K the maximal compact subgroup defined as in (1.2).

Let V_{n+1} be the $n+1$ irreducible representation of K, we assume that $n+1 > 0$. In §3, given an orthonormal base of p_o it was defined the Dirac operator $\tilde{\mathbf{D}}$. If we take $\{X_i\}_{i=1}^2$ an orthonormal base of the complexification p of p_o, another expresion of $\tilde{\mathbf{D}}$ is

$$\tilde{\mathbf{D}} = \sum_{i=1}^{2} (1 \otimes c(X_i))\, \bar{X}_i \tag{4.1}$$

where bar is conjugation with respect to g_o.

One form to obtain the representations $S^\pm$ is choosing the left minimal ideals of the Clifford algebra of p,

$$S^+ = \mathbf{C}E_+ \qquad\qquad S^- = \mathbf{C}E_-E_+$$

where the product is Clifford multiplication. In $Cliff(p)$ the following set of relations holds:

(4.2) $$E_+^2 = E_-^2 = 0 \qquad E_+E_-E_+ = -E_+$$

Hence $S = V_{-1} \oplus V_1$. Thus, we have that

$$V_{n+1} \otimes S = V_n \oplus V_{n+2}$$

The set of K-finite elements of a principal serie $\mathrm{I}_{MAN}^G(\epsilon \otimes e^{\lambda\delta} \otimes 1)$ defined in (2.4), is the representation of K induced by ϵ of M, hence

$$I_M^K(\epsilon) = \underset{i \in \hat{K}}{\oplus} V_i \otimes \mathrm{Hom}_M(V_i, \epsilon)$$

So, if the representation ϵ occur at V_n and V_{n+2} as M-submodule, then $\epsilon = (-1)^n$. We denote by i_j the inclusions

$$i_j : (\epsilon, W_\epsilon) \quad \longrightarrow \quad (\tau_j, V_j) \qquad j = n, n+2$$

As W_ϵ and V_j are one dimensional

$$W_\epsilon = \mathbf{C}w \qquad V_j = \mathbf{C}\, v \otimes u$$

where $w \in W_\epsilon$, $v \in V_{n+1}$ and $u \in S^{\pm}$.

Then the inclusions i_j are determined by the constants a_j such that

(4.3) $$i_j(w) = a_j\, v \otimes u \qquad \text{where } u = \begin{cases} E_+ & j = n \\ E_-E_+ & j = n+2 \end{cases}$$

If $sg\,\alpha$ is the sign of the real number α, fix

$$a_n = \left(\frac{\lambda + n + 1}{-\lambda + n + 1}\right)^{\frac{1}{2}} sg\,\alpha \qquad \text{con } 0 \neq \lambda \in \mathbf{Z}, |\lambda| \leq n$$

$$a_{n+2} = 1$$

Let $G = KAN$ be the Iwasawa decomposition of G. According to this decomposition we write an element of G by

$$x = \kappa(x)e^{H(x)}n(x)$$

Let $S(x,t)$ be the function on $G \times K$ defined by

(4.4) $$S(x,t) = e^{(\lambda-1)\delta H(x^{-1}t)} \left(\tau_n(\kappa(x^{-1}t))i_n + \tau_{n+2}(\kappa(x^{-1}t))i_{n+2}\right)$$

Let $\tau = \tau_n + \tau_{n+2}$ on $V_n \oplus V_{n+2}$, so (4.4) implies

(4.5) $$S(xk, t) = \tau(k)^{-1}S(x,t) \qquad \text{for all } k \in K$$

We will call $S(x,t)$ the Szegö kernel associated to the parameters $(\lambda, n+1)$. If $f \in \mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$, the Szegö map associated to the parameters $(\lambda, n+1)$ is

(4.6) $$\begin{aligned} S(f)(x) &= \int_K S(x,t)\, f(t)\, dt \\ &= \int_K e^{(\lambda-1)\delta H(x^{-1}t)}\tau(\kappa(x^{-1}t))(i_n + i_{n+2})\, f(t)\, dt \end{aligned}$$

The equation (4.5) ensure that the image of the Szegö map is in $C^\infty(G/K, V_n \oplus V_{n+2})$.

Let $\tilde{\mathbf{D}}$ defined as in §3

PROPOSITION 4.1.

Given $n \in \mathbf{Z}$, α a non zero eigenvalue of $\tilde{\mathbf{D}}$, and λ a negative integer which satisfies the equality

$$\alpha = \tfrac{1}{8}\left(-\lambda^2 + (n+1)^2\right)^{\frac{1}{2}} sg\, \alpha$$

Then, the Szegö map of parameters $(\lambda, n+1)$ is a G-invariant operator onto the eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$.

Before proving this result we will see that Szegö map is not the zero map. Let $f \in C^\infty(K/M, W_\epsilon)$ where $\epsilon = (-1)^n$, given by

$$f(k) = i^{-1}\, \tau_n(k)^{-1}\, i_n w$$

Extend f to G so that $f \in \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$.

$$\begin{aligned}(S(f)(1), i_n w) &= \int_K \left(\tau(t)(i_n + i_{n+2})\,(i_n^{-1}\, \tau_n(t)^{-1}\, i_n w), i_n w\right) dt \\ &= \int_K \left(i_n w + \tau_{n+2}(t) i_{n+2}\,(i^{-1}\, \tau_n(t)^{-1}\, i_n w), i_n w\right) dt \\ &= \int_K \|i_n w\|^2\, dt \\ &\neq 0\end{aligned}$$

because $\tau_{n+2}(t) i_{n+2}\,(i^{-1}\, \tau_n(t)^{-1}\, i_n w) \in V_{n+2}$ which is orthogonal to V_n.

To see that the Szegö map is G-invariant we need next lemma

Lemma 4.2.

Let S be the Szegö map with parameters $(\lambda, n+1)$. If $f \in \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ then

$$S(f)(x) = \int_K \tau(t)(i_n + i_{n+2})\, f(xt)\, dt$$

Proof of Lemma 4.2. Using the change of variable

$$\int_K h(k)\, dk = \int_K h(\kappa(x^{-1}t)) e^{-2\delta H(x^{-1}t)}\, dt$$

for $h(k) = \tau(k)(i_n + i_{n+2})\, f(xk)$ the following equality holds

$$\int_K \tau(k)(i_n + i_{n+2})\, f(xk) dk =$$

$$= \int_K \tau(\kappa(x^{-1}t)) e^{-2\delta H(x^{-1}t)} (i_n + i_{n+2})\, f(x\kappa(x^{-1}t))\, dt$$

As A normalize N,

$$\begin{aligned} x^{-1}t &= \kappa(x^{-1}t)e^{H(x^{-1}t)}n(x^{-1}t) \\ x\kappa(x^{-1}t) &= tn(x^{-1}t)^{-1}e^{-H(x^{-1}t)} \\ &= te^{-H(x^{-1}t)}n' \qquad \text{with } n' \in N \end{aligned}$$

So, $f\left(x\kappa(x^{-1}t)\right) = f\left(te^{-H(x^{-1}t)}n'\right) = e^{(\lambda+1)\delta H(x^{-1}t)}f(t)$. And

$$\begin{aligned} \int_K \tau(k)(i_n + i_{n+2})\, f(xk)dk &= \int_K \tau(\kappa(x^{-1}t))e^{(\lambda-1)\delta H(x^{-1}t)}(i_n + i_{n+2})\, f(t)\, dt \\ &= \int_K S(x,t)\, f(t)\, dt \qquad \square \end{aligned}$$

Proof of the Proposition 4.1. By the lemma 4.2 the Szegö map is G-equivariant for left regular actions. As $\tilde{\mathbf{D}}$ also commute with the action of G, it is enough to see that if $f \in \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$

$$\tilde{\mathbf{D}}(Sf)(1) = \alpha Sf(1)$$

If $f \in \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$, the image of f is in $W_\epsilon = \mathbf{C}w$ with $\epsilon = (-1)^n$, then $f(t) = h(t)w$ with h a complex valued function. So,

$$\begin{aligned} Sf(x) &= \int_K S(x,t)wh(t)\, dt \\ \tilde{\mathbf{D}}Sf(1) &= \int_K \tilde{\mathbf{D}}(S(x,t)w)_{x=1} h(t)\, dt \end{aligned}$$

from which we only need prove that

$$\begin{aligned} D(S(x,t)w)_{x=1} &= \alpha S(1,t)w \\ &= \alpha\, \tau(t)(i_n w + i_{n+2} w) \end{aligned}$$

Let X_1, X_2 be an orthonormal base of p. Then,

$$\begin{aligned} &\tilde{\mathbf{D}}(S(x,t)w)_{x=1} = \\ &= (I \otimes c)\left(\sum_{i=1}^{2}(X_i S(x,t)w)_{x=1} \otimes \bar{X}_i\right) \\ &= (I \otimes c)\left(\sum_{i=1}^{2} \frac{d}{du}\Big|_{u=0} e^{(\lambda-1)\delta H(\exp(-uX_i)t)}\tau(\kappa(\exp(-uX_i)t))\right)(i_n + i_{n+2})w \otimes \bar{X}_i \\ &= (I \otimes c)\left(\sum_{i=1}^{2} \frac{d}{du}\Big|_{u=0} e^{(\lambda-1)\delta H(exp(-u\mathrm{Ad}(t^{-1})X_i)}\tau(\kappa(t\exp(-uAd(t^{-1})X_i))\right) \\ &\qquad (i_n + i_{n+2})w \otimes \bar{X}_i \\ &= (I \otimes c)\left(\tau(t) \otimes Ad(t)\sum_{i=1}^{2}(Ad(t^{-1})X_i)S(1,1)w \otimes \overline{Ad(t^{-1})X_i}\right) \end{aligned}$$

As $\{Ad(t^{-1})X_i\}_{i=1,2}$ is another orthonormal base of p, and

$$\tau(t)(I\otimes c)=(I\otimes c)(\tau(t)\otimes Ad(t))$$

then

$$\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=\tau(t)\tilde{\mathbf{D}}(S(x,1)w)_{x=1}$$

So we must prove

$$\begin{aligned}\tilde{\mathbf{D}}(S(x,1)w)_{x=1}&=\alpha S(1,1)w\\&=\alpha(i_n+i_{n+2})w\end{aligned}$$

Let $\frac{1}{2}E_-,\frac{1}{2}E_+$ be the orthonormal base of p given in §1, then

$$\begin{aligned}&\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=\\&=(I\otimes c)\left(\frac{d}{du}\bigg|_{u=0}e^{(\lambda-1)\delta H(\exp(-u\frac{1}{2}E_-))}\tau(\kappa(\exp(-u\tfrac{1}{2}E_-)))(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+\right.\\&\left.\quad+\frac{d}{du}\bigg|_{u=0}e^{(\lambda-1)\delta H(\exp(-u\frac{1}{2}E_+))}\tau(\kappa(\exp(-u\tfrac{1}{2}E_+)))(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+\right)\end{aligned}$$

By (1.7)

$$\begin{aligned}\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=(I\otimes c)\Bigg(&-(\lambda-1)\delta\tfrac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+-\\&-(\lambda-1)\delta\tfrac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+-\\&-\tau\left(\tfrac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+-\\&-\tau\left(-\tfrac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_-\Bigg)\end{aligned}$$

By (4.2) and (4.3) applying $I\otimes c$, the following holds

$$c(\tfrac{1}{2}E_+)i_nw=c(\tfrac{1}{2}E_-)i_{n+2}w=0$$

and by (4.4)

$$\begin{aligned}c(\tfrac{1}{2}E_-)i_nw&=\tfrac{1}{2}a_n\,i_{n+2}w\\c(\tfrac{1}{2}E_+)i_{n+2}w&=-\tfrac{1}{2}\tfrac{1}{a_n}i_w\end{aligned}$$

So that

$$\begin{aligned}&\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=\\&=-\tfrac{1}{8}(-\lambda+1)\tfrac{1}{a_n}i_nw+\tfrac{1}{8}(-\lambda+1)a_ni_{n+2}w+\tfrac{1}{8}(n+2)\tfrac{1}{a_n}i_nw+\tfrac{1}{8}na_ni_{n+2}w\\&=\tfrac{1}{8}(\lambda+n+1)\tfrac{1}{a_n}i_nw+\tfrac{1}{8}(-\lambda+n+1)a_ni_{n+2}w\end{aligned}$$

because

$$\delta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 1$$

$$\tau_j\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} v = jv \quad \text{si } v \in V_{j\delta} \qquad j = n, n+2$$

The coefficients of $i_n w$ and $i_{n+2} w$ are

$$\begin{aligned}\tfrac{1}{8}(\lambda+n+1)\tfrac{1}{a_n} &= \tfrac{1}{8}(\lambda+n+1)\left(\frac{-\lambda+n+1}{\lambda+n+1}\right)^{\frac{1}{2}} sg\,\alpha\\ &= \tfrac{1}{8}\left(-\lambda^2+(n+1)^2\right)^{\frac{1}{2}} sg\,\alpha\\ &= \alpha\end{aligned}$$

$$\begin{aligned}\tfrac{1}{8}(-\lambda+n+1)a_n &= \tfrac{1}{8}\left(-\lambda^2+(n+1)^2\right)^{\frac{1}{2}} sg\,\alpha\\ &= \alpha\end{aligned}$$

That is,

$$\tilde{\mathbf{D}}(S(x,1)w)_{x=1} = \alpha\, S(1,1)w$$

Now, we will prove that the Sezgö map of parameters $(\lambda, n+1)$ for negative λ maps onto $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$. We know by proposition 3.1 that $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is irreducible. As S is non zero, if $\mathrm{Im}(S)$ is square integrable, then $\mathrm{Im}(S) = \mathrm{W}_\alpha(\tilde{\mathbf{D}})$. $\mathrm{Im}(S)$ is a subset of the eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ of the Dirac operator $\tilde{\mathbf{D}}$. But $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is a subset of $\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2)$. According with the notation of §2, as $\tilde{\mathbf{D}}^2$ differ with the Casimir operator Ω by a constant, $\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2)$ is isomorphic to $A_\lambda^n \oplus A_\lambda^{n+2}$. But the only quotient of $\mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ isomorphic to a subspace of $A_\lambda^n \oplus A_\lambda^{n+2}$ is infinitesimally equivalent to a discrete serie. Let $\phi \in \mathrm{Im}(S)$ in a non zero K-type, as the action of this K-type is one and the set of K-finite elements of the square integrable function space is a subset of the K-finite elements of the C^∞, then ϕ is square integrable. So $\mathrm{Im}(S)$ is a subset of $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$. The irreducibility concludes the proof. □

REFERENCES.

[A] M.Atiyah - "Elliptic Operators, Discrete Groups and von Neumann Algebras" Astérisque - Vol.32-33, 1976.

[A-S] M.Atiyah and W.Schmid - "A Geometric Construction of the Discrete Series for Semisimple Lie groups" Inventiones Mathematicae - Vol.42, 1977.

[B] W.Barker -"L^p Harmonic Analysis in $SL(2, \mathbf{L})$" Memoirs of the American Mathematical Society.

[C] Coddington y Levinson -"Theory of Ordinary Differencial Equations" Mc Graw Hill, New York, 1955.

[C-M] A.Connes and H.Moscovici - "The L^2-index Theorem for Homogeneous Spaces of Lie groups" Annals of Mathematics -Vol. 115, N.2, 1982.

[Ca-M] W.Casselman and D.Milicic - "Asymtotic Behavior of Matrix Coefficients of Admissible Representations" Duke Math.- J.49, 1982,869-930.

[G-V] E.Galina and J.Vargas - Eigenvalues and eigenvectors for the twisted Dirac operator over $SU(n,1)$ and $Spin(2n,1)$ accepted by Transactions of the American Math. Soc.

[H-1] S.Helgason -"A Duality for Simetric Spaces with Applications to Group Representations, II. Differencial Equations and Eigenspace Representations" Advances in Mathematics - Vol.22, N.2, 1976, 187-219.

[H-2] S.Helgason -"Group and Geometric Analysis" Academic Press,1984. Inc.

[K] A.Knapp -"Representation Theory of Real Reductive Groups" Princeton University Press. 1986.

[K-W] A.W.Knapp and N.R.Wallach -"Sezgö Kernels Associated with Discrete Series" Inventiones Mathematicae - Vol.34, F.3, 1976, 163-200.

[S] W.Schmid - "Boudary Value Problems for Group Invariant Differential Equations" Astérisque, hors serie, 1985.

[V] D.Vogan - "Representations of Real Reductive Lie Groups" Birkhäuser, Boston, 1981.

[Wa] Wallach, Harmonic analysis on homogeneous spaces, Marcell Deker, 1974.

[W] J.A.Wolf - "Essential Self Adjointness for the Dirac Operator and its Square" Indiana University Mathematical Journal-Vol 22, N.7 , Jan. 1973.

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