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COMPARISON OF TWO WEAK VERSIONS OF THE ORLICZ SPACES

Bibiana Iaffei

Presentado por Carlos Segovia Fernández

Abstract: In this work two versions of weak Orlicz spaces that appear in the literature, $\mathcal{M}_A$ and M_A, are analyzed. One of those include the weak Lebesgue spaces for $1 \leq p < \infty$, while the other version gives these spaces only for $p > 1$, resulting the stronger space L^1 in the extrem case $p = 1$. Necessary and sufficient conditions about the growth function A in order that both spaces coincide are given. Moreover we prove that these same conditions characterize the normability of the $\mathcal{M}_A$ space.

1.INTRODUCTION.

We shall denote by M_A the weak Orlicz space associated to A, defined as in the work of O'Neil, [O], where he makes use of this kind of functions to obtain a generalization of the Hardy-Littlewood-Sobolev's theorem on fractional integration into the context of Orlicz spaces. This version of weak Orlicz spaces generalizes the weak L^p spaces, L^p_*, but only for $p > 1$. In fact the class M_A for A the identity function gives a proper subspace of L^1_*.
Our aim in this work is to present an alternative definition of a weak Orlicz space associated to the function A, denoted by $\mathcal{M}_A$, in order to include all L^p_* for $1 \leq p < \infty$. In this way our spaces $\mathcal{M}_A$ give L^1_* for A the identity function and they coincide with M_A for $A(t) = t^p$, $p > 1$. Moreover we shall prove that both spaces are exactly the same as long as A keeps a "little bit away" from the identity. In fact we establish in theorem (4.8) the necessary and sufficient conditions on A to guarantee the equality $M_A = \mathcal{M}_A$.
We would like to point out that the spaces M_A are easier to handle since they are defined in terms of a norm while in turn, $\mathcal{M}_A$ is given by means of a quantity which is not necessarily a norm. It is well known that the weak L^p spaces are

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normable for $p > 1$ while L^1_* is not. Following this line we shall give in theorem (4.11) the necessary and sufficient conditions on A for $\mathcal{M}_A$ to be normable.
As a last remark we may say that the usefulness of one version or the other it would depend on the type of problem we are dealing with. On one side the spaces M_A seem to be the appropriate ones when generalizing the Hardy- Littlewood-Sobolev's theorem, while on the other side the spaces $\mathcal{M}_A$ would fit better for a theorem on interpolation of operators for example.

2.THE ORLICZ SPACES.

(2.1)Definition: *Along this work, for a Young function A we shall mean a non-negative, convex and non decreasing function defined on $[0, \infty]$ with $A(0) = 0$, $A(\infty) = \infty$ and such that it is neither identically zero nor identically infinity. We notice that A may have an jump at some $x_1 > 0$, but in this case $\lim_{x \to x_1^-} A(x) = \infty$ and $A(x) = \infty$ for $x \geq x_1$. Under these assumptions the inverse function A^{-1} is well defined and it is also increasing and continuous.*

We introduce now some notions related to the role of growth of non-negative functions as above.

(2.2)Definitions: *We shall say that two non-negative functions are equivalent if and only if their ratio is bounded above and bellow by two positive constants.*
A non negative function A defined on $\mathbb{R}^+$ is of lower type p (upper type p) if $A(st) \leq Cs^p A(t)$ for any $s \leq 1$ $(s \geq 1)$.

We notice that lower and upper types are preserved by equivalence of functions and also for any function we may choose another for which the definition of type is satisfied with $C = 1$. In particular A is of lower type zero if and only if is equivalent to a non decreasing function.

(2.3)Definition: *For a Young function A we define the Orlicz space $L_A = L_A(X)$ as the linear space of those measurable functions acting on the measure space (X, μ) for which there is a finite number $K > 0$ such that*

$$\int_X A\left(\frac{|f(x)|}{K}\right) d\mu \leq 1 \quad .$$

The infimun of such K is a norm which will be denoted by $\|f\|_A$.

3.WEAK ORLICZ SPACES.

For a complex or real valued and measurable f, defined on a measure space (X, μ), we will denote by $\mu_f(t)$ the distribution function of f given by

$$\mu_f(t) = \mu(\{x : |f(x)| > t\}).$$

Then for $t \in [0, \infty)$, $\mu_f(t)$ is a non increasing function taking non-negative values. Therefore we may define its inverse f^* by

$$f^*(s) = \inf\{t : \mu_f(t) \leq s\}$$

where $s \geq 0$. This function f^* usually called the non-increasing rearrangement of f, has the of property being equimeasurable with f in the sense that they share the distribution function.

By f^{**} we shall denote the average of f^* over the interval $[0, x]$, that is

$$f^{**}(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x f^*(t)dt & x > 0 \\ f^*(0) & x = 0. \end{cases}$$

Given a Young function A, it is possible to define a class of functions M_A in terms of the size of the f^{**}, wider than the Orlicz space L_A. The following definition of a version of weak Orlicz spaces is taken from the work of O'Neil [O], where the author used this class in connection with the boundedness of convolution operators on strong Orlicz spaces.

(3.1)Definition: *For a Young function A we will say that f defined on (X, μ) belongs to M_A if and only if there exists a real number λ large enough so that for $x > 0$*

$$f^{*}(x) \leq \lambda A^{-1}\left(\frac{1}{x}\right).$$

We define $\|f\|_{M_A}$ as the infimum of such λ. Therefore

$$\|f\|_{M_A} = \sup_{s>0} \frac{f^{**}(s)}{A^{-1}(1/s)} \quad .$$

In [O], O'Neil shows that the quantity $\|f\|_{M_A}$ is indeed a norm wich makes M_A a Banach space.

For $A(t) = t^p$ with $p > 1$, it is well known that M_A agrees with the space L^p_* or weak L^p, defined as those functions satisfying

$$\|f\|^*_p = \sup_{t>0} t^{1/p} f^*(t) < \infty$$

since for $1 < p \leq \infty$ both quantities $\|f\|^*_p$ and $\|f\|_{M_{t^p}}$, are in fact equivalent. Moreover it is known that for $p \geq 1$ the Lebesgue spaces $L^p(\mathbb{R}^n)$ are proper subspaces of $L^p_*(\mathbb{R}^n)$ (see for example [SW]. However the situation changes for $A(t) = t$, that is for $p = 1$. In this case the O'Neil version of weak L^1 is no longer the same that L^1_*; it rather coincides with the strong L^1 space. In fact if $A(t) = t$, $f \in M_A$ if and only if for some λ

$$f^{**}(x) \leq \lambda \frac{1}{x}$$

which means

$$\int_0^x f^*(t)dt \leq \lambda.$$

This is equivalent to f^* being integrable, that is, f in L^1.
At this point it appears in a natural way another version of weak Orlicz spaces as to include all the L^p_* spaces for $p \geq 1$.

(3.2)Definition: *We will say that a μ-measurable function f defined on X belongs to the weak Orlicz space $\mathcal{M}_A$ if and only if there is a constant C so that for $t > 0$*

$$A(t)\mu(\{x : |f(x)| > t\}) \leq C.$$

This definition implies that the quantity

$$\|f\|_{\mathcal{M}_A} = \inf\left\{\lambda > 0 / \sup_{t>0} \mu_f(\lambda t)A(t) \leq 1\right\}$$

is finite. Moreover the following properties hold

a) $\|cf\|_{\mathcal{M}_A} = |c| \quad \|f\|_{\mathcal{M}_A}$

b) $\|f + g\|_{\mathcal{M}_A} \leq 2(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})$

We notice that the factor 2 in b) does not allow to say that $\| \ \|_{\mathcal{M}_A}$ is a norm.
The proof of a) is immediate. On the other hand we observe that b) will follow if we are able to prove the inequality

$$\mu\left(\left\{\frac{|f(x) + g(x)|}{c(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t) \leq 1$$

for all $t > 0$. But

$$\mu\left(\left\{\frac{|f(x) + g(x)|}{c(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t) \leq \mu\left(\left\{\frac{|f(x)| + |g(x)|}{c(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t)$$

$$= \mu\left(\left\{\frac{|f(x)|}{c\|f\|_{\mathcal{M}_A}} \frac{\|f\|_{\mathcal{M}_A}}{(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} + \frac{|g(x)|}{c\|g\|_{\mathcal{M}_A}} \frac{\|g\|_{\mathcal{M}_A}}{(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t)$$

$$= \mu\left(\left\{\frac{|f(x)|}{c\|f\|_{\mathcal{M}_A}}\theta_1 + \frac{|g(x)|}{c\|g\|_{\mathcal{M}_A}}\theta_2 > t\right\}\right) A(t)$$

$$\leq \mu\left(\left\{\frac{|f(x)|}{c\|f\|_{\mathcal{M}_A}} > t\right\}\right) A(t) + \mu\left(\left\{\frac{|g(x)|}{c\|g\|_{\mathcal{M}_A}} > t\right\}\right) A(t)$$

since $\theta_1 + \theta_2 = 1$. The convexity of A implies $A(st) \leq sA(t)$ for $0 \leq s \leq 1$. Then, if $c \geq 1$, we can bound the above sum by

$$\mu\left(\left\{\frac{|f(x)|}{\|f\|_{\mathcal{M}_A}} > ct\right\}\right) \frac{A(ct)}{c} + \mu\left(\left\{\frac{|g(x)|}{\|g\|_{\mathcal{M}_A}} > ct\right\}\right) \frac{A(ct)}{c}$$

$$\leq \frac{1}{c} + \frac{1}{c}$$

which in turn is bounded by one as long as we take $c \geq 2$.■

4.RELATIONSHIP BETWEEN THE TWO DEFINITIONS.

As we already apointed out $L^1(\mathbb{R}^n)$ is a proper subespace of $L^1_*(\mathbb{R}^n)$. Consequently the spaces M_A and $\mathcal{M}_A$ are not always the same. Indeed when A is the identity function there are functions on $\mathbb{R}^n$ for which

$$\mu(\{x : |f(x)| > t\}) \leq \frac{C}{t}$$

for some finite constant C, even though they are not integrable. Such is the case of for example $f(x) = \frac{1}{|x|^n}$. However M_A is always a subspace of $\mathcal{M}_A$. In fact we have the following result.

(4.1)Lemma: *For any Young function A, we have*

$$M_A \subset \mathcal{M}_A \quad .$$

Moreover we have the inequality

$$\|f\|_{\mathcal{M}_A} \leq \|f\|_{M_A} \quad .$$

First we will find an expression for $\|f\|_{\mathcal{M}_A}$ in terms of the non increasing rearrengement of f. From this lemma (4.1) will be an obvious consequence.

(4.2)Lemma: *If f is a measurable function and by $\mu_f(t)$ and $f^*(s)$ we denote its distribution and rearrengement function, then the following identity holds*

$$\sup_{t>0} \mu_f(\lambda t)A(t) = \sup_{s>0} sA\left(\frac{f^*(s)}{\lambda}\right),$$

and hence

$$\|f\|_{\mathcal{M}_A} = \sup_{s>0} \frac{f^*(s)}{A^{-1}(1/s)}.$$

Proof:

First, let us assume that f is a non-negative simple function. Then it may be written as

$$f = \sum_{j=1}^{n} c_j \chi_{E_j},$$

where $\mu(E_j) > 0, E_j \cap E_k = \emptyset$ if $j \neq k$ and $c_1 > c_2 > ... > c_n > 0$. Set $d_j = \mu(E_1) + ... + \mu(E_j)$, $1 \leq j \leq n$, and let us define $d_0 = 0, c_{n+1} = 0$. Then, if we set $\mu_f(t) = |\{x : |f(x)| > t\}|$, this function and its inverse f^* are given by

$$\mu_f(\lambda t) = \begin{cases} d_j & \frac{c_{j+1}}{\lambda} \leq t < \frac{c_j}{\lambda} \\ 0 & t \geq c_1 \end{cases}$$

$$f^*(s) = \begin{cases} c_j & d_{j-1} \le s < d_j \\ 0 & s \ge d_n. \end{cases}$$

Therefore, using that A is non-decreasing we have

$$\sup_{t>0} A(t)\mu_f(\lambda t) = \sup_{j>0} A\left(\frac{c_j}{\lambda}\right) d_j = \sup_{s>0} A\left(\frac{f^*(s)}{\lambda}\right) s.$$

Now, for a general measurable function f, we can find a non-decreasing sequence of non-negative simple functions f_n such that $\lim_{n\to\infty} f_n(x) = |f(x)|$, for each x in the domain of f. Therefore, for each $t > 0$, the sequence $\{\mu_n(t)\}$ is non-decreasing and $\lim_{n\to\infty} \mu_n(t) = \mu(t)$, where μ_n and μ denote the distribution functions of f_n and f respectively. Likewise, for each $s > 0$ we also have that $f_n^*(s)$ increases to $f^*(s)$ and the first claim of the lemma follows immediately.
As for the second equality we just notice that

$$\|f\|_{\mathcal{M}_A} = \inf\left\{\lambda > 0 / \sup_{t>0} \mu_f(\lambda t) A(t) \le 1\right\}$$

$$(4.3) \qquad = \inf\left\{\lambda > 0 / \sup_{s>0} sA\left(\frac{f^*(s)}{\lambda}\right) \le 1\right\}$$

$$= \sup_{s>0} \frac{f^*(s)}{A^{-1}(1/s)} \quad .$$

where in the last equality we have used that $sA(\frac{f^*(s)}{\lambda}) \le 1$ is equivalent to $f^*(s) \le \lambda A^{-1}(\frac{1}{s})$.■

Proof of lemma (4.1):
From of definition of f^{**} it follows that for any $s > 0$ we have $f^*(s) \le f^{**}(s)$. This observation together with lemma (4.2) give the desired conclusion.■

As we shall see the difference between the spaces M_A and $\mathcal{M}_A$ may appear in other cases besides $A(t) = t$. In fact if for $x > 0$ we denote by $\log^+ x$ the maximum between $\log t$ and zero and for $x \in \mathbb{R}^n$ we take the function

$$f(x) = \frac{9e^2}{\omega_n |x|^n (3 + \log^+(\frac{1}{\omega_n |x|^n}))^2}$$

then f belongs to the space $\mathcal{M}_A$ for $A(t)$ such that $A^{-1}(t) = 9e^2 t(3 + \log^+ t)^{-2}$. First, $A(t)$ is a Young function because we have chosen the constants in such a way that A^{-1} is increasing, continuous and concave on $[0, \infty]$. Also,it is not hard to check that $A(t)$ behaves at infinity like $t(\log^+ t)^2$. Second, for any increasing function A^{-1}, the function defined on $I\!R^n$ by $f(x) = A^{-1}\left(\frac{1}{\omega_n |x|^n}\right)$ is such that $f^*(s) = A^{-1}(1/s)$ proving our assertion that $f \in \mathcal{M}_A$. Finally let us see that f is not in M_A. If it were, there would be a constant $\lambda > 0$ such that

$$\frac{1}{s}\int_0^s A^{-1}(1/t)dt \leq \lambda A^{-1}(1/s).$$

But then, for any $s < 1$ we have

$$\int_0^s \frac{9e^2}{t(3+\log(1/t))^2} dt = -9e^2 \int_\infty^{-\log s} \frac{1}{(3+u)^2} du$$

$$= 9e^2(3 - \log s)^{-1}.$$

This together with our assumption would lead to

$$9e^2(3+\log(1/s))^{-1} < \lambda 9e^2 (3+\log(1/s))^{-2}$$

for some $\lambda > 0$. But this impossible because it would imply that $-\log s$ is a bounded function on (0,1).
This example shows that when $X = I\!R^n$ and μ is the Lebesgue measure there are other Young functions different from $A(t) = t$ for which the space M_A is strictly contained in $\mathcal{M}_A$. In our next step we will characterize all the Young functions for which both spaces are exactly the same. In what follows we shall restrict ourselves to the case of $X = I\!R^n$ with μ the Lebesgue measure. Nevertheless the main results contained in theorems (4.8) and (4.11) could also be derived working in more general measure spaces.

We start by giving two real functions lemmas; the first can be found in [M], and the second is an stronger version of a result proved by Viviani in [V]. This last result will be an essential tool in looking for necessary and sufficient conditions on A to ensure that $M_A = \mathcal{M}_A$.

(4.4)Lemma: *Let $h(t)$ be a non negative and non decreasing function on $[0, j]$ for which there exists a constant D such that for $0 \leq s \leq j/20$, $\int_0^s h(t)dt \leq Dsh(s)$. Then if $1 \leq r < D/(D-1)$,*

$$\int_0^j [h(t)]^r dt \leq \frac{(20)^r j^{1-r} D}{D - r(D-1)} \left[\int_0^j h(t)dt\right]^r. \tag{4.5}$$

(4.6)Lemma: *Let η be a non negative function such that $\frac{\eta(t)}{t}$ is non increasing. Then $\eta(t)$ is equivalent to $\tilde{\eta}(t) = \int_0^t \frac{\eta(s)}{s} ds$ if and only if η has a positive lower type.*

Proof:

Since $\frac{\eta(t)}{t}$ is non increasing the inequality $\eta(t) \leq \int_0^t \frac{\eta(s)}{s} ds$ is always true no matter what the lower type of η is. Also, the fact that the inequality $\int_0^t \frac{\eta(s)}{s} ds \leq C\eta(t)$ holds whenever η is of positive lower type is proved in [V]. Conversely the equivalence between η and $\tilde{\eta}$ implies that $\int_0^t h(s)ds \leq Cth(t)$, for $h(t) = \frac{\eta(t)}{t}$ and $\forall t > 0$. This allows us to use (4.5) from Muckenhoupt lemma for any finite interval in order to obtain that η is of positive lower type. In fact, if $r > 1$, as in the conclusion of the previous lemma, $0 < u \leq 1$ and $s > 0$ we have

$$ush^r(us) \leq \int_0^{us} h^r(t)dt \leq \int_0^s h^r(t)dt \leq Cs^{1-r}\left[\int_0^s h(t)dt\right]^r \leq Csh^r(s)$$

Therefore

$$h(us) \leq C\left(\frac{1}{u}\right)^{\frac{1}{r}} h(s)$$

$$\frac{\eta(us)}{us} \leq C\left(\frac{1}{u}\right)^{\frac{1}{r}} \frac{\eta(s)}{s}.$$

Since $r > 1$ we arrive to the desired conclusion.■

Now we make an useful remark on the relationship between the types of a Young function and its inverse.

(4.7)Lemma: *Let A be a Young function. Then A has a lower type m if and only if A^{-1} has an upper type $1/m$.*

Proof:

The Young function A has a lower type m if and only if there is a constant $C > 0$ such that

$$A(st) \leq Ct^m A(s) \quad \text{for any} \quad 0 < t \leq 1 \ .$$

Now taking a pair $t \leq s$ the latter inequality can be written

$$A(t) = A(s\frac{t}{s}) \leq C\left(\frac{t}{s}\right)^m A(s)$$

which is equivalent to say

$$\frac{A(t)}{t^m} \leq C\frac{A(s)}{s^m}$$

for any $t \leq s$. Setting $\alpha = A(t)$ and $\beta = A(s)$, by the continuity of A the above inequality can be written

$$\frac{\alpha}{[A^{-1}(\alpha)]^m} \leq C\frac{\beta}{[A^{-1}(\beta)]^m} \quad .$$

Since A is non decreasing we get that the inequality

$$\frac{A^{-1}(\beta)}{\beta^{\frac{1}{m}}} \leq C\frac{A^{-1}(\alpha)}{\alpha^{\frac{1}{m}}}$$

holds for any $\alpha \leq \beta$, but this is to say that A^{-1} has an upper type $\frac{1}{m}$.■

Now we are in position to state and prove the anounced characterization.

(4.8)Theorem: *Let A be a Young function. Then the following statements are equivalent*

i) $M_A = \mathcal{M}_A$,
ii) $\frac{1}{s}\int_0^s A^{-1}(1/t)dt$ is equivalent with $A^{-1}(1/s)$,
iii) A has a lower type greater than one 1.

Proof:

Let us assume i) is true. Since by (4.1) $M_A \subset \mathcal{M}_A$ always holds, we must obtain ii) from $\mathcal{M}_A \subset M_A$. Take the function $f(x) = A^{-1}\left(\frac{1}{\omega_n|x|^n}\right)$; since it is radial and non increasing it is easy to check that its rearrengement is $f^*(s) = A^{-1}(\frac{1}{s})$ and hence $f \in \mathcal{M}_A$. Now, our hypothesis implies that f belongs also to M_A which means that for some $\lambda > 0$ the inequality

$$\frac{1}{s}\int_0^s A^{-1}\left(\frac{1}{t}\right) = f^{**}(s) \leq \lambda A^{-1}\left(\frac{1}{s}\right)$$

holds for any $s > 0$ giving one of the inequalities in ii). Finally, the other inequality follows using that $A^{-1}(1/t)$ is a non increasing function.
To check that ii) $\Rightarrow$ iii) we set $\eta(t) = tA^{-1}(1/t)$ and we make use of lemma (4.6) to conclude that η has a positive lower type, say a. Therefore we have

$$\eta(ut) = utA^{-1}\left(\frac{1}{ut}\right) \leq Cu^a tA^{-1}\left(\frac{1}{t}\right) \quad (0 < u \leq 1, t > 0 \text{ y } y\ a > 0)$$

which implies

$$A^{-1}\left(\frac{1}{ut}\right) \leq Cu^{a-1}A^{-1}\left(\frac{1}{t}\right) \quad (0 < u \leq 1, t > 0 \text{ and } a > 0)$$

setting $\sigma = \frac{1}{u}$ and $z = \frac{1}{t}$ the above expresion is equivalent to

$$A^{-1}(\sigma z) \leq C\sigma^{1-a}A^{-1}(z) \quad (\sigma \geq 1, z > 0 \text{ y } a > 0)$$

which means that A^{-1} has an upper type less than one. By using now Lemma 4.7 we may conclude that A has lower type greater than one.
In order to prove iii) $\Rightarrow$ ii) we use again lemma (4.7) to conclude that A^{-1} has an upper type, say b, less than one and that, in consequence, the function $\eta(t) = tA^{-1}(1/t)$ has a positive lower type. In fact, if $0 < u \leq 1$ and $t > 0$ we have

$$\eta(ut) = utA^{-1}\left(\frac{1}{ut}\right) \leq Cut\left(\frac{1}{u}\right)^{b} A^{-1}\left(\frac{1}{t}\right) = Cu^{1-b}\eta(t).$$

Since $1 - b > 0$ we may apply lemma (4.6) to get ii).
It remains to prove that ii)$\Rightarrow$i). First we observe that by lemma 4.1 it is enough to check $\mathcal{M}_A \subset M_A$. Let us assume $f \in \mathcal{M}_A$, that is $f^*(s) \leq \lambda A^{-1}\left(\frac{1}{s}\right)$. Then we have

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt \leq \frac{\lambda}{x}\int_0^x A^{-1}\left(\frac{1}{s}\right) ds.$$

But, using ii) we get

$$f^{**}(x) \leq KA^{-1}\left(\frac{1}{x}\right)$$

and hence $f \in M_A$.■

(4.9)Corollary: *If A has a lower type greater than one, then there exists a constant C such that*

$$\|f\|_{M_A} \leq C\|f\|_{\mathcal{M}_A}$$

holds for any $f \in \mathcal{M}_A$ and moreover $\mathcal{M}_A$ is normable.

Proof:

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt$$

$$= \frac{1}{x}\int_0^x \frac{f^*(t)}{A^{-1}(1/t)}A^{-1}(1/t)dt$$

$$\leq C\|f\|_{\mathcal{M}_A}\frac{1}{x}\int_0^x A^{-1}(1/t)dt$$

$$\leq C\|f\|_{\mathcal{M}_A}A^{-1}(1/x),$$

where we have used iii) $\Rightarrow$ ii) from theorem (4.8). Taking supremum over all x, we get

$$\|f\|_{M_A} \leq C\|f\|_{\mathcal{M}_A}.$$

Finally, since by lemma (4.1) the reverse inequality between $\|f\|_{M_A}$ and $\|f\|_{\mathcal{M}_A}$ always holds, our space $\mathcal{M}_A$ is normable so that the proof of the corollary is complete.■

(4.10) Remark: As we have just seen the space $\mathcal{M}_A$ is normable, with the norm $\|.\|_{M_A}$, whenever A has a lower type greater than one. For A a Young function without this property (i.e. A has lower type one and no greater than) we already know that our space $\mathcal{M}_A$ is much bigger than M_A and consequently the quantity $\|.\|_{\mathcal{M}_A}$ is not longer equivalent to the norm $\|.\|_{M_A}$. A natural question then arises: is there a norm on the space $\mathcal{M}_A$ equivalent to the quantity $\|.\|_{\mathcal{M}_A}$?. In other words we would like to know whether or not this spaces $\mathcal{M}_A$ are normable for Young functions A without a lower type greater than one. It is known that the space L^1_* is not normable. Our next result shows that this situation extends to all $\mathcal{M}_A$ with A having a lower type at most one.

(4.11) Theorem: *Let A be a Young function. Then the weak Orlicz space $\mathcal{M}_A$ is normable, with a norm equivalent to $\|.\|_{\mathcal{M}_A}$ if and only if A has a lower type greater than one.*

Proof:

By corollary (4.9) we only have to show that $\mathcal{M}_A$ normable implies that A must have a lower type greater than one. For simplicity we will work out the proof only in the one dimensional case. For higher dimensions it follows the same lines. For given $s > 0$ and $N \in \mathbb{N}$ we define the function

$$f(x) = \sum_{k=1}^{N} A^{-1}\left(\frac{1}{2|x - \frac{ks}{N}|}\right) \quad .$$

If we call $f_{k,s}(x) = A^{-1}\left(\frac{1}{2|x-\frac{ks}{N}|}\right)$ it is easy to check that they all belong to $\mathcal{M}_A$ for any $1 \leq k \leq N$ and $s > 0$ and moreover we have $\|f_{k,s}\|_{\mathcal{M}_A} \leq 1$ since all of these functions sheare the same distribution $\frac{1}{A(t)}$. Therefore, if by $\|.\|$ we denote a norm equivalent to the quantity $\|.\|_{\mathcal{M}_A}$, we get

$$\|f\| \leq \sum_{k=1}^{N} \|f_{k,s}\| \leq C_1 \sum_{k=1}^{N} \|f_{k,s}\|_{\mathcal{M}_A} \leq C_1 N \quad .$$

However, elementary computations show that the derivative of f is negative on $[0, s + \frac{s}{N}]$ which implies that $f(x) \leq f(0)$ for $x \in [0, s + \frac{s}{N}]$, . Then if we set

$$H_{N,s} = f(0) = A^{-1}(\frac{1}{2}\frac{N}{s}) + A^{-1}(\frac{1}{2}\frac{N}{2s}) + ... + A^{-1}(\frac{1}{2}\frac{1}{s})$$

we obtain

$$1 < \mu_f(H_{N,s})\frac{1}{s} = \mu_f(\lambda_{N,s} t_s) A(t_s)$$

where $t_s = A^{-1}(\frac{1}{s})$ y $\lambda_{N,s} = \frac{H_{N,s}}{A^{-1}(\frac{1}{s})}$. Then

$$\lambda_{N,s} \leq \|f\|_{\mathcal{M}_A} \leq C_2 N \ .$$

Thus

$$H_{N,s} \leq C_2 N A^{-1}(\frac{1}{s})$$

and

$$H_{N,s} = \sum_{k=1}^{N} A^{-1}(\frac{1}{2}\frac{N}{ks}) \geq \frac{N}{s} \sum_{k=1}^{N} \int_{\frac{s}{N}k}^{\frac{s}{N}(k+1)} A^{-1}(\frac{1}{2u})du.$$

Since A^{-1} is non decreasing we obtain

$$\frac{N}{s} \int_{\frac{s}{N}}^{\frac{s}{N}(N+1)} A^{-1}(1/2u)du \leq C_2 N A^{-1}(1/s) \ .$$

Letting N go to infinity we get that for any fixed $s > 0$

$$\int_0^s A^{-1}(1/2u)du \leq C_2 s A^{-1}(1/s).$$

Changing variables $v = 2u$ we get

$$\int_0^{2s} A^{-1}(1/v)dv \leq C s A^{-1}(1/s).$$

Finally since A^{-1} is non negative we arrived to the inequality

$$\int_0^s A^{-1}(1/v)dv \leq C s A^{-1}(1/s)$$

which by theorem (4.8), implies that A has a lower type greater than one.∎

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